Mean-field Games for Fun and Profit

Santa Barbara Control Workshop
Decision, Dynamics and Control in Multi-Agent Systems

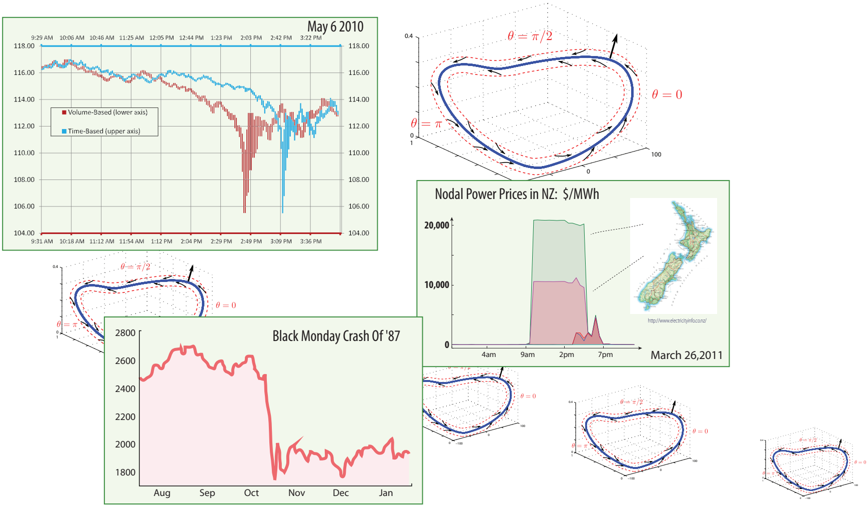
Sean P. Meyn

Joint work with: H. Yin, T. Yang, P. G. Mehta, and U. V. Shanbhag,

Coordinated Science Laboratory
and the Department of Electrical and Computer Engineering
University of Illinois at Urbana-Champaign, USA

Thanks to NSF & AFOSR

Outline



Who Cares About Oscillators?

Background: Economics

Can you spot the lunacy?

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A New Era For Control

We should remove derivative control from our engineering curriculum!

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Fundamental Theorem of Calculus: If the airplane is flying at level height, then the ultimate contribution of the derivative is *zero*:

$$0 = y(T) - y(0) = \int_0^T \dot{y}(t)\, dt$$

Background: Economics

Can you spot the lunacy?

A New Era For Government

We should remove government spending from our economics curriculum!

Barro-Ricardo Equivalence Proposition: Government budget imbalances are irrelevant to resource allocation. Every dollar of taxes postponed today must be paid with interest tomorrow by the exact same group of taxpayers alive today.

Background: Economics

Saner voices

Ericson and Pakes [3]

This paper provides a model of firm and industry dynamics that allows for entry, exit and firm-specific uncertainty generating variability in the fortunes of firms. It focuses on the impact of uncertainty arising from investment in research and exploration-type processes. ...

Coupled Markov models to address transients.

—Transients are *everything* in both business and economics

[3] Markov-perfect industry dynamics: A framework for empirical work, Rev. of Econ. Studies 1995

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Computation of Nash equilibria for coupled MDP models?

[3] Markov-perfect industry dynamics: A framework for empirical work, Rev. of Econ. Studies 1995

Background: Economics

Greater sanity

Weintraub, Benkard, and Van Roy [17]

… oblivious equilibrium (OE) is an approximation in which each player makes decisions based on his own state and the "average" state of the other players. …

Some aspects of dynamics and uncertainty are preserved.

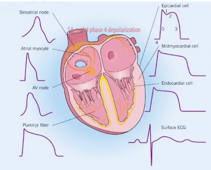
Computation of Nash equilibria is possible.

[17] Markov perfect industry dynamics with many firms, Econometrica 2008; [6] Huang et al., TAC, 2007

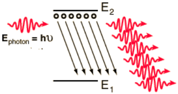
Background: Synchrony in Nature

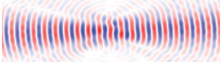
Synchrony is Good

Pacemaker cells



Laser light





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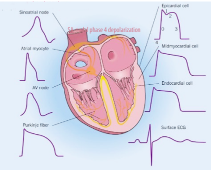

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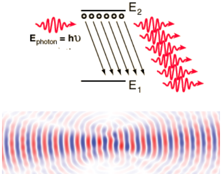
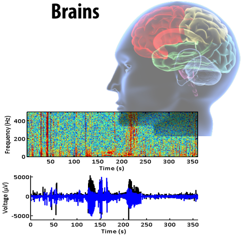


Synchrony is Not Good

Bridges

Brains



Question (Fundamental question in Neuroscience)

Why is synchrony (neural rhythms) useful?
Does it have a functional role?

