Contents

Series preface vii

Preface ix

Part I Modeling of mechanical systems

1 Introductory examples and problems 3
1.1 Rigid body systems 4
1.2 Manipulators and multi-body systems 6
1.3 Constrained mechanical systems 8
1.4 Bibliographical notes 10

2 Linear and multilinear algebra 15
2.1 Basic concepts and notation 15
2.1.1 Sets and set notation 16
2.1.2 Number systems and their properties 16
2.1.3 Maps 17
2.1.4 Relations 19
2.1.5 Sequences and permutations 19
2.1.6 Zorn's Lemma 20
2.2 Vector spaces 21
2.2.1 Basic definitions and concepts 21
2.2.2 Linear maps 24
2.2.3 Linear maps and matrices 26
2.2.4 Invariant subspaces, eigenvalues, and eigenvectors 29
2.2.5 Dual spaces 30
2.3 Inner products and bilinear maps 33
2.3.1 Inner products and norms 33
2.3.2 Linear maps on inner product spaces 35
2.3.3 Bilinear maps 36

2.3.4 Linear maps associated with bilinear maps 39
2.4 Tensors 40
2.4.1 Basic definitions 41
2.4.2 Representations of tensors in bases 42
2.4.3 Behavior of tensors under linear maps 43
2.5 Convexity 44

3 Differential geometry 49
3.1 The prelude to differential geometry 50
3.1.1 Topology 51
3.1.2 Calculus in $\mathbb{R}^n$ 56
3.1.3 Convergence of sequences of maps 59
3.2 Manifolds, maps, and submanifolds 60
3.2.1 Charts, atlases, and differentiable structures 60
3.2.2 Maps between manifolds 66
3.2.3 Submanifolds 68
3.3 Tangent bundles and more about maps 70
3.3.1 The tangent bundle 70
3.3.2 More about maps 73
3.4 Vector bundles 77
3.4.1 Vector bundles 78
3.4.2 Tensor bundles 83
3.5 Vector fields 84
3.5.1 Vector fields as differential operators 85
3.5.2 Vector fields and ordinary differential equations 89
3.5.3 Lifts of vector fields to the tangent bundle 94
3.6 Tensor fields 95
3.6.1 Covector fields 96
3.6.2 General tensor fields 98
3.7 Distributions and codistributions 104
3.7.1 Definitions and basic properties 104
3.7.2 Integrable distributions 105
3.7.3 The Orbit Theorem for distributions 108
3.7.4 Codistributions 110
3.8 Affine differential geometry 111
3.8.1 Definitions and general concepts 112
3.8.2 The Levi-Civita affine connection 114
3.8.3 Coordinate formulae 116
3.8.4 The symmetric product 118
3.9 Advanced topics in differential geometry 119
3.9.1 The differentiable structure of an immersed submanifold 120
3.9.2 Comments on smoothness, in particular analyticity 121
3.9.3 Properties of generalized subbundles 123
3.9.4 An alternative notion of distribution 125
3.9.5 Fiber bundles 130

3.9.6 Additional topics in affine differential geometry 131

4 Simple mechanical control systems 141
4.1 The configuration manifold 143
4.1.1 Interconnected mechanical systems 143
4.1.2 Finding the configuration manifold 146
4.1.3 Choosing coordinates 152
4.1.4 The forward kinematic map 155
4.1.5 The tangent bundle of the configuration manifold 157
4.2 The kinetic energy metric 162
4.2.1 Rigid bodies 162
4.2.2 The kinetic energy of a single rigid body 166
4.2.3 From kinetic energy to a Riemannian metric 168
4.3 The Euler–Lagrange equations 172
4.3.1 A problem in the calculus of variations 173
4.3.2 Necessary conditions for minimization—the Euler–Lagrange equations 174
4.3.3 The Euler–Lagrange equations and changes of coordinate 176
4.3.4 The Euler–Lagrange equations on a Riemannian manifold 178
4.3.5 Physical interpretations 182
4.4 Forces 187
4.4.1 From rigid body forces and torques to Lagrangian forces 188
4.4.2 Definitions and examples of forces in Lagrangian mechanics 189
4.4.3 The Lagrange–d'Alembert Principle 193
4.4.4 Potential forces 195
4.4.5 Dissipative forces 198
4.5 Nonholonomic constraints 198
4.5.1 From rigid body constraints to a distribution on Q 199
4.5.2 Definitions and basic properties 200
4.5.3 The Euler–Lagrange equations in the presence of constraints 204
4.5.4 Simple mechanical systems with constraints 207
4.5.5 The constrained connection 209
4.5.6 The Poincaré representation of the equations of motion 213
4.5.7 Special features of holonomic constraints 215
4.6 Simple mechanical control systems and their representations . . 218
4.6.1 Control-affine systems 218
4.6.2 Classes of simple mechanical control systems 221
4.6.3 Global representations of equations of motion 224
4.6.4 Local representations of equations of motion 225
4.6.5 Linear mechanical control systems 227
4.6.6 Alternative formulations 229

