1

Introductory examples and problems

To motivate the kinds of problems we consider in this book, we use this introductory chapter to present some simple examples for which the problems are fairly easily understood. The short presentation in this chapter is more informal than will be encountered in the remainder of the book. We draw the examples in this chapter from three loosely defined collections of physical systems: aerospace and underwater vehicles, robotic manipulators and multi-body systems, and constrained systems. These examples will give us an opportunity to discuss a variety of topics, and motivate the introduction of appropriate mathematical tools.

For each example, we will introduce the notion of degrees-of-freedom, configuration, velocity, state variables, forces, and constraints. We also pose some natural control theoretic questions that arise naturally for the example systems. This allows us to introduce some of the sorts of questions we address in the book, although we delay answering these questions for the actual text of the book.

Many readers might be familiar with "vector mechanics." This sort of mechanics is useful for modeling, say, a point mass moving in the plane $\mathbb{R}^2$ or in the three-dimensional space $\mathbb{R}^3$. We refer to these vector spaces as Euclidean spaces. However, only in exceptional circumstances can the configuration of a Lagrangian system be described by a vector in a vector space. In the natural mathematical setting, the system's configuration space is described loosely as a curved space, or more accurately as a differentiable manifold. In Chapter 4 we shall be precise about what we mean by this, and about how the mathematical objects of differential geometry represent the physical objects of mechanics. In this chapter we will be a little vague and descriptive about what we might mean by this correspondence. We hope, however, that the illustrative character of the examples we give exhibits the value of taking an approach that will unify all of these examples in one framework.

We close the chapter with a broad overview of the research literature on topics related to those we cover. A reader who is new to the subject can look here to get started in reading the literature. In the text of the book are contained many more references.

1.1 Rigid body systems

The first example we consider is from the class of systems modeled by rigid bodies. In applications, such systems include many aerospace and marine vehicle systems. The system we consider here is about the simplest example in this class, and can be thought of as a model for a simplified hovercraft, as depicted in Figure 1.1. Let us go through the elements of the model. The planar

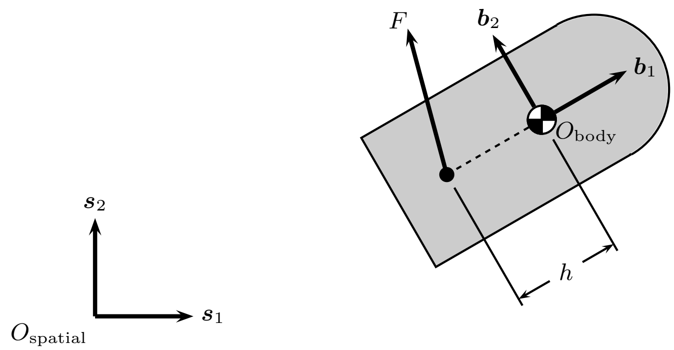

Figure 1.1. Planar rigid body

system has obviously three degrees-of-freedom, since it can translate in the plane, and rotate about its center of mass. The configuration is given by the following variables: θ describes the relative orientation of the body reference frame $\Sigma_{\text{body}} = (O_{\text{body}}, \{\boldsymbol{b}_1, \boldsymbol{b}_2\})$ with respect to the inertial reference frame $\Sigma_{\text{spatial}} = (O_{\text{spatial}}, \{\boldsymbol{s}_1, \boldsymbol{s}_2\})$. The vector (x, y) denotes the position of the center of mass measured with respect to the inertial reference frame Σ_{spatial}. We shall write $q = (\theta, x, y)$, but sometimes it will also be convenient to represent q as the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}.$$

This representation emphasizes the matrix group structure that the configuration space enjoys; we refer the reader to the discussion in Chapter 5 where this structure is examined systematically under the name of "homogeneous representation."