Destexhe & Marder, Nature, 2004; Kopell et al., Neuroscience, 2009; Lee & Mumford, J. Opt. Soc, 2003

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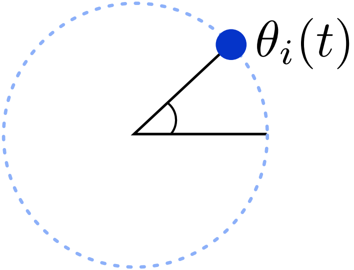
3. Neuronal computations
   - Bayesian inference
   - Neural circuits as particle filters (Lee & Mumford)
   - T. Yang, P. G. Mehta and S. P. Meyn, " A Control-oriented Approach for Particle Filtering," ACC&CDC 2011

Destexhe & Marder, Nature, 2004; Kopell et al., Neuroscience, 2009; Lee & Mumford, J. Opt. Soc, 2003

Background: Kuramoto model

$$\mathrm{d}\theta_i(t) = \left(\omega_i + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(\theta_j(t) - \theta_i(t)) \right) \mathrm{d}t + \sigma \, \mathrm{d}\xi_i(t), \quad i = 1, \ldots, N$$



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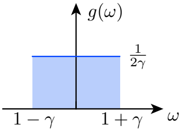
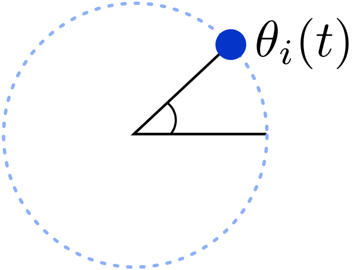
- ω_i: taken from distribution $g(\omega)$ over $[1-\gamma, 1+\gamma]$

  γ: measures the heterogeneity of the population





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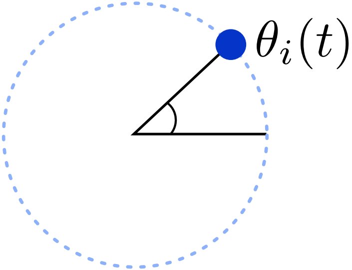
- $\boxed{\kappa\text{: measures the strength of coupling}}$



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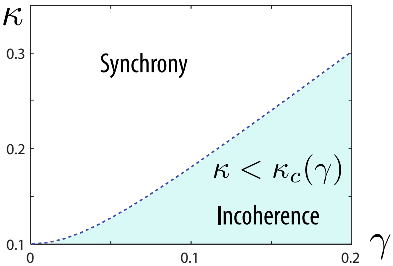

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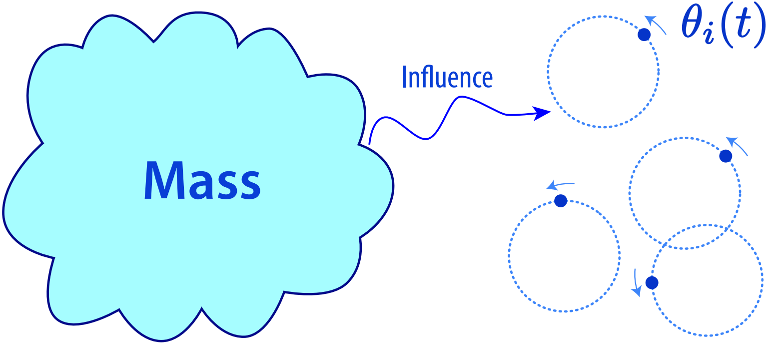
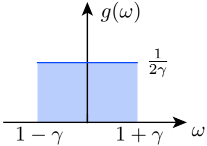


Oscillator Games

Oscillator Game

N oscillators with natural frequency ω_i, chosen from distribution $g(\cdot)$



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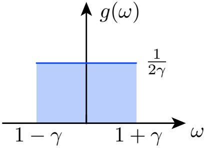

