Viewing networks as systems: adventures in search of a system theory of networks

Mehran Mesbahi
Aeronautics & Astronautics
University of Washington

(joint work with Airlie Chapman, Marzieh Nabi Abdolyousefi, Amir Rahmani, Magnus Egerstedt)

The 2011 Santa Barbara Control Workshop

June 24, 2011

basic premise

1. there is a need to adopt an input-output point view on autonomous networks
2. applications
   - human-swarm interaction; security and semi-autonomy
   - biological networks
   - quantum networks
   - applications in mathematics
3. theory
   - controllability and observability
   - influencing network behavior
   - fundamental bounds on performance

what is meant by autonomous networks?

- microscopic/local interactions between dynamic elements lead to macroscopic behavior
- no joysticks or adversaries

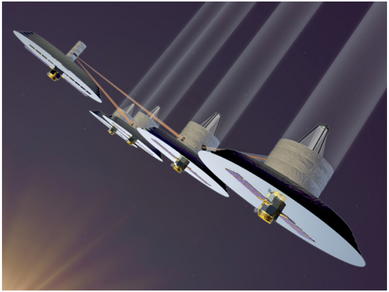
popular models include: diffusion, advection, diffusion-advection, etc.

some advantages

- component-level fault-tolerance
- reconfigurable, less costly to manufacture, more design and operational flexibility

some disadvantages

- interaction overhead
- interaction-induced complexity, robustness, security, external interactions

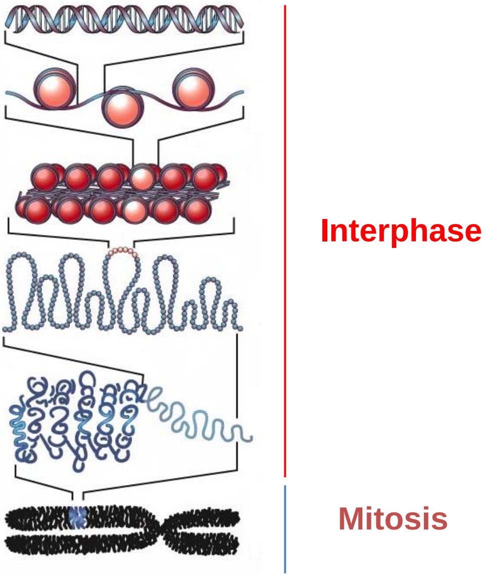


Quaternionic networks

Unicycle networks

Unicycle networks with collision avoidance

Packaging DNA

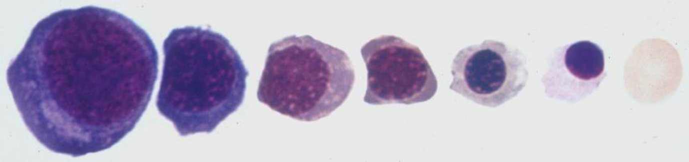


Felsenfeld G, Groudine M. 2003. Controlling the double helix. *Nature* 421:448–53

Chromosomal network

Courtesy of Tom Misteli (2011)

Genomic Reorganization



Differentiation

Hematopoietic stem cell

Red blood cell
(no nucleus)

quantum networks

- motion of a quantum particle on a manifold is governed by the Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi = H\Psi$$

  where Ψ is quantum state of the particle, and H is the Hamiltonian $H = L - K$; L is the Laplace operator and K is the potential
- Spatial discretization/normalization on the graph $G = (V, E)$:

$$i\frac{\partial}{\partial t}\Psi = (\mathcal{L}(\mathcal{G}) - K)\Psi$$

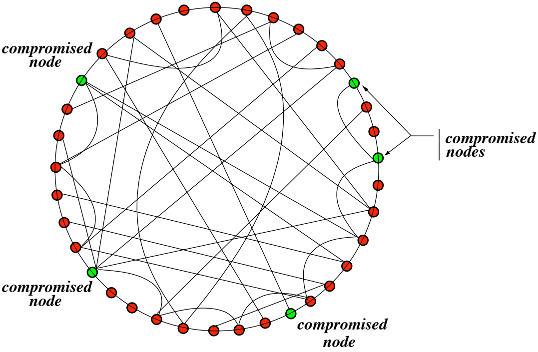
  where $\mathcal{L}$ is the normalized Laplacian. For simplicity we let $K = qB$, $q \in \Re$, for some fixed B and vary q only
- $|\Psi_t(x)|^2$ can be interpreted as the probability that the particle is in vertex x at time t

