Lecture #4: Deployment via Geometric Optimization

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Summary introduction

- Another motion coordination objective: deployment
- optimal task allocation and space partitioning, optimal placement and tuning of sensors
- Connection with geometric optimization and basic behaviors
- Formal definition and analysis of tasks and algorithms

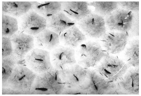
Coverage optimization

DESIGN of performance metrics

1. how to cover a region with n minimum-radius overlapping disks?
2. how to design a minimum-distortion (fixed-rate) vector quantizer? (Lloyd '57)
3. where to place mailboxes in a city / cache servers on the internet?

ANALYSIS of cooperative distributed behaviors

4. how do animals share territory? what if every fish in a swarm goes toward center of own dominance region?



Barlow, Hexagonal territories, *Animal Behavior*, 1974

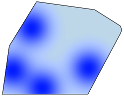
5. what if each vehicle goes to center of mass of own Voronoi cell?
6. what if each vehicle moves away from closest vehicle?

Expected-value multicenter function

Objective: Given sensors/nodes/robots/sites $(p_1, \dots, p_n)$ moving in environment Q achieve **optimal coverage**

$\phi : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ density

$f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ non-increasing and piecewise continuously differentiable, possibly with finite jump discontinuities



$$\text{maximize} \quad \mathcal{H}_{\exp}(p_1, \dots, p_n) = E_\phi \left[\max_{i \in \{1,\dots,n\}} f(\|q - p_i\|) \right]$$

$\mathcal{H}_{\text{exp}}$-optimality of the Voronoi partition

Alternative expression in terms of Voronoi partition,

$$\mathcal{H}_{\text{exp}}(p_1,\dots,p_n) = \sum_{i=1}^{n} \int_{V_i(\mathcal{P})} f(\|q-p_i\|_2)\phi(q)dq$$

for $(p_1,\dots,p_n)$ distinct

Proposition

Let $\mathcal{P} = \{p_1,\dots,p_n\} \in \mathbb{F}(S)$. For any performance function f and for any partition $\{W_1,\dots,W_n\} \subset \mathbb{P}(S)$ of S,

$$\mathcal{H}_{\text{exp}}(p_1,\dots,p_n,V_1(\mathcal{P}),\dots,V_n(\mathcal{P})) \geq \mathcal{H}_{\text{exp}}(p_1,\dots,p_n,W_1,\dots,W_n),$$

and the inequality is strict if any set in $\{W_1,\dots,W_n\}$ differs from the corresponding set in $\{V_1(\mathcal{P}),\dots,V_n(\mathcal{P})\}$ by a set of positive measure

Distortion problem

$f(x) = -x^2$

$$\mathcal{H}_{\text{distor}}(p_1,\dots,p_n) = -\sum_{i=1}^{n} \int_{V_i(P)} \|q-p_i\|_2^2\phi(q)dq = -\sum_{i=1}^{n} \mathsf{J}_\phi(V_i(\mathcal{P}),p_i)$$

($\mathsf{J}_\phi(W,p)$ is moment of inertia). Note

$$\begin{aligned}&\mathcal{H}_{\text{distor}}(p_1,\dots,p_n,W_1,\dots,W_n)\\ &\quad= -\sum_{i=1}^{n} \mathsf{J}_\phi(W_i,\mathsf{CM}_\phi(W_i)) - \sum_{i=1}^{n} \mathsf{A}_\phi(W_i)\|p_i - \mathsf{CM}_\phi(W_i)\|_2^2\end{aligned}$$

Proposition

Let $\{W_1,\dots,W_n\} \subset \mathbb{P}(S)$ be a partition of S. Then,

$$\begin{aligned}\mathcal{H}_{\text{distor}}\big(&\mathsf{CM}_\phi(W_1),\dots,\mathsf{CM}_\phi(W_n),W_1,\dots,W_n\big)\\ &\geq \mathcal{H}_{\text{distor}}(p_1,\dots,p_n,W_1,\dots,W_n),\end{aligned}$$

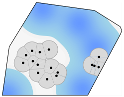
and the inequality is strict if there exists $i \in \{1,\dots,n\}$ for which W_i has non-vanishing area and $p_i \neq \mathsf{CM}_\phi(W_i)$