The velocity of the system can be written either with respect to the inertial coordinate system Σ_{spatial}, or with respect to the body-fixed coordinate system

Σ_{body}. If we denote the components of the spatial velocity as $\dot{q} = (\dot{\theta}, \dot{x}, \dot{y})$ and the components of the body velocity as (ω, v_x, v_y), we have

$$\begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \omega \\ v_x \\ v_y \end{bmatrix}.$$

The state of the system is given by q along with its spatial velocity $\dot{q}$. However, it will sometimes be convenient to work with body rather than spatial velocities, although the reasons for this choice are not obvious, at this time.

Having determined the system's state, it is possible now to present the system's total energy as the sum of kinetic and potential energy. The kinetic energy is equal to

$$\mathrm{KE} = \frac{1}{2}\dot{q}^T \mathbb{G}(q)\dot{q}, \quad \text{where} \quad \mathbb{G}(q) = \begin{bmatrix} J & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix},$$

and where m is the mass of the body and J is its moment of inertia about the center of mass. If we assume that the body moves in a plane perpendicular to the direction of the gravitational forces, the potential energy is zero. As we shall see in Section 4.3, the kinetic and potential energies together allow us to write the equations of motion for the system, at least in the absence of external forces.

Since our interest is primarily with control systems, we will certainly have external forces, and these will be forces that the user can specify. For the planar body, the force we consider is applied to a point on the body that is a distance $h > 0$ from the center of mass, along the body $\boldsymbol{b}_1$-axis, as shown in Figure 1.1. Physically, this force might be thought of as being supplied by a variable-direction thruster on the body. By resolving the force into components in the body $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ directions, we consider this as a two-input system. In this case, one can readily ascertain, either "by hand," or by applying the methods of Section 4.3, that the equations of motion are

$$\begin{aligned} J\ddot{\theta} &= -hu_2, \\ m\ddot{x} &= u_1\cos\theta - u_2\sin\theta, \\ m\ddot{y} &= u_1\sin\theta + u_2\cos\theta, \end{aligned}$$

where u_a is the component of F in the body $\boldsymbol{b}_a$-direction, $a \in \{1, 2\}$. These equations provide a model for planar vehicles, for example, a hovercraft that glides on the surface of a body of water with negligible friction.

For these innocuous looking equations, one can ask the following control-theoretic questions.

1. Is it possible to steer from a given initial state to any desired final state?
2. Is it possible to steer the system from rest at an initial configuration q_1 to a final configuration q_2, also at rest?

3. If the answer to either of the first two questions is, “Yes,” how does one accomplish the stated objective?

Questions 1 and 2 are referred to in the control theory literature as controllability questions. These and related issues shall be addressed in Chapter 7, and also in Chapter 8. Question 3 is a design question, dealing with what is called motion planning in the literature. We shall provide partial answers to this question in Chapter 13.

One may also wish to consider these questions in the event that the direction of the force is fixed to the body, i.e., the ratio between u_1 and u_2 is specified. Physically, this may happen if one is no longer able to vary the direction of the thruster. We note that in the original case there are two inputs, while in the latter case, there is only one. In either case, there are fewer inputs than degrees-of-freedom. As we shall see, that this has the effect of making the problem significantly more difficult than a system that has as many actuators as degrees-of-freedom. A system is “fully actuated” if each degree-of-freedom is actuated, and is otherwise “underactuated.” (This is slightly imprecise, and we shall be more precise about this in Section 4.6.)