Dynamics of i^{th} oscillator, $\quad \mathrm{d}\theta_i = (\omega_i + u_i(t))\,\mathrm{d}t + \sigma\,\mathrm{d}\xi_i$

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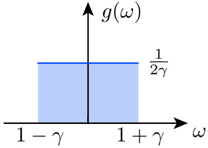

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Oscillator seeks control $u_i(\cdot)$ to minimize,

$$\eta_i(u_i; u_{-i}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathsf{E}[\ \underbrace{c(\theta_i;\theta_{-i})}_{\text{cost of anarchy}} + \underbrace{\tfrac{1}{2}Ru_i^2}_{\text{cost of control}}\ ]\,\mathrm{d}s$$

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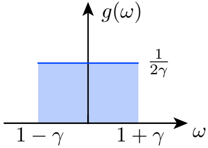

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Cost of anarchy,

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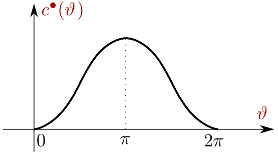

Mean-field model

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$$c(\theta_i;\theta_{-i}) = \frac{1}{N}\sum_{j\neq i} c^\bullet(\theta_i, \theta_j(t)) \xrightarrow{N\to\infty} \bar{c}(\theta_i, t)$$

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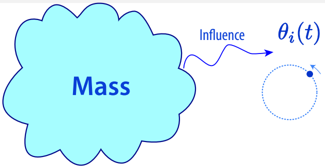

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HJB: $\partial_t h + \omega\partial_\theta h = \dfrac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta,t) + \eta^* - \dfrac{\sigma^2}{2}\partial^2_{\theta\theta} h \quad \Rightarrow \quad h(\theta,t,\omega)$

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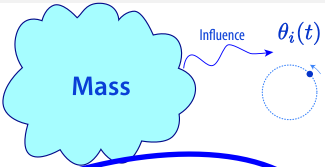

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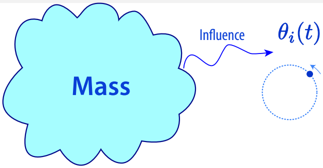

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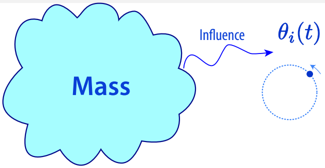

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ε-Nash equilibrium

Solution to PDE $\Longrightarrow$ Oblivious control for ith oscillator,

$$\boxed{u_i^o = -\frac{1}{R}\partial_\theta h(\theta(t), t, \omega)\Big|_{\omega=\omega_i}}$$

Theorem: ε-Nash equilibrium property,

$$\eta_i(u_i^o; u_{-i}^o) \leq \eta_i(u_i; u_{-i}^o) + O(\frac{1}{\sqrt{N}}), \quad i = 1, \ldots, N,$$

for any adapted control u_i.

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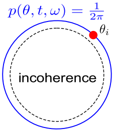
Solution to PDE?

Incoherent Solution

$$h(\theta, t, \omega) \equiv 0\,, \qquad p(\theta, t, \omega) \equiv \frac{1}{2\pi}$$



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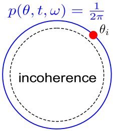

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$$= \frac{1}{4}$$

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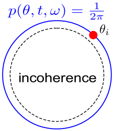

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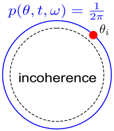


$$h(\theta,t,\omega)=0 \quad \Rightarrow \quad \partial_t h+\omega\partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta,t)+\eta^* - \frac{\sigma^2}{2}\partial^2_{\theta\theta}h$$

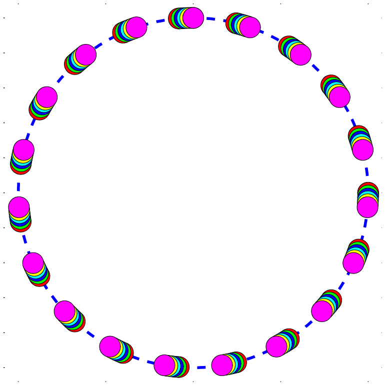
$$p(\theta,t,\omega)=\frac{1}{2\pi} \quad \Rightarrow \quad \partial_t p+\omega\partial_\theta p = \frac{1}{R}\partial_\theta[p(\partial_\theta h)] + \frac{\sigma^2}{2}\partial^2_{\theta\theta}p$$

$$p(\theta,t,\omega)=\frac{1}{2\pi} \quad \Rightarrow \quad \bar{c}(\theta,t) = \int_\Omega\int_0^{2\pi} c^\bullet(\theta,\vartheta)p(\vartheta,t,\omega)g(\omega)\,\mathrm{d}\vartheta\,\mathrm{d}\omega$$