Area problem

$f(x) = 1_{[0,a]}(x)$, $a \in \mathbb{R}_{>0}$

$$\begin{aligned}\mathcal{H}_{\text{area},a}(p_1,\dots,p_n) &= \sum_{i=1}^{n} \int_{V_i(\mathcal{P})} 1_{[0,a]}(\|q-p_i\|_2)\phi(q)dq\\ &= \sum_{i=1}^{n} \int_{V_i(\mathcal{P})\cap\overline{B}(p_i,a)} \phi(q)dq\\ &= \sum_{i=1}^{n} \mathsf{A}_\phi(V_i(\mathcal{P})\cap\overline{B}(p_i,a)) = \mathsf{A}_\phi(\cup_{i=1}^{n}\overline{B}(p_i,a)),\end{aligned}$$

Area, measured according to ϕ, covered by the union of the n balls $\overline{B}(p_1,a),\dots,\overline{B}(p_n,a)$



Mixed distortion-area problem

$f(x) = -x^2\,1_{[0,a]}(x) + b\cdot 1_{]a,+\infty[}(x)$, with $a \in \mathbb{R}_{>0}$ and $b \leq -a^2$

$$\mathcal{H}_{\text{distor-area},a,b}(p_1,\dots,p_n) = -\sum_{i=1}^{n} \mathsf{J}_\phi(V_{i,a}(\mathcal{P}),p_i) + b\,\mathsf{A}_\phi(Q \setminus \cup_{i=1}^{n}\overline{B}(p_i,a)),$$

If $b = -a^2$, performance f is continuous, and we write $\mathcal{H}_{\text{distor-area},a}$. Extension to sets of points and partitions reads

$$\begin{aligned}&\mathcal{H}_{\text{distor-area},a}(p_1,\dots,p_n,W_1,\dots,W_n)\\ &\quad= -\sum_{i=1}^{n}\Big(\mathsf{J}_\phi(W_i\cap\overline{B}(p_i,a),p_i) + a^2\,\mathsf{A}_\phi(W_i\cap(S\setminus\overline{B}(p_i,a)))\Big).\end{aligned}$$

Proposition ($\mathcal{H}_{\text{distor-area},a}$-optimality of centroid locations)

Let $\{W_1,\dots,W_n\} \subset \mathbb{P}(S)$ be a partition of S. Then,

$$\begin{aligned}\mathcal{H}_{\text{distor-area},a}\big(&\mathsf{CM}_\phi(W_1\cap\overline{B}(p_1,a)),\dots,\mathsf{CM}_\phi(W_n\cap\overline{B}(p_n,a)),W_1,\dots,W_n\big)\\ &\geq \mathcal{H}_{\text{distor}}(p_1,\dots,p_n,W_1,\dots,W_n),\end{aligned}$$

and the inequality is strict if there exists $i \in \{1,\dots,n\}$ for which W_i has non-vanishing area and $p_i \neq \mathsf{CM}_\phi(W_i\cap\overline{B}(p_i,a))$.

Smoothness properties of $\mathcal{H}_{\text{exp}}$

$\mathsf{Dscn}(f)$ (finite) discontinuities of f
f_- and f_+, limiting values from the left and from the right

Theorem

Expected-value multicenter function $\mathcal{H}_{\text{exp}} : S^n \to \mathbb{R}$ is

1. *globally Lipschitz on S^n; and*
2. *continuously differentiable on $S^n \setminus \mathcal{S}_{\text{coinc}}$, where*

$$\frac{\partial \mathcal{H}_{\text{exp}}}{\partial p_i}(P) = \int_{V_i(\mathcal{P})} \frac{\partial}{\partial p_i} f(\|q - p_i\|_2)\phi(q)dq$$
$$+ \sum_{a \in \mathsf{Dscn}(f)} (f_-(a) - f_+(a)) \int_{V_i(\mathcal{P}) \cap \partial \overline{B}(p_i,a)} \mathsf{n}_{\text{out},\overline{B}(p_i,a)}(q)\phi(q)dq$$
$$= \textit{integral over } V_i + \textit{integral along arcs in } V_i$$

Therefore, the gradient of $\mathcal{H}_{\text{exp}}$ is spatially distributed over $\mathcal{G}_{\text{D}}$