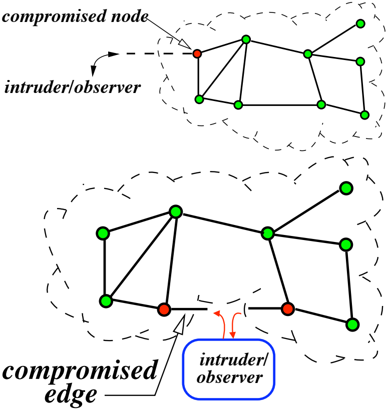
Human swarm interaction

semi-autonomy, security,
and input-state-output analysis of networks

consider following question ...

how easy is it to **control** or **observe** an autonomous network via a small subset of its nodes or edges?





consider certain nodes in diffusion-based protocols on graphs as being compromised ...

... our system now looks like

$$\dot{q}(t) = A(\mathcal{G}, \mathcal{R})q(t) + B(\mathcal{G}, \mathcal{R})u(t), \quad y(t) = C(\mathcal{G}, \mathcal{S})q(t)$$

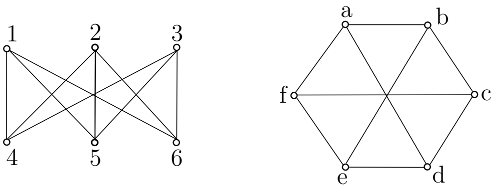
- "$A(\mathcal{G}, \mathcal{R})$" (system matrix) is a perturbation of the Laplacian matrix associated with the floating nodes
- "$B(\mathcal{G}, \mathcal{R})$" (the input matrix) is an indicator function, i.e., $(\delta_a)_i = 1$ if a is connected to i, zero otherwise
- possible "$C(\mathcal{G}, \mathcal{S})$" (the observation matrix) can also be included, e.g., $C(\mathcal{G}) = \delta_a^T$
- setup resembles the Dirichlet boundary condition on diffusion PDEs or advection-diffusion PDEs on manifolds

question: is this "compromised" diffusion controllable/observable from the infiltration point(s)? how does the structure of the graph enter the analysis?

partial answer: symmetry structure & equitable partitions

graph isomorphism and automorphism

- In the world of unlabeled graphs, there is an equivalency relation that is NOT easy to check!
- $\mathcal{G}, \mathcal{H}$ are isomorphic if there adjacency preserving bijection between them



$$\theta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & c & e & d & f & b \end{pmatrix}$$

- it is know that two graphs $\mathcal{G}$ and $\mathcal{H}$ are isomorphic iff there exists a permutation matrix P such that

$$PA(\mathcal{G}) = A(\mathcal{H})P$$

automorphism group of the graph

A more precise way to characterize internal symmetry is via the graph automorphism: a mapping

$$\alpha : V(\mathcal{G}) \to V(\mathcal{G})$$

such that

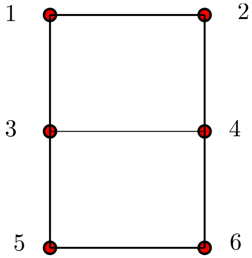
$$uv \in E(\mathcal{G}) \quad \text{if and only if} \quad \alpha(u)\alpha(v) \in E(\mathcal{G})$$

Algebraically

$$\mathbf{AUT}(\mathcal{G}) = \{P \in \mathrm{sym}\,(\mathcal{G}) \,|\, PA(\mathcal{G}) = A(\mathcal{G})P\}$$

with $A(\mathcal{G})$ as the adjacency matrix of the graph.

automorphism: example



$$\mathbf{AUT}(\mathcal{G})$$

$$\{1, 2, 3, 4, 5, 6\}$$

$$\{2, 1, 4, 3, 6, 5\}$$

$$\{5, 6, 3, 4, 1, 2\}$$

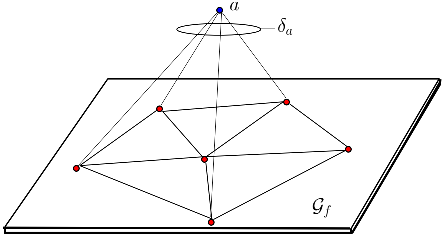
$$\{6, 5, 5, 3, 2, 1\}$$

AUT(G) has a group structure with respect to composition: trivial permutation is identity, composition is associative, inverses exist

graph-theoretic means of viewing uncontrollability and unobservability

Theorem (single infiltrator case)

The compromised diffusion is uncontrollable/unobservable if $\mathcal{G}_f$ admits an automorphism that leaves the indicator function invariant under its action.