5 Lie groups, systems on groups, and symmetries 247
5.1 Rigid body kinematics 248
5.1.1 Rigid body transformations 249
5.1.2 Infinitesimal rigid body transformations 252
5.1.3 Rigid body transformations as exponentials of twists . . . 254
5.1.4 Coordinate systems on the group of rigid displacements 255
5.2 Lie groups and Lie algebras 258
5.2.1 Groups 258
5.2.2 From one-parameter subgroups to matrix Lie algebras . 261
5.2.3 Lie algebras 263
5.2.4 The Lie algebra of a Lie group 265
5.2.5 The Lie algebra of a matrix Lie group 268
5.3 Metrics, connections, and systems on Lie groups 271
5.3.1 Invariant metrics and connections 271
5.3.2 Simple mechanical control systems on Lie groups 275
5.3.3 Planar and three-dimensional rigid bodies as systems on Lie groups 277
5.4 Group actions, isometries, and symmetries 283
5.4.1 Group actions and infinitesimal generators 283
5.4.2 Isometries 288
5.4.3 Symmetries and conservation laws 290
5.4.4 Examples of mechanical systems with symmetries 293
5.5 Principal bundles and reduction 296
5.5.1 Principal fiber bundles 297
5.5.2 Reduction by an infinitesimal isometry 298

Part II Analysis of mechanical control systems

6 Stability 313
6.1 An overview of stability theory for dynamical systems 315
6.1.1 Stability notions 315
6.1.2 Linearization and linear stability analysis 317
6.1.3 Lyapunov Stability Criteria and LaSalle Invariance Principle 319
6.1.4 Elements of Morse theory 325
6.1.5 Exponential convergence 327
6.1.6 Quadratic functions 329
6.2 Stability analysis for equilibrium configurations of mechanical systems 331
6.2.1 Linearization of simple mechanical systems 331
6.2.2 Linear stability analysis for unforced systems 334
6.2.3 Linear stability analysis for systems subject to Rayleigh dissipation 336
6.2.4 Lyapunov stability analysis 340

6.2.5 Global stability analysis . . . 344
6.2.6 Examples illustrating configuration stability results . . . 345
6.3 Relative equilibria and their stability . . . 349
6.3.1 Existence and stability definitions . . . 349
6.3.2 Lyapunov stability analysis . . . 351
6.3.3 Examples illustrating existence and stability of relative equilibria . . . 355
6.3.4 Relative equilibria for simple mechanical systems on Lie groups . . . 357

7 Controllability . . . 367
7.1 An overview of controllability for control-affine systems . . . 368
7.1.1 Reachable sets . . . 369
7.1.2 Notions of controllability . . . 371
7.1.3 The Sussmann and Jurdjevic theory of attainability . . . 372
7.1.4 From attainability to accessibility . . . 374
7.1.5 Some results on small-time local controllability . . . 377
7.2 Controllability definitions for mechanical control systems . . . 387
7.3 Controllability results for mechanical control systems . . . 389
7.3.1 Linearization results . . . 390
7.3.2 Accessibility of affine connection control systems . . . 392
7.3.3 Controllability of affine connection control systems . . . 394
7.4 Examples illustrating controllability results . . . 398
7.4.1 Robotic leg . . . 398
7.4.2 Planar body with variable-direction thruster . . . 400
7.4.3 Rolling disk . . . 402

8 Low-order controllability and kinematic reduction . . . 411
8.1 Vector-valued quadratic forms . . . 412
8.1.1 Basic definitions and properties . . . 412
8.1.2 Vector-valued quadratic forms and affine connection control systems . . . 414
8.2 Low-order controllability results . . . 415
8.2.1 Constructions concerning vanishing input vector fields . 416
8.2.2 First-order controllability results . . . 417
8.2.3 Examples and discussion . . . 420
8.3 Reductions of affine connection control systems . . . 422
8.3.1 Inputs for dynamic and kinematic systems . . . 422
8.3.2 Kinematic reductions . . . 424
8.3.3 Maximally reducible systems . . . 429
8.4 The relationship between controllability and kinematic controllability . . . 432
8.4.1 Implications . . . 433
8.4.2 Counterexamples . . . 434