Some proof ideas

Consider the case of smooth performance f,

$$\frac{\partial \mathcal{H}_{\text{exp}}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f\left(\|q - p_i\|\right) \phi(q)dq$$
$$+ \int_{\partial V_i(P)} f\left(\|q - p_i\|\right) \langle n_i(q), \frac{\partial q}{\partial p_i}\rangle \phi(q)dq$$
$$+ \sum_{j \text{ neigh } i} \int_{V_j(P) \cap V_i(P)} f\left(\|q - p_j\|\right) \langle n_{ji}(q), \frac{\partial q}{\partial p_i}\rangle \phi(q)dq$$

Some proof ideas

$$\frac{\partial \mathcal{H}_{\text{exp}}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f\left(\|q - p_i\|\right) \phi(q)dq \underbrace{= 2\,\mathsf{A}_\phi(V_i(\mathcal{P}))(\mathsf{CM}_\phi(V_i(\mathcal{P})) - p_i)}_{\text{for } f(x)=x^2}$$
$$+ \int_{\partial V_i(P)} f\left(\|q - p_i\|\right) \langle n_i(q), \frac{\partial q}{\partial p_i}\rangle \phi(q)dq$$
$$- \int_{\partial V_i(P)} f\left(\|q - p_i\|\right) \langle n_i(q), \frac{\partial q}{\partial p_i}\rangle \phi(q)dq$$

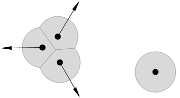
Particular gradients

Distortion problem: continuous performance,

$$\frac{\partial \mathcal{H}_{\text{distor}}}{\partial p_i}(P) = 2\,\mathsf{A}_\phi(V_i(\mathcal{P}))(\mathsf{CM}_\phi(V_i(\mathcal{P})) - p_i)$$

Area problem: performance has single discontinuity,

$$\frac{\partial \mathcal{H}_{\text{area},a}}{\partial p_i}(P) = \int_{V_i(\mathcal{P}) \cap \partial \overline{B}(p_i,a)} \mathsf{n}_{\text{out},\overline{B}(p_i,a)}(q)\phi(q)dq$$



Mixed distortion-area: continuous performance ($b = -a^2$),

$$\frac{\partial \mathcal{H}_{\text{distor-area},a}}{\partial p_i}(P) = 2\,\mathsf{A}_\phi(V_{i,a}(\mathcal{P}))(\mathsf{CM}_\phi(V_{i,a}(\mathcal{P})) - p_i)$$

Tuning the optimization problem

Gradients of $\mathcal{H}_{\text{area},a}$, $\mathcal{H}_{\text{distor-area},a,b}$ are distributed over $\mathcal{G}_{\text{LD}}(2a)$

Robotic agents with range-limited interactions can compute gradients of $\mathcal{H}_{\text{area},a}$ and $\mathcal{H}_{\text{distor-area},a,b}$ as long as $r \geq 2a$

Proposition (Constant-factor approximation of $\mathcal{H}_{\text{distor}}$)

Let $S \subset \mathbb{R}^d$ be bounded and measurable. Consider the mixed distortion-area problem with $a \in]0, \text{diam}\, S]$ and $b = -\text{diam}(S)^2$. Then, for all $P \in S^n$,

$$\mathcal{H}_{\text{distor-area},a,b}(P) \leq \mathcal{H}_{\text{distor}}(P) \leq \beta^2\, \mathcal{H}_{\text{distor-area},a,b}(P) < 0,$$

where $\beta = \frac{a}{\text{diam}(S)} \in [0, 1]$

Similarly, constant-factor approximations of $\mathcal{H}_{\text{exp}}$

Geometric-center laws

Uniform networks $\mathcal{S}_{\text{D}}$ and $\mathcal{S}_{\text{LD}}$ of locally-connected first-order agents in a polytope $Q \subset \mathbb{R}^d$ with the Delaunay and r-limited Delaunay graphs as communication graphs

All laws share similar structure

At each communication round each agent performs the following tasks:

- *it transmits its position and receives its neighbors' positions;*
- *it computes a notion of geometric center of its own cell determined according to some notion of partition of the environment*

Between communication rounds, each robot moves toward this center

VRN-CNTRD ALGORITHM

Optimizes distortion $\mathcal{H}_{\text{distor}}$

`Robotic Network:` $\mathcal{S}_{\text{D}}$in Q, with absolute sensing of own position