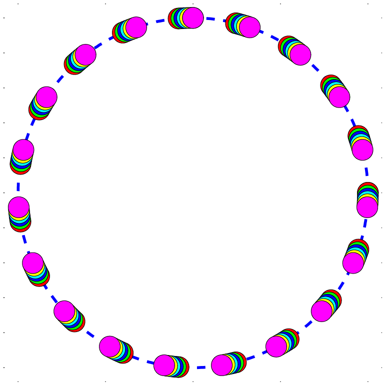
$$= \int_\Omega\int_0^{2\pi} \tfrac{1}{2}\sin^2\left(\frac{\theta-\vartheta}{2}\right)\frac{1}{2\pi}g(\omega)\,\mathrm{d}\vartheta\,\mathrm{d}\omega$$

$$= \frac{1}{4}$$

Examples of Solutions: Incoherence and Synchrony

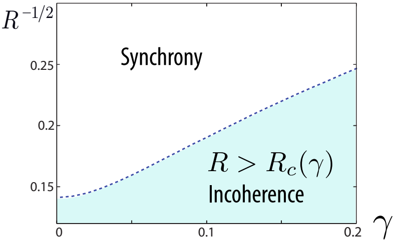


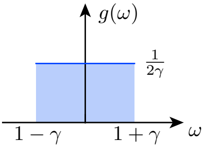
Incoherence



Synchrony

Bifurcation



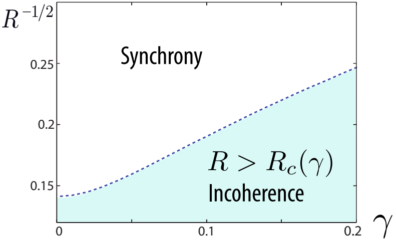


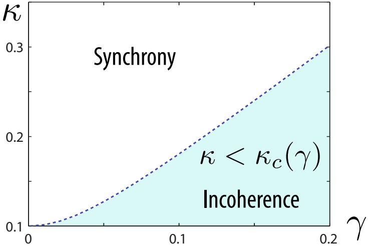
$$\mathrm{d}\theta_i = (\omega_i + u_i)\,\mathrm{d}t + \sigma\,\mathrm{d}\xi_i$$

$$\eta_i(u_i; u_{-i}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathsf{E}[c(\theta_i;\theta_{-i}) + \tfrac{1}{2}Ru_i^2]\,\mathrm{d}s$$

Yin et al., ACC 2010

Bifurcation





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Yin et al., ACC 2010

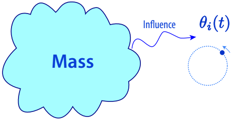
$$\mathrm{d}\theta_i = \left(\omega_i + \frac{\kappa}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\right)\mathrm{d}t + \sigma\,\mathrm{d}\xi_i$$

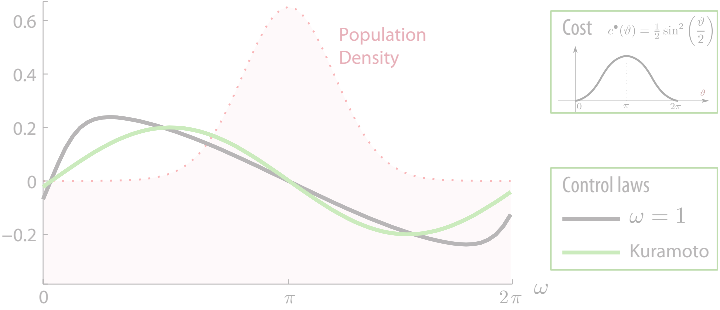
Strogatz et al., J. Stat. Phy., 1991

Comparison of controls

Control law

$$u_i = \varphi(\theta, t, \omega_i) := -\frac{1}{R}\partial_\theta h(\theta, t, \omega)\Big|_{\omega=\omega_i}$$