1.2 Manipulators and multi-body systems

The next example is drawn from robotic manipulation. These problems generally include robotic arms composed of rigid links and joints. Joints can be revolute, prismatic, or spherical. We present a planar chain for simplicity; a treatment of manipulators in three-dimensional Euclidean space is provided via the product of exponentials formula [Murray, Li, and Sastry 1994]. The system we consider is a planar two-link manipulator; see Figure 1.2. For $i \in \{1, 2\}$,

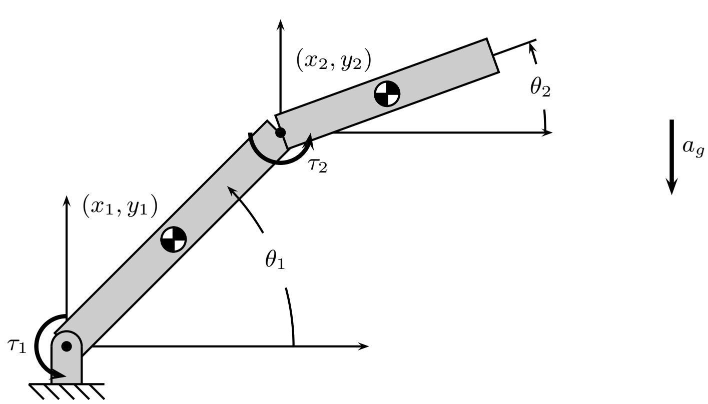

Figure 1.2. A two-link planar manipulator

we let θ_i denote the orientation of the ith link measured counterclockwise from the positive horizontal axis. If we suppose the bottom of the first link to be stationary in an inertial reference frame, the configuration of the system is specified by $q = (\theta_1, \theta_2)$. Thus this system has two degrees-of-freedom. The kinetic and potential energy of the ith link are

$$\mathrm{KE}_i = \tfrac{1}{2} J_i \dot{\theta}_i^2 + \tfrac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2), \quad \mathrm{PE}_i = m_i a_g \, y_i, \qquad i \in \{1, 2\}, \tag{1.1}$$

where (x_i, y_i) is the position of the center of mass, J_i is the moment of inertia of the ith link about its center of mass, and m_i is the mass of the ith link. Also, a_g denotes the acceleration due to gravity. While these equalities for kinetic and potential energy are easy to write down, they involve quantities such as (x_1, y_1, x_2, y_2) that are related explicitly to the actual configuration $q = (\theta_1, \theta_2)$. It is important to emphasize that, for any mechanical system, given the configuration q, one must be able to uniquely compute the position and orientation of each component of the system as a function of q. For the two-link manipulator, we must be able to write (x_1, y_1, x_2, y_2) as a function of (θ_1, θ_2). Indeed, we clearly have

$$\begin{aligned} x_1 &= \tfrac{1}{2}\ell_1 \cos\theta_1, \qquad & x_2 &= \ell_1 \cos\theta_1 + \tfrac{1}{2}\ell_2 \cos\theta_2, \\ y_1 &= \tfrac{1}{2}\ell_1 \sin\theta_1, \qquad & y_2 &= \ell_1 \sin\theta_1 + \tfrac{1}{2}\ell_2 \sin\theta_2, \end{aligned} \tag{1.2}$$

where ℓ_1 and ℓ_2 are the lengths of the links. Furthermore, it is possible to differentiate these relationships with respect to time and obtain an expression for $(\dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2)$ as a function of the state $(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$.

In summary, the only necessary variables to describe the system's kinetic and potential energy are its configuration variables. After some simplification, one can show that

$$\mathrm{KE} = \tfrac{1}{2} \dot{q}^T \mathbb{G}(q) \dot{q}, \tag{1.3}$$

where

$$\mathbb{G}(q) = \begin{bmatrix} J_1 + \frac{1}{4}(m_1 + 4m_2)\ell_1^2 & \frac{1}{2} m_2 \ell_1 \ell_2 \cos(\theta_1 - \theta_2) \\ \frac{1}{2} m_2 \ell_1 \ell_2 \cos(\theta_1 - \theta_2) & J_2 + \frac{1}{4} m_2 \ell_2^2 \end{bmatrix}.$$

The explicit expression for the total gravitational potential energy can also be easily written, although we do not give it here. As with the planar rigid body of Section 1.1, the total kinetic and potential energies suffice to ascertain the unforced equations of motion. We do not provide these equations here, since they are a bit cumbersome. The reader is asked to derive the forced version of these equations using Newtonian mechanics in Exercise E1.3.