`Distributed Algorithm:` VRN-CNTRD

`Alphabet:` $\mathbb{A} = \mathbb{R}^d \cup \{\texttt{null}\}$

`function` $\text{msg}(p, i)$

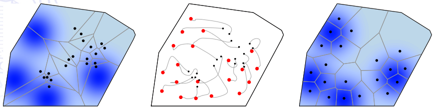
1: **return** p

`function` $\text{ctl}(p, y)$

1: $V := Q \cap \left(\bigcap \{H_{p,p_{\text{rcvd}}} \mid \text{for all non-null } p_{\text{rcvd}} \in y\} \right)$

2: **return** $\text{CM}_{\phi}(V) - p$

Simulation



initial configuration — gradient descent — final configuration

For $\varepsilon \in \mathbb{R}_{>0}$, the *$\varepsilon$-distortion deployment task*

$$\mathcal{T}_{\varepsilon\text{-distor-dply}}(P) = \begin{cases} \texttt{true}, & \text{if } \left\| p^{[i]} - \text{CM}_{\phi}(V^{[i]}(P)) \right\|_2 \leq \varepsilon,\ i \in \{1, \dots, n\}, \\ \texttt{false}, & \text{otherwise}, \end{cases}$$

Voronoi-centroid law on planar vehicles

Robotic Network: $\mathcal{S}_{\text{vehicles}}$ in Q with absolute sensing of own position

Distributed Algorithm: VRN-CNTRD-DYNMCS

Alphabet: $\mathbb{A} = \mathbb{R}^2 \cup \{\texttt{null}\}$

function $\mathrm{msg}((p,\theta), i)$

1: **return** p

function $\mathrm{ctl}((p,\theta), (p_{\mathrm{smpld}}, \theta_{\mathrm{smpld}}), y)$

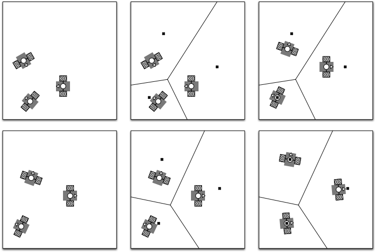
1: $V := Q \cap \left(\bigcap\{H_{p_{\mathrm{smpld}}, p_{\mathrm{rcvd}}} \mid \text{for all non-null } p_{\mathrm{rcvd}} \in y\}\right)$

2: $v := -k_{\mathrm{prop}}(\cos\theta, \sin\theta) \cdot (p - \mathrm{CM}_\phi(V))$

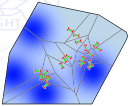
3: $\omega := 2k_{\mathrm{prop}} \arctan \dfrac{(-\sin\theta, \cos\theta) \cdot (p - \mathrm{CM}_\phi(V))}{(\cos\theta, \sin\theta) \cdot (p - \mathrm{CM}_\phi(V))}$

4: **return** (v, ω)

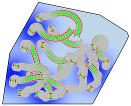
Algorithm illustration



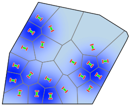
Simulation



initial configuration



gradient descent



final configuration

LMTD-VRN-NRML algorithm

Optimizes area $\mathcal{H}_{\text{area},\frac{r}{2}}$

Robotic Network: $\mathcal{S}_{\mathrm{LD}}$ in Q with absolute sensing of own position and with communication range r

Distributed Algorithm: LMTD-VRN-NRML

Alphabet: $\mathbb{A} = \mathbb{R}^d \cup \{\texttt{null}\}$

function $\mathrm{msg}(p, i)$

1: **return** p

function $\mathrm{ctl}(p, y)$

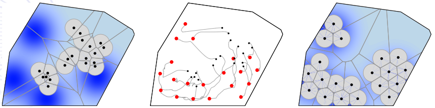
1: $V := Q \cap \left(\bigcap\{H_{p, p_{\mathrm{rcvd}}} \mid \text{for all non-null } p_{\mathrm{rcvd}} \in y\}\right)$

2: $v := \int_{V \cap \partial \overline{B}(p, \frac{r}{2})} \mathsf{n}_{\mathrm{out}, \overline{B}(p, \frac{r}{2})}(q)\phi(q)dq$