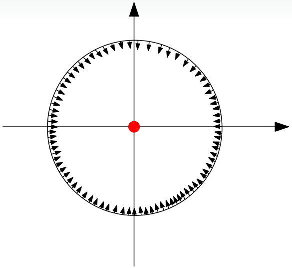
This result has been extended for the multi-infiltrator case using equitable partitions ... see **Rahmani, Ji, M, & Egerstedt,** *SIAM J. Control and Opt., January 2009*

performance measures and adaptive topologies

Two Candidate Influence Measures

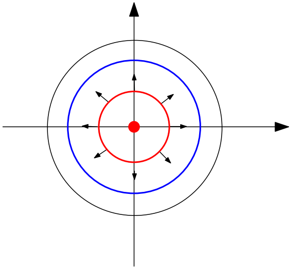
Mean tracking measure is the *average mean of the error* to steer the entire network to the origin over an infinite time horizon:

$$J_\mu(\mathcal{G}, \mathcal{R}) = 2\,\mathbb{E}_{\|z(0)\|=1} \int_0^\infty z(t)^T z(t) dt.$$



Variance damping measure is the *average variance of the error* due to external agents injecting zero-mean, unit covariance Gaussian noise, as $t \to \infty$,

$$J_\sigma(\mathcal{G}, \mathcal{R}) = \frac{2}{n} \lim_{t\to\infty} \mathbb{E}\, z(t)^T z(t).$$



Mean Tracking Measure

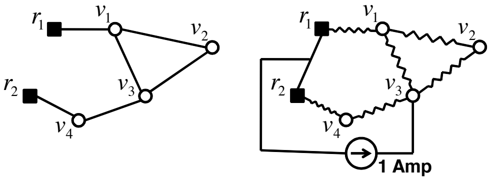
Equivalent expressions for J_μ, one algebraic:

$$J_\mu(\mathcal{G},\mathcal{R}) = \frac{1}{n}\mathbf{tr}\left(-A(\mathcal{G},\mathcal{R})\right)^{-1}$$

and one, graph-theoretic/electric networks:

$$J_\mu(\mathcal{G},\mathcal{R}) = \frac{1}{n}\sum_{i=1}^{n} E_{\text{eff}}(v_i)$$

- $E_{\text{eff}}(v_i)$ is the effective resistance of the corresponding resistive electrical network as viewed from node node v_i



More graph theoretic connection for trees ... and one external node

$$J_\mu(\mathcal{T}, \mathcal{R}^i) = \frac{1}{n}\left(\sum_{j=1}^{n} d\left(v_i, v_j\right) + n\right).$$

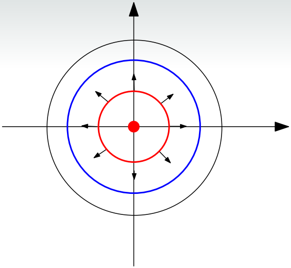
$$J_\mu(\mathcal{T}, \mathcal{R}^i) = \frac{n-1}{n} c\left(v_i, \mathcal{T}\right) + 1.$$

where $c\left(v_i, \mathcal{G}\right)$ is the **closeness centrality** of node v_i, i.e., the average distance between v_i and all other nodes

More generally, for n-node trees,

$$2 - \frac{1}{n} \leq J_\mu(\mathcal{T}, \mathcal{R}^i) \leq \frac{1}{2}\left(n+1\right)$$

Variance Damping Measure



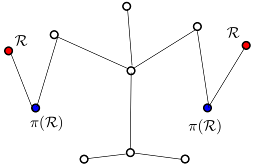
Recall: the variance damping is defined as

$$J_\sigma(\mathcal{G},\mathcal{R}) = \frac{2}{n}\lim_{t\to\infty} \mathbb{E}\, z(t)^T z(t).$$

We have

$$J_\sigma(\mathcal{G},\mathcal{R}) = \frac{2}{n}\mathbf{tr}(P(\mathcal{G},\mathcal{R}))$$

$$J_\sigma(\mathcal{G},\mathcal{R}) = \frac{1}{n}\sum_{v_i\in\pi(\mathcal{E}_R)} E_{\text{eff}}(v_i)$$