Again, we are interested in applying forces to the system. For the system of Figure 1.2, natural input forces to the system are torques applied at the base of the first link, and/or at the joint between the two links (in our general setup, forces and torques are both called "forces"). This allows a possibility of at least three natural input configurations for the system. For any of the input configurations, we may ask the questions posed at the end of Section 1.1. Furthermore, we will also be interested in the following analysis and design questions.

1. Does the system, subject to no control forces, have any equilibrium configurations?
2. When is an equilibrium configuration stable? More precisely, for all initial conditions near the equilibrium configuration, when can it be guaranteed that the trajectories of the mechanical system remain near the equilibrium configuration?
3. Which configurations can be turned into stable equilibrium configurations by means of control forces? Can this be done in such a manner that the mechanical structure of the system is preserved?
4. Is it possible to specify the controls as functions of q and $\dot{q}$ in order to ensure that the resulting set of second-order differential equations will asymptotically approach the equilibrium configuration as $t \to +\infty$?
5. Given a reference trajectory, is it possible to specify the controls as functions of q and $\dot{q}$, as well as of the reference trajectory, in order to ensure that the resulting set of second-order differential equations will asymptotically follow the reference trajectory as $t \to +\infty$?

Questions 1 and 2 fall under the umbrella of stability theory, and are dealt with in Chapter 6. Questions 3, 4, and 5 are design questions. The first two of these problems are dealt with in Chapters 10, 11, and 12. Question 5 is considered in Chapters 11 and 12.

1.3 Constrained mechanical systems

The third class of systems from which we pull an example is those systems subject to velocity constraints. In particular, we focus on systems with rolling or skating constraints, that is, constraints on the instantaneous velocity of the system. In many physical applications, such systems arise in the case of vehicles with wheels. In this case, the constraint imposed is that the wheel roll without sliding on a horizontal plane.

The simple system we consider here is a disk of radius ρ shown in Figure 1.3. One may wish to think of this as a simple model for a unicycle. We assume that the rider is able to perfectly maintain their balance, so that the disk always remains exactly upright. The coordinates $q = (x, y, \theta, \phi)$ as shown in Figure 1.3 describe the configuration of the system ((x, y) being taken relative to the coordinate frame $\{\boldsymbol{s}_1, \boldsymbol{s}_2\}$ in the plane). Since these four coordinates uniquely characterize the position of the disk, the system has four degrees-of-freedom. Note, however, that not all velocities of the system are admissible. Indeed, the constraint that the disk roll without slipping means exactly that

$$\dot{x} - \rho \cos\theta \dot{\phi} = 0, \quad \dot{y} - \rho \sin\theta \dot{\phi} = 0.$$

This can be determined from Figure 1.4. One may also express this by saying that every admissible velocity is a linear combination of velocities of the form

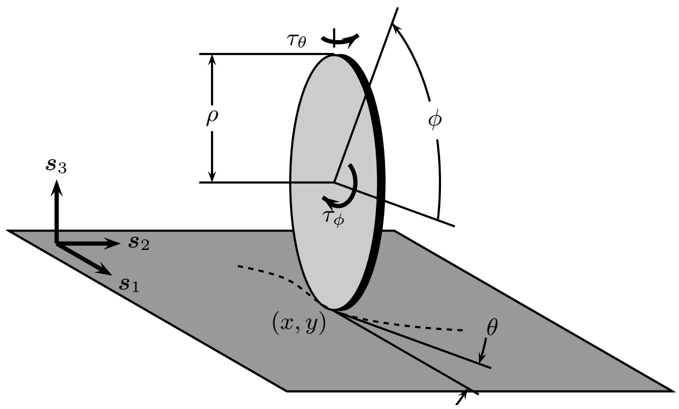

Figure 1.3. The rolling disk

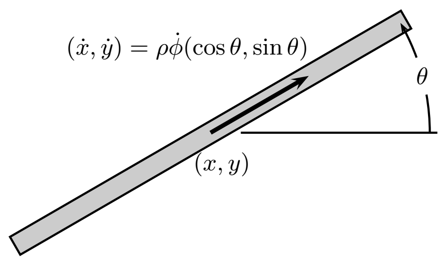

Figure 1.4. Depiction of constraints for rolling disk (the view is from above)