3: $\lambda_* := \max\{\lambda \mid \delta \mapsto \int_{V \cap \overline{B}(p+\delta v, \frac{r}{2})} \phi(q)dq$ is strictly increasing on $[0, \lambda]\}$

4: **return** $\lambda_* v$

Simulation



initial configuration | gradient descent | final configuration

For $r, \varepsilon \in \mathbb{R}_{>0}$,

$$\mathcal{T}_{\varepsilon\text{-}r\text{-area-dply}}(P) = \begin{cases} \texttt{true}, & \text{if } \left\| \int_{V^{[i]}(P)\cap\partial\overline{B}(p^{[i]},\frac{r}{2})} \mathsf{n}_{\text{out},\overline{B}(p^{[i]},\frac{r}{2})}(q)\phi(q)dq \right\|_2 \le \varepsilon,\ i\in\{1,\dots,n\} \\ \texttt{false}, & \text{otherwise.}\end{cases}$$

LMTD-VRN-CNTRD algorithm

Optimizes $\mathcal{H}_{\text{distor-area},\frac{r}{2}}$

`Robotic Network:` $\mathcal{S}_{\text{LD}}$ in Q with absolute sensing of own position, and with communication range r

`Distributed Algorithm:` LMTD-VRN-CNTRD

`Alphabet:` $\mathbb{A} = \mathbb{R}^d \cup \{\texttt{null}\}$

`function` msg(p, i)

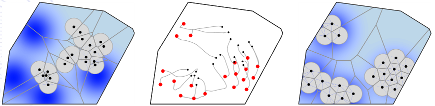
1: **return** p

`function` ctl(p, y)

1: $V := Q \cap \overline{B}(p, \frac{r}{2}) \cap \left(\bigcap\{H_{p,p_{\text{rcvd}}} \mid \text{for all non-null } p_{\text{rcvd}} \in y\}\right)$

2: **return** $\mathsf{CM}_\phi(V) - p$

Simulation



initial configuration | gradient descent | final configuration

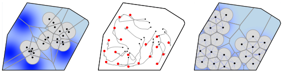
For $r, \varepsilon \in \mathbb{R}_{>0}$,

$$\mathcal{T}_{\varepsilon\text{-}r\text{-distor-area-dply}}(P) = \begin{cases} \texttt{true}, & \text{if } \left\|p^{[i]} - \mathsf{CM}_\phi(V^{[i]}_{\frac{r}{2}}(P))\right\|_2 \le \varepsilon,\ i\in\{1,\dots,n\}, \\ \texttt{false}, & \text{otherwise.}\end{cases}$$

Optimizing $\mathcal{H}_{\text{distor}}$ via constant-factor approximation

Limited range

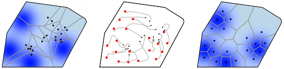
run #1: 16 agents, density ϕ is sum of 4 Gaussians, time invariant, 1st order dynamics



initial configuration | gradient descent of $\mathcal{H}_{\frac{r}{2}}$ | final configuration

Unlimited range

run #2: 16 agents, density ϕ is sum of 4 Gaussians, time invariant, 1st order dynamics



initial configuration | gradient descent of $\mathcal{H}_{\text{exp}}$ | final configuration

Correctness of the geometric-center algorithms

Theorem

For $d \in \mathbb{N}$, $r \in \mathbb{R}_{>0}$ and $\varepsilon \in \mathbb{R}_{>0}$, the following statements hold.