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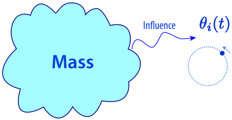

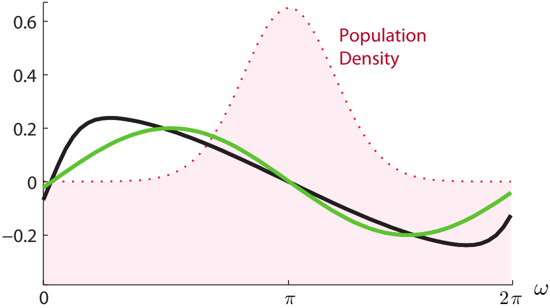

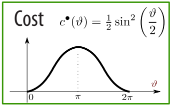

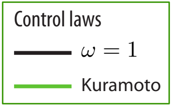

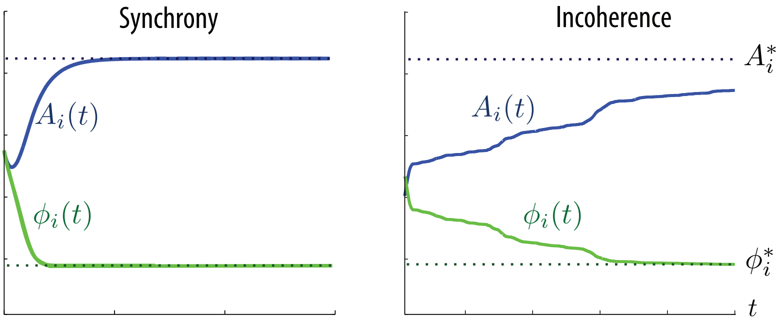

Mean-field Filter

Learning to Control

Approximate Dynamic Programming

- Optimality equation $\displaystyle \min_{u_i}\{\underbrace{c(\theta;\theta_{-i}(t))+\tfrac{1}{2}Ru_i^2+\mathscr{D}_{u_i}h_i(\theta,t)}_{=:\,H_i(\theta,u_i;\theta_{-i}(t))}\} = \eta_i^*$

Watkins & Dayan, Q-learning, 1992; Bertsekas & Tsitsiklis, NDP, 1996; Mehta & Meyn, CDC 2009

Approximate Dynamic Programming

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- Optimal control law

$$u_i^* = -\frac{1}{R}\partial_\theta h_i(\theta,t)$$

Watkins & Dayan, Q-learning, 1992; Bertsekas & Tsitsiklis, NDP, 1996; Mehta & Meyn, CDC 2009

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- Optimal control law

$$u_i^* = -\frac{1}{R}\partial_\theta h_i(\theta,t)$$

- Parameterization for approximation:

$$H_i^{(A_i,\phi_i)}(\theta,u_i;\theta_{-i}(t)) = c(\theta;\theta_{-i}(t)) + \tfrac{1}{2}Ru_i^2 + (\omega_i - 1 + u_i)A_i S^{(\phi_i)} + \frac{\sigma^2}{2}A_i C^{(\phi_i)}$$

where

$$S^{(\phi)}(\theta,\theta_{-i}) = \frac{1}{N}\sum_{j\neq i}\sin(\theta-\theta_j-\phi), \quad C^{(\phi)}(\theta,\theta_{-i}) = \frac{1}{N}\sum_{j\neq i}\cos(\theta-\theta_j-\phi)$$

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- Approx. optimal control:

$$u_i^{(A_i,\phi_i)} = \arg\min_{u_i}\{H_i^{(A_i,\phi_i)}(\theta,u_i;\theta_{-i}(t))\} = -\frac{A_i}{RN}\sum_{j\neq i}\sin(\theta-\theta_j(t)-\phi_i)$$

Watkins & Dayan, Q-learning, 1992; Bertsekas & Tsitsiklis, NDP, 1996; Mehta & Meyn, CDC 2009

Learning algorithm

- Bellman error:

$$\text{Pointwise:} \quad \mathscr{L}^{(A_i,\phi_i)}(\theta,t) = \min_{u_i}\{H_i^{(A_i,\phi_i)}\} - \eta_i^{(A_i^*,\phi_i^*)}$$