$$(0, 0, 1, 0), \quad (\rho\cos\theta, \rho\sin\theta, 0, 1).$$

This means that at each configuration (x, y, θ, ϕ), there are only two of the possible four directions available for the system to move. Note, however, that this does *not* mean that the configurations of the system are restricted. In Section 4.5 we will see how to systematically derive the equations of motion for systems with velocity constraints.

For the rolling disk, the kinetic energy is easy to compute, and is given by

$$\mathrm{KE} = \frac{1}{2}\dot{q}^T\mathbb{G}(q)\dot{q}, \quad \text{where} \quad \mathbb{G} = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & J_{\mathrm{spin}} & 0 \\ 0 & 0 & 0 & J_{\mathrm{roll}} \end{bmatrix}.$$

Here m is the mass of the disk, J_{roll} is the moment of inertia of the disk about its center, and J_{spin} is the moment of inertia of the disk about the vertical axis. Note that this function is defined without regard for the velocity constraints.

As forces applied to the rolling disk, one may consider combinations of two torques, one that "rolls" the disk, and the other that "spins" the disk. With these inputs, one can then consider the control-theoretic questions posed at the end of Section 1.1. Let us also throw in another sort of question, so that we can early on address a somewhat common (but thankfully decreasingly so) misconception about control theory for mechanical systems. For the rolling disk, one can ask the following question, which has nothing to do with the mechanical nature of the problem, and is only concerned with the constraints.

1. Is it possible to connect two configurations with a sequence of curves that satisfy the constraint of rolling without slipping?

Related to this is the following question, which brings us back to the mechanical realm.

2. Is it possible to follow, with a trajectory of the forced mechanical system, any curve in the set of configurations that satisfies the velocity constraints?

This latter question is dealt with systematically in our framework in Chapter 8, using the language of kinematic reductions.

1.4 Bibliographical notes

Both mechanics and control theory have rich histories. We intend to restrict ourselves to how these two overlap. We refer the reader to [Dugas 1957] for a history of mechanics. One of the first books devoted to an uncompromisingly geometric treatment of mechanics is the classic of Abraham and Marsden [1967], a substantial revision of which appeared in 1978. Another classic text in geometric methods in mechanics is that of Arnol'd [1978]. The history of feedback control is surveyed in [Mayr 1970]. Mathematical control theory is a younger subject, and a good source with which to judge its progress is the collection of seminal papers in [Başar 2001]. Recent differential geometric treatments of control theory may be found in the books [Agrachev and Sachkov 2004, Bloch 2003, Isidori 1995, 1999, Jurdjevic 1997, Nijmeijer and van der Schaft 1990, Sastry 1999]. In particular, the book of Agrachev and Sachkov [2004] provides a rather thorough discussion of many core ideas in mathematical geometric control theory.

An early paper that explicitly identifies the differential geometric bond between mechanics and control theory is that of Brockett [1977]. In this paper, Brockett looks at the forced spherical pendulum and a few single degree-of-freedom systems as motivational examples. Control problems in both the Lagrangian and Hamiltonian framework are identified. On the Hamiltonian side, some connections are drawn between Hamiltonian and gradient systems, something followed up on by Crouch [1981]. Brockett also considers passivity methods for stabilization, these having been introduced by Willems [1972]. The importance of the controllability problem, made more appealing for mechanical systems by their differential geometric structure, was also commented upon by Brockett.