9 Perturbation analysis 441
9.1 An overview of averaging theory for oscillatory control systems 442
9.1.1 Iterated integrals and their averages 443
9.1.2 Norms for objects defined on complex neighborhoods . . 446
9.1.3 The variation of constants formula 447
9.1.4 First-order averaging 451
9.1.5 Averaging of systems subject to oscillatory inputs 454
9.1.6 Series expansion results for averaging 459
9.2 Averaging of affine connection systems subject to oscillatory controls 463
9.2.1 The homogeneity properties of affine connection control systems 463
9.2.2 Flows for homogeneous vector fields 466
9.2.3 Averaging analysis 466
9.2.4 Simple mechanical control systems with potential control forces 471
9.3 A series expansion for a controlled trajectory from rest 473

Part III A sampling of design methodologies

10 Linear and nonlinear potential shaping for stabilization 481
10.1 An overview of stabilization 482
10.1.1 Defining the problem 483
10.1.2 Stabilization using linearization 485
10.1.3 The gaps in linear stabilization theory 487
10.1.4 Control-Lyapunov functions 489
10.1.5 Lyapunov-based dissipative control 490
10.2 Stabilization problems for mechanical systems 493
10.3 Stabilization using linear potential shaping 495
10.3.1 Linear PD control 495
10.3.2 Stabilization using linear PD control 497
10.3.3 Implementing linear control laws on nonlinear systems . 501
10.3.4 Application to the two-link manipulator 505
10.4 Stabilization using nonlinear potential shaping 507
10.4.1 Nonlinear PD control and potential energy shaping 507
10.4.2 Stabilization using nonlinear PD control 509
10.4.3 A mathematical example 515
10.5 Notes on stabilization of mechanical systems 515
10.5.1 General linear techniques 516
10.5.2 Feedback linearization and partial feedback linearization 517
10.5.3 Backstepping 517
10.5.4 Passivity-based methods 518
10.5.5 Sliding mode control 518
10.5.6 Total energy shaping methods 519

10.5.7 When stabilization by smooth feedback is not possible . 520

11 Stabilization and tracking for fully actuated systems 529
11.1 Configuration stabilization for fully actuated systems 530
11.1.1 Stabilization via configuration error functions 530
11.1.2 PD control for a point mass in three-dimensional Euclidean space 532
11.1.3 PD control for the spherical pendulum 533
11.2 Trajectory tracking for fully actuated systems 534
11.2.1 Time-dependent feedback control and the tracking problem 534
11.2.2 Tracking error functions 535
11.2.3 Transport maps 536
11.2.4 Velocity error curves 538
11.2.5 Proportional-derivative and feedforward control 540
11.3 Examples illustrating trajectory tracking results 542
11.3.1 PD and feedforward control for a point mass in three-dimensional Euclidean space 542
11.3.2 PD and feedforward control for the spherical pendulum 543
11.4 Stabilization and tracking on Lie groups 546
11.4.1 PD control on Lie groups 547
11.4.2 PD and feedforward control on Lie groups 548
11.4.3 The attitude tracking problem for a fully actuated rigid body fixed at a point 552

12 Stabilization and tracking using oscillatory controls 559
12.1 The design of oscillatory controls 560
12.1.1 The averaging operator 560
12.1.2 Inverting the averaging operator 563
12.2 Stabilization via oscillatory controls 567
12.2.1 Stabilization with the controllability assumption 568
12.2.2 Stabilization without the controllability assumption 571
12.3 Tracking via oscillatory controls 574

13 Motion planning for underactuated systems 583
13.1 Motion planning for driftless systems 584
13.1.1 Definitions 584
13.1.2 A brief literature survey of synthesis methods 587
13.2 Motion planning for mechanical systems 589
13.2.1 Definitions 589
13.2.2 Kinematically controllable systems 590
13.2.3 Maximally reducible systems 591
13.3 Motion planning for two simple systems 593
13.3.1 Motion planning for the planar rigid body 593
13.3.2 Motion planning for the robotic leg 596

13.4 Motion planning for the snakeboard 598
13.4.1 Modeling 598
13.4.2 Motion planning on $\mathsf{SE}(2)$ for the snakeboard 605
13.4.3 Simulations 612

A Time-dependent vector fields 619
A.1 Measure and integration 619
A.1.1 General measure theory 619
A.1.2 Lebesgue measure 621
A.1.3 Lebesgue integration 622
A.2 Vector fields with measurable time-dependence 624
A.2.1 Carathéodory sections of vector bundles and bundle maps 624
A.2.2 The time-dependent Flow Box Theorem 625

B Some proofs 627
B.1 Proof of Theorem 4.38 627
B.2 Proof of Theorem 7.36 629
B.3 Proof of Lemma 8.4 635
B.4 Proof of Theorem 9.38 638
B.5 Proof of Theorem 11.19 648
B.6 Proof of Theorem 11.29 652
B.7 Proof of Proposition 12.9 654

References 657

Symbol index 689

Subject index 705