- Stochastic approximation based on ODE,

$$\tilde{e}(A_i,\phi_i) = \sum_{k=1}^{2} |\langle \mathscr{L}^{(A_i,\phi_i)}, \tilde{\varphi}_k(\theta)\rangle|^2$$

$$\frac{\mathrm{d}A_i}{\mathrm{d}t} = -\varepsilon \frac{\mathrm{d}\tilde{e}(A_i,\phi_i)}{\mathrm{d}A_i}, \quad \frac{\mathrm{d}\phi_i}{\mathrm{d}t} = -\varepsilon \frac{\mathrm{d}\tilde{e}(A_i,\phi_i)}{\mathrm{d}\phi_i}$$

Yin et al., CDC 2010

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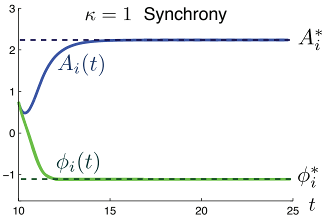

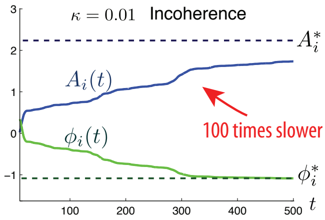

Yin et al., CDC 2010

Comparison of average cost

$$\mathrm{d}\theta_i = (\omega_i + u_i)\,\mathrm{d}t + \sigma\,\mathrm{d}\xi_i \qquad u_i = -\frac{A_i^*}{RN}\sum_{j\neq i}\sin(\theta_i - \theta_j(t) - \phi_i^*)$$

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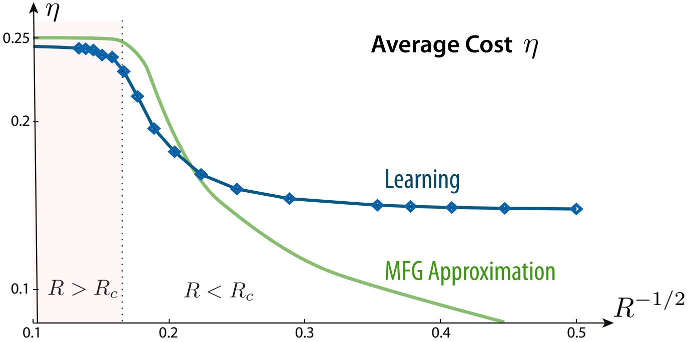
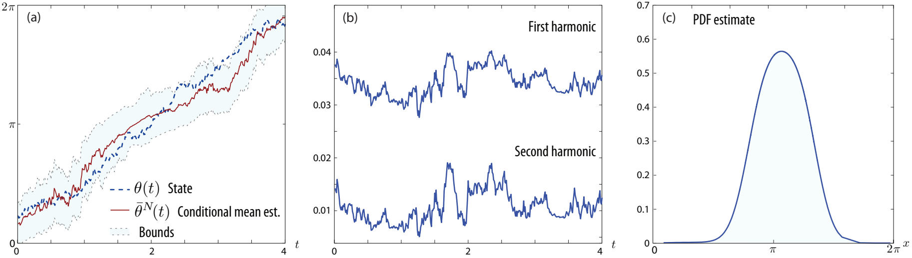


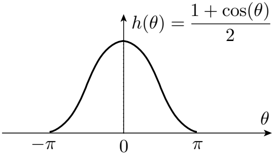


Particle Filter Games

Filtering problem

Signal, observation processes:

$$\mathrm{d}\theta_t = \omega\,\mathrm{d}t + \sigma_B\,\mathrm{d}B_t \quad \text{mod } 2\pi$$
$$\mathrm{d}Z_t = h(\theta_t)\,\mathrm{d}t + \sigma_W\,\mathrm{d}W_t$$



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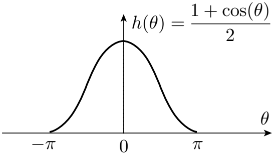

Nonlinear Filtering

Objective: estimate the posterior distribution p^* of θ_t given $\mathscr{Z}^t$.