During the course of the next fifteen years, there was only essentially isolated activity in terms of the development of a general theory of mechanics and control theory. An important contribution is that by Takegaki and Arimoto [1981] on the stabilization via so-called proportional-derivative control of certain robotic manipulators. These results led to a successful line of research, and we refer the reader to Chapters 10, 11, and 12 for details concerning this literature. One paper of a general nature was that of Bonnard [1984], which touches on two important subjects, one being controllability of mechanical systems, and the other being mechanical systems on Lie groups (considered in detail by us in Chapter 5). The work of Crouch [1981] also addresses the relationships between ideas in control theory and differential geometry, with many of the geometric ideas being those that arise in mechanics. Another early paper where geometric techniques are prominent is that of [Crouch 1984], where stabilization is addressed. Constituting one of the few fully developed forays into the theory of control theory and mechanics during this period is the sequence of papers, [van der Schaft 1981/82, 1982, 1983, 1985, 1986], that developed a fairly complete picture of Hamiltonian control theory. A treatment of Hamiltonian control systems also appears as part of the work of Willems [1979] on physical systems modeling.

Around 1990 there began to appear some concentrated interest in developing the theory of mechanical control systems. This activity came from two directions.

On the one hand, in the geometric mechanics community there arose an interest in understanding the role of external forces and constraints in geometric mechanics, since these had largely been ignored in the geometrization of mechanics (this was pointed out by Brockett [1977]). Papers dealing with the inclusion of constraints in the modern geometric framework include [Bloch, Krishnaprasad, Marsden, and Murray 1996, Koiller 1992] in the Lagrangian setting, and [Bates and Śniatycki 1993, van der Schaft and Maschke 1994] in the Hamiltonian setting. When one considers external forces for a mechanical system, questions from control theory arise quite naturally. An early representative of this development is the work of van der Schaft on Hamiltonian control systems mentioned above, and the work of Bloch and Crouch [1992] (developed further in [Bloch and Crouch 1995]) on control for mechanical systems with nonholonomic constraints. This latter work is, as far as we know, the first place where the affine connection features prominently in the development of control theory for mechanical systems. Work on controllability of mechanical systems was done by Bloch, Reyhanoglu, and McClamroch [1992b], a geometric mechanics based treatment of stabilization appears in the paper of Bloch, Krishnaprasad, Marsden, and Sánchez de Alvarez [1992a], and an approach to stabilization of mechanical systems using vibrational methods is undertaken by Baillieul [1993]. From this point on, there has been a fairly steady growth in the development of the geometric features of control theory for mechanical systems, including, for example, the plenary presentations of Murray [1995] and Leonard [1998], a new series of IFAC Workshops on Lagrangian and Hamiltonian Methods for Nonlinear Control, and the PhD theses of the authors [Bullo 1998, Lewis 1995]. Indeed, since about 1995, there has been a significant growth in the research literature on the subject of geometric control theory for mechanical systems. We shall not comment at this point on specific research papers; this shall be done in the text at relevant junctures. However, we do point out the recent appearance of a few books in the area that represent at least partial culminations of efforts by various groups. An early such representative is Chapter 12 of Nijmeijer and van der Schaft [1990], which gives a rather complete picture of Hamiltonian control theory from the point of view of Poisson geometry. A recent book with a treatment similar in spirit with ours is the book by Bloch [2003]. This book represents the work of Bloch with various coworkers, principally Baillieul, Crouch, Krishnaprasad, Marsden, Murray, and Zenkov. As such, it covers a variety of perspectives in the control theory of mechanical systems, particularly those with nonholonomic constraints. Systems are treated from both the Hamiltonian and Lagrangian points of view, and special emphasis is given to systems with symmetry. The aims of our book are less sweeping, generally focusing on the geometric perspective offered by the simple mechanical structure.