1. *on the network $\mathcal{S}_{\mathrm{D}}$, the law $\mathcal{CC}_{\text{VRN-CNTRD}}$ and on the network $\mathcal{S}_{\text{vehicles}}$, the law $\mathcal{CC}_{\text{VRN-CNTRD-DYNMCS}}$ both achieve the ε-distortion deployment task $\mathcal{T}_{\varepsilon\text{-distor-dply}}$. Moreover, any execution of $\mathcal{CC}_{\text{VRN-CNTRD}}$ and $\mathcal{CC}_{\text{VRN-CNTRD-DYNMCS}}$ monotonically optimizes the multicenter function $\mathcal{H}_{\text{distor}}$;*
2. *on the network $\mathcal{S}_{\mathrm{LD}}$, the law $\mathcal{CC}_{\text{LMTD-VRN-NRML}}$ achieves the ε-r-area deployment task $\mathcal{T}_{\varepsilon\text{-}r\text{-area-dply}}$. Moreover, any execution of $\mathcal{CC}_{\text{LMTD-VRN-NRML}}$ monotonically optimizes the multicenter function $\mathcal{H}_{\text{area},\frac{r}{2}}$; and*
3. *on the network $\mathcal{S}_{\mathrm{LD}}$, the law $\mathcal{CC}_{\text{LMTD-VRN-CNTRD}}$ achieves the ε-r-distortion-area deployment task $\mathcal{T}_{\varepsilon\text{-}r\text{-distor-area-dply}}$. Moreover, any execution of $\mathcal{CC}_{\text{LMTD-VRN-CNTRD}}$ monotonically optimizes the multicenter function $\mathcal{H}_{\text{distor-area},\frac{r}{2}}$.*

Time complexity of $\mathcal{CC}_{\text{LMTD-VRN-CNTRD}}$

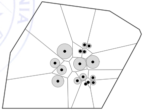
Assume $\mathrm{diam}(Q)$ is independent of n, r and ε

Theorem (Time complexity of LMTD-VRN-CNTRD law)

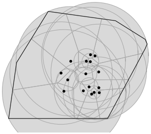
Assume the robots evolve in a closed interval $Q \subset \mathbb{R}$, that is, $d = 1$, and assume that the density is uniform, that is, $\phi \equiv 1$. For $r \in \mathbb{R}_{>0}$ and $\varepsilon \in \mathbb{R}_{>0}$, on the network $\mathcal{S}_{\mathrm{LD}}$

$$\mathsf{TC}(\mathcal{T}_{\varepsilon\text{-}r\text{-distor-area-dply}}, \mathcal{CC}_{\text{LMTD-VRN-CNTRD}}) \in O(n^3 \log(n\varepsilon^{-1}))$$

Deployment: basic behaviors



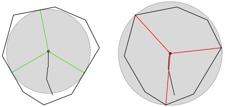
"move away from closest"



"move towards furthest"

Equilibria? Asymptotic behavior?
Optimizing network-wide function?

Deployment: 1-center optimization problems



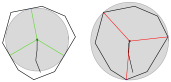
$\mathsf{sm}_Q(p) = \min\{\|p - q\| \mid q \in \partial Q\}$ Lipschitz $0 \in \partial\, \mathsf{sm}_Q(p) \Leftrightarrow p \in \mathsf{IC}(Q)$

$\mathsf{lg}_Q(p) = \max\{\|p - q\| \mid q \in \partial Q\}$ Lipschitz $0 \in \partial\, \mathsf{lg}_Q(p) \Leftrightarrow p = \mathsf{CC}(Q)$

Locally Lipschitz function V are differentiable a.e.
Generalized gradient of V is

$$\partial V(x) = \text{convex closure}\{\lim_{i\to\infty} \nabla V(x_i) \mid x_i \to x\,,\ x_i \notin \Omega_V \cup S\}$$

Deployment: 1-center optimization problems



+ gradient flow of sm_Q	$\dot{p}_i = +\mathsf{Ln}[\partial\, \mathsf{sm}_Q](p)$	"move away from closest"
− gradient flow of lg_Q	$\dot{p}_i = -\mathsf{Ln}[\partial\, \mathsf{lg}_Q](p)$	"move toward furthest"

For X essentially locally bounded, **Filippov solution** of $\dot{x} = X(x)$ is absolutely continuous function $t \in [t_0, t_1] \mapsto x(t)$ verifying

$$\dot{x} \in K[X](x) = \mathsf{co}\{\lim_{i\to\infty} X(x_i) \mid x_i \to x\,,\ x_i \notin S\}$$

For V locally Lipschitz, gradient flow is $\dot{x} = \mathsf{Ln}[\partial V](x)$

Ln = least norm operator

Nonsmooth LaSalle Invariance Principle

Evolution of V along Filippov solution $t \mapsto V(x(t))$ is differentiable a.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \in \underbrace{\widetilde{\mathcal{L}}_X V(x(t)) = \{a \in \mathbb{R} \mid \exists v \in K[X](x) \text{ s.t. } \zeta \cdot v = a\,,\ \forall \zeta \in \partial V(x)\}}_{\text{set-valued Lie derivative}}$$