Filtering problem

Signal, observation processes:

$$d\theta_t = \omega\, dt + \sigma_B\, dB_t \quad \text{mod } 2\pi$$
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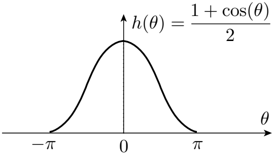

Nonlinear Filtering

Objective: estimate the posterior distribution p^* of θ_t given $\mathscr{Z}^t$.

Solution approaches:

- Linear system: Kalman filter (R. E. Kalman, 1960)
- Nonlinear system: Wonham filter (W. M. Wonham, 1965)
- Numerical Methods: Particle filter (N. J. Gordon et al., 1993)

Feedback Particle Filter

Signal, observation processes:

$$\mathrm{d}\theta_t = \omega\,\mathrm{d}t + \sigma_B\,\mathrm{d}B_t \quad \text{mod } 2\pi$$
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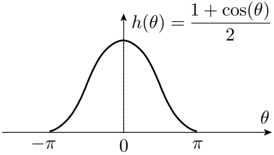


Feedback Particle Filter

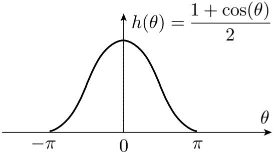
Particles evolve as controlled SDEs with independent noise,

$$\mathrm{d}\theta_t^i = \omega\,\mathrm{d}t + \sigma_B\,\mathrm{d}B_t^i + \mathrm{d}U_t^i \quad \text{mod } 2\pi, \qquad i = 1, \ldots, N.$$

Feedback Particle Filter

Signal, observation processes:

$$\begin{aligned} \mathrm{d}\theta_t &= \omega\,\mathrm{d}t + \sigma_B\,\mathrm{d}B_t \quad \text{mod } 2\pi \\ \mathrm{d}Z_t &= h(\theta_t)\,\mathrm{d}t + \sigma_W\,\mathrm{d}W_t \end{aligned}$$



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Objective: Choose control U_t^i so that,

$$\mathsf{P}\{\theta_t^i \in \cdot \mid Z_0^t\} = p^* = \mathsf{P}\{\theta_t \in \cdot \mid Z_0^t\}$$

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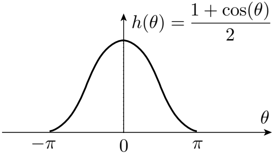

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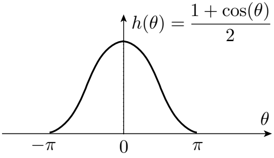
$\Longrightarrow$ Empirical distribution of particles approximates p^*.

Filtering for Oscillator

Signal, observation processes:



$$
\begin{aligned}
\mathrm{d}\theta_t &= \omega\,\mathrm{d}t + \sigma_B\,\mathrm{d}B_t \quad \text{mod } 2\pi \\
\mathrm{d}Z_t &= h(\theta_t)\,\mathrm{d}t + \sigma_W\,\mathrm{d}W_t
\end{aligned}
$$

Particle evolution,

$$
\begin{aligned}
\mathrm{d}\theta_t^i = \omega\,\mathrm{d}t + \sigma_B\,\mathrm{d}B_t^i + v(\theta_t^i)[\mathrm{d}Z_t - \frac{1}{2}(h(\theta_t^i) + \hat{h})\,\mathrm{d}t] \\
+ \frac{1}{2}\sigma_W^2 v v'\,\mathrm{d}t \quad \text{mod } 2\pi, \qquad i = 1, \ldots, N.
\end{aligned}
$$

Observer gain $v(\theta_t^i)$ is obtained via the solution of an E-L equation,

$$
-\frac{\partial}{\partial\theta}\left(\frac{1}{p(\theta,t)}\frac{\partial}{\partial\theta}\{p(\theta,t)v(\theta,t)\}\right) = -\frac{\sin\theta}{\sigma_W^2}
$$

Filtering for Oscillator

Fourier form of $p(\theta, t)$,

$$p(\theta, t) = \frac{1}{2\pi} + P_s(t)\sin\theta + P_c(t)\cos\theta$$

Approx. solution of E-L equation, using a perturbation method:

$$v(\theta, t) = \frac{1}{2\sigma_W^2}\left\{-\sin\theta + \frac{\pi}{2}[P_c(t)\sin 2\theta - P_s(t)\cos 2\theta]\right\},$$
$$v'(\theta, t) = \frac{1}{2\sigma_W^2}\{-\cos\theta + \pi[P_c(t)\cos 2\theta + P_s(t)\sin 2\theta]\}$$