A second point of view from which arose the increased interest in control theory for mechanical systems is from the applications side, with a focus on stabilization and passivity techniques. Representative of this, the books [Arimoto 1996, Ortega, Loria, Nicklasson, and Sira-Ramirez 1998] provide a thorough account of stabilization for Lagrangian systems using passivity methods. The methodology in these books differs from ours in that there is less reliance on differential geometry and more focus on electrical and electromechanical systems. This makes the books amenable to researchers looking to quickly apply tools to problems.

The above cited works form an extremely incomplete overview of the existing and ongoing research in the area of mechanical control systems. In the body of the text we have attempted to provide references to specific papers that are related to topics in the book. As such, the bibliography at the end of the book is incomplete. We hope that the interested reader can use it as a starting point, albeit a biased one, for entering the research literature.

Exercises

E1.1 Use Newton's and Euler's laws to derive the equations of motion for the planar body subject to the variable-direction force in Section 1.1.

E1.2 Suppose that the force applied to the planar body of Section 1.1 is restricted so as to always point in the direction of the body $\boldsymbol{b}_1$-axis, making the system a single-input system.

(a) Try to deduce the character of the motion of the body subjected to such a force. Test your intuition with numerical simulations using various inputs $u_1(t)$, including $u_1(t) = 1$ and $u_1(t) = \cos(t)$, for $t \in [0, 2\pi]$. (Assume $J = m = 1$.)

(b) Next, do the same when the force applied to the planar body is restricted so as to always point in the direction of the body $\boldsymbol{b}_2$-axis. Can you detect any qualitative difference between the two cases?

E1.3 Use Newton's and Euler's laws to derive the equations of motion for the two-link manipulator subjected to two torques in Section 1.2.

E1.4 For the two-link manipulator of Section 1.2, consider the question, "Can one steer the system from a given configuration at rest to another configuration, also at rest?"
(a) If the system is fully actuated, why is this question trivial?
(b) What do you think is the answer to the question when the system is underactuated?

E1.5 Use Newton's and Euler's laws to derive the equations of motion for the rolling disk subjected to two torques in Section 1.3.

E1.6 Consider a spherical pendulum; that is a point mass in three-dimensional space constrained to move on a spherical surface. Let (x, y, z) denote Cartesian coordinates relative to the orthonormal frame $\{\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3\}$ in the figure. The system has two degrees-of-freedom with coordinates (θ, ϕ) as in Figure E1.1 (the distance from the origin to the point mass is equal to 1). Answer the following questions.

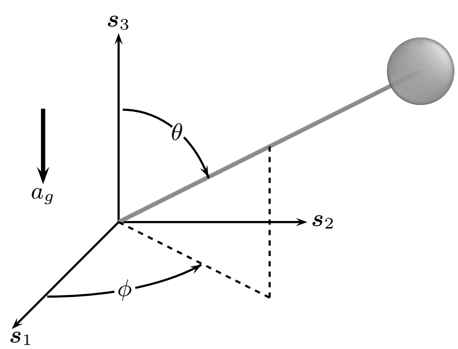

Figure E1.1. Coordinates for spherical pendulum

(a) Write the (x, y, z) position of the point mass in terms of the coordinates (θ, ϕ).
(b) Compute the kinetic energy in terms of the coordinates (θ, ϕ) and their time derivatives.
(c) Compute the potential energy assuming a gravitational field along the $\boldsymbol{s}_3$-axis.
(d) Show that the system satisfies the constraint $\dot{x}x + \dot{y}y + \dot{z}z = 0$. Is this a constraint only on velocities, or does it also constrain the evolution of (x, y, z)?

E1.7 For a realistic model of a bicycle, answer the following questions.

(a) Determine the configuration space.
(b) Write down a set of coordinates.
(c) Determine the velocity constraints that result from the wheels rolling without slipping.

E1.8 For which (if any) of the examples presented in Sections 1.1, 1.2, and 1.3 is it true that the coordinate “vector” q describing the configuration of the system can be legitimately thought of as being an element of Euclidean space? For those systems for which this is not true, indicate why.