LaSalle Invariance Principle

For S compact and strongly invariant with $\max \widetilde{\mathcal{L}}_X V(x) \le 0$

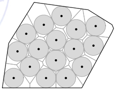
Any Filippov solution starting in S converges to largest weakly invariant set contained in $\overline{\{x \in S \mid 0 \in \widetilde{\mathcal{L}}_X V(x)\}}$

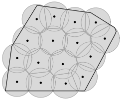
E.g., **nonsmooth gradient flow** $\dot{x} = -\mathsf{Ln}[\partial V](x)$ converges to critical set

Deployment: multi-center optimization

sphere packing and disk covering

"move away from closest":	$\dot{p}_i = +\mathsf{Ln}(\partial\, \mathsf{sm}_{V_i(P)})(p_i)$	— at fixed $V_i(P)$
"move towards furthest":	$\dot{p}_i = -\mathsf{Ln}(\partial\, \mathsf{lg}_{V_i(P)})(p_i)$	— at fixed $V_i(P)$





Aggregate objective functions!

$$\mathcal{H}_{\mathrm{sp}}(P) = \min_i \ \mathsf{sm}_{V_i(P)}(p_i) = \min_{i \neq j} \left[\tfrac{1}{2}\|p_i - p_j\|, \mathsf{dist}(p_i, \partial Q)\right]$$

$$\mathcal{H}_{\mathrm{dc}}(P) = \max_i \ \mathsf{lg}_{V_i(P)}(p_i) = \max_{q \in Q} \left[\min_i \|q - p_i\|\right]$$

Deployment: multi-center optimization

Critical points of $\mathcal{H}_{\mathrm{sp}}$ and $\mathcal{H}_{\mathrm{dc}}$ (locally Lipschitz)

- If $0 \in \mathsf{int}(\partial\mathcal{H}_{\mathrm{sp}}(P))$, then P is strict local maximum, all agents have same cost, and P is **incenter Voronoi configuration**
- If $0 \in \mathsf{int}(\partial\mathcal{H}_{\mathrm{dc}}(P))$, then P is strict local minimum, all agents have same cost, and P is **circumcenter Voronoi configuration**

Aggregate functions **monotonically optimized** along evolution

$\min \widetilde{\mathcal{L}}_{\mathsf{Ln}(\partial\, \mathsf{sm}_{\mathcal{V}()})} \mathcal{H}_{\mathbf{sp}}(P) \ge 0$ $\qquad$ $\max \widetilde{\mathcal{L}}_{-\mathsf{Ln}(\partial\, \mathsf{lg}_{\mathcal{V}()})} \mathcal{H}_{\mathbf{dc}}(P) \le 0$

Asymptotic convergence to center Voronoi configurations via nonsmooth LaSalle

Voronoi-circumcenter algorithm

Robotic Network: $\mathcal{S}_{\mathrm{D}}$ in Q with absolute sensing of own position
Distributed Algorithm: VRN-CRCMCNTR
Alphabet: $\mathbb{A} = \mathbb{R}^d \cup \{\texttt{null}\}$

function msg(p, i)
1: **return** p

function ctl(p, y)
1: $V := Q \cap \left(\bigcap\{H_{p,p_{\mathrm{rcvd}}} \mid \text{for all non-null } p_{\mathrm{rcvd}} \in y\}\right)$
2: **return** $\mathsf{CC}(V) - p$

Voronoi-incenter algorithm

Robotic Network: $\mathcal{S}_{\mathrm{D}}$ in Q, with absolute sensing of own position
Distributed Algorithm: VRN-NCNTR
Alphabet: $\mathbb{A} = \mathbb{R}^d \cup \{\texttt{null}\}$

function msg(p, i)
1: **return** p

function ctl(p, y)
1: $V := Q \cap \left(\bigcap\{H_{p,p_{\mathrm{rcvd}}} \mid \text{for all non-null } p_{\mathrm{rcvd}} \in y\}\right)$
2: **return** $x \in \mathsf{IC}(V) - p$