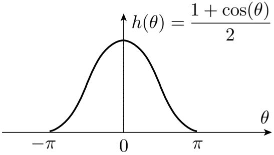
where

$$P_c(t) \approx \bar{P}_c^{(N)}(t) = \frac{1}{\pi N}\sum_{j=1}^{N}\cos\theta_t^j, \quad P_s(t) \approx \bar{P}_s^{(N)}(t) = \frac{1}{\pi N}\sum_{j=1}^{N}\sin\theta_t^j.$$

Simulation Results

Signal, observation processes:

$$\mathrm{d}\theta_t = 1\,\mathrm{d}t + 0.5\,\mathrm{d}B_t \quad \text{mod } 2\pi$$
$$\mathrm{d}Z_t = h(\theta_t)\,\mathrm{d}t + 0.4\,\mathrm{d}W_t$$



$N = 100$ particles,

$$\mathrm{d}\theta_t^i = 1\,\mathrm{d}t + 0.5\,\mathrm{d}B_t^i + U(\theta_t^i; \bar{P}_c^{(N)}(t), \bar{P}_s^{(N)}(t)) \quad \text{mod } 2\pi$$

Simulation Results

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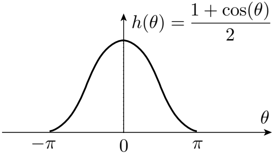

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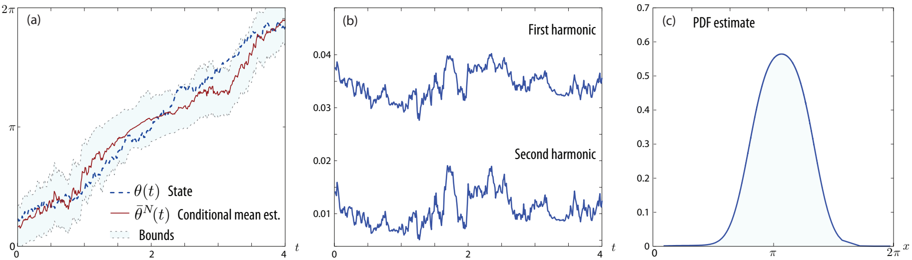

Simulation Results

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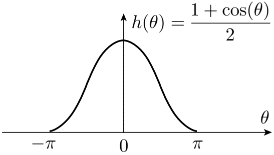

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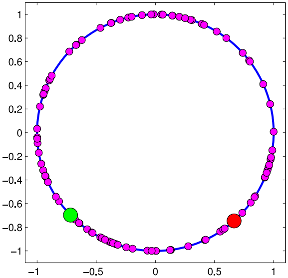
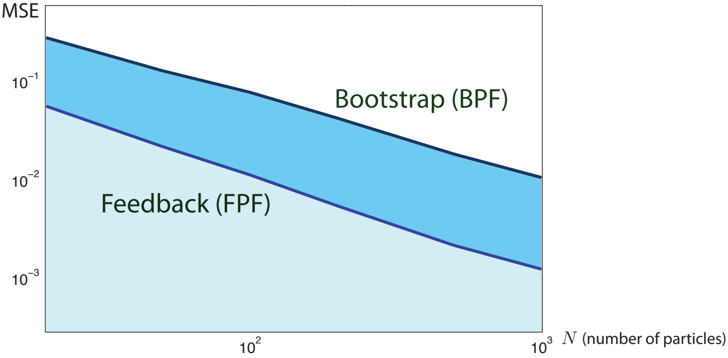


Variance Reduction

Filtering for simple linear model.

Mean-square error: $$\frac{1}{T}\int_0^T \left(\frac{\Sigma_t^{(N)} - \Sigma_t}{\Sigma_t}\right)^2 \mathrm{d}t$$



Conclusions

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The Feedback Particle Filter is a great playground, and has enormous potential for approximate nonlinear filtering in practice.

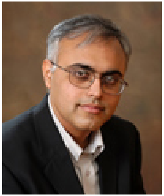
Perhaps this is where the profit lies?

Thank you!

Collaborators

Huibing Yin

Tao Yang



Prashant Mehta

Uday Shanbhag

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