Correctness of the geometric-center algorithms

For $\varepsilon \in \mathbb{R}_{>0}$, the ε-*disk-covering deployment task*

$$\mathcal{T}_{\varepsilon\text{-dc-dply}}(P) = \begin{cases} \texttt{true}, & \text{if } \|p^{[i]} - \mathsf{CC}(V^{[i]}(P))\|_2 \leq \varepsilon,\ i \in \{1, \dots, n\}, \\ \texttt{false}, & \text{otherwise,} \end{cases}$$

For $\varepsilon \in \mathbb{R}_{>0}$, the ε-*sphere-packing deployment task*

$$\mathcal{T}_{\varepsilon\text{-sp-dply}}(P) = \begin{cases} \texttt{true}, & \text{if } \mathsf{dist}_2(p^{[i]}, \mathsf{IC}(V^{[i]}(P))) \leq \varepsilon,\ i \in \{1, \dots, n\}, \\ \texttt{false}, & \text{otherwise,} \end{cases}$$

Theorem

For $d \in \mathbb{N}$, $r \in \mathbb{R}_{>0}$ and $\varepsilon \in \mathbb{R}_{>0}$, the following statements hold.

1. *on the network $\mathcal{S}_{\mathrm{D}}$, any execution of the law $\mathcal{CC}_{\text{VRN-CRCMCNTR}}$ monotonically optimizes the multicenter function $\mathcal{H}_{\mathrm{dc}}$;*
2. *on the network $\mathcal{S}_{\mathrm{D}}$, any execution of the law $\mathcal{CC}_{\text{VRN-NCNTR}}$ monotonically optimizes the multicenter function $\mathcal{H}_{\mathrm{sp}}$.*

Summary and conclusions

Aggregate objective functions

1. variety of scenarios: expected-value, disk-covering, sphere-packing
2. smoothness properties and gradient information
3. geometric-center control and communication laws

Technical tools

1. Geometric optimization
2. Geometric models, proximity graphs, spatially-distributed maps
3. Systems theory, nonsmooth stability analysis

References

Deployment scenarios and algorithms:

- J. Cortés, S. Martínez, T. Karatas, and F. Bullo. Coverage control for mobile sensing networks. *IEEE Transactions on Robotics and Automation*, 20(2):243--255, 2004
- J. Cortés, S. Martínez, and F. Bullo. Spatially-distributed coverage optimization and control with limited-range interactions. *ESAIM. Control, Optimisation & Calculus of Variations*, 11:691--719, 2005

Nonsmooth stability analysis:

- J. Cortés. Discontinuous dynamical systems -- a tutorial on solutions, nonsmooth analysis, and stability. *IEEE Control Systems Magazine*, 28(3):36--73, 2008

Geometric and combinatorial optimization:

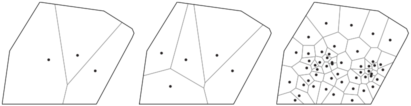
- P. K. Agarwal and M. Sharir. Efficient algorithms for geometric optimization. *ACM Computing Surveys*, 30(4):412--458, 1998

Voronoi partitions

Let $(p_1, \dots, p_n) \in Q^n$ denote the positions of n points

The **Voronoi partition** $\mathcal{V}(P) = \{V_1, \dots, V_n\}$ generated by $(p_1, \dots, p_n)$

$$V_i = \{q \in Q \mid \|q - p_i\| \leq \|q - p_j\|, \forall j \neq i\}$$
$$= Q \cap_j \mathcal{HP}(p_i, p_j) \qquad \text{where } \mathcal{HP}(p_i, p_j) \text{ is half plane } (p_i, p_j)$$

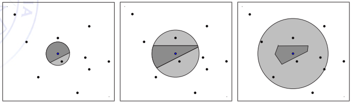


3 generators — 5 generators — 50 generators

Return

Distributed Voronoi computation

Assume: agent with sensing/communication radius R_i
Objective: smallest R_i which provides sufficient information for V_i



For all i, agent i performs:

1: initialize R_i and compute $\widehat{V}_i = \cap_{j:\|p_i - p_j\| \leq R_i} \mathcal{HP}(p_i, p_j)$
2: **while** $R_i < 2\max_{q \in \widehat{V}_i} \|p_i - q\|$ **do**
3: $R_i := 2R_i$
4: detect vehicles p_j within radius R_i, recompute $\widehat{V}_i$

Return