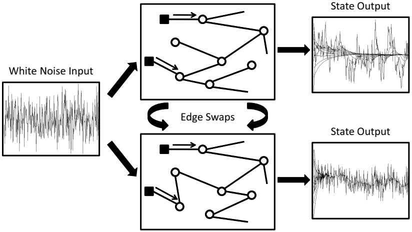
For a single external agent attached to v_i: $J_\sigma\left(\mathcal{G},\mathcal{R}^i\right) = \frac{1}{n}$

Network Synthesis via Local Edge Swaps

- For a secure network: large $J_\mu(\mathcal{G}, \mathcal{R})$ (resistance to external influence) and small $J_\sigma(\mathcal{G}, \mathcal{R})$ (external noise damping)
- In terms of effective resistance

$$J_\mu(\mathcal{G}, \mathcal{R}) = J_\sigma(\mathcal{G}, \mathcal{R}) + \frac{1}{n} \sum_{v_i \notin \pi(\mathcal{E}_R)} E_{\text{eff}}(v_i)$$

- **approach:** adaptively alter the tree graph structure to increase $\sum_{v_i \notin \pi(\mathcal{E}_R)} E_{\text{eff}}(v_i)$ while keeping $J_\sigma(\mathcal{G}, \mathcal{R})$ small



adaptive trees

Price of Stability (PoS) and Anarchy (PoA)

Protocol setup can be considered in the context of a potential game ... and examine the properties of the corresponding Nash equilibria

Price of Stability (PoS) and Price of Anarchy (PoA) are defined as:

$$\text{PoS} = \frac{\text{Cost of Best Equilibrium Network}}{\text{Cost of Optimal Network}}$$

$$\text{PoA} = \frac{\text{Cost of Worst Equilibrium Network}}{\text{Cost of Optimal Network}}$$

One of our typical observations: for protocol "5":

- With cost $1/J_\mu(\mathcal{T},\mathcal{R})$ the PoS$=1$ and PoA$\leq r$.
- With cost $J_\sigma(\mathcal{T},\mathcal{R})$ the PoS$=1$ and PoA$<\frac{11\sqrt{5}}{20}\approx 1.23$.

Consequence: For $r=1$, protocol 5 will always reach the optimal value for $1/J_\mu(\mathcal{T},\mathcal{R})$.

identify and infiltrate

Controllability/observability over Random Networks

are “most” diffusion-based networks controllable/observable?

- we argued that symmetry of the network provides insights into its controllability and observability, along with notions such as effective resistance
- it can be shown that almost all graphs have no automorphism group as $n \rightarrow \infty$
- **a natural questions is whether random networks are controllable or observable?**

why study random interactions ...

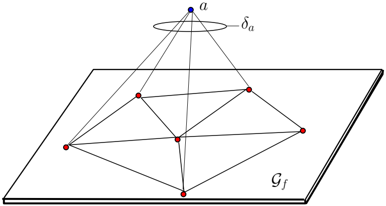
- robustness
- design, e.g., prolonging battery life-time
- approximating state dependency
- sometimes it leads to more streamlined analysis

influencing diffusion-based networks over random networks

Consider again our Dirichlet dynamics over graphs

$$\begin{aligned} \dot{q}(t) &= A(\mathcal{G},\mathcal{R})q(t) + B(\mathcal{G},\mathcal{R})u(t) \\ z(t) &= C(\mathcal{G},\mathcal{S})q(t) \end{aligned}$$

- '$\mathcal{G}$' is a realization of random graph $G(n,p)$ over Δ time steps
- $B(\mathcal{G},\mathcal{R})$ is a realization of random indicator over Δ time steps
- $C(\mathcal{G},\mathcal{S})$ is random observation matrix for each node



Can such a random observation of a compromised diffusion over a random network be observed?

Controlled Diffusion over Random Networks ...

- Let
$$R(k) = C(\mathcal{G}(k), \mathcal{S}(k))\, C(\mathcal{G}(k), \mathcal{S}(k))^T$$
- Let
$$A(k) = e^{\Delta A(\mathcal{G}(k), \mathcal{R}(k))}$$

and consider the event:

$$\begin{aligned}\Omega_k &= \{\mathbf{det}(R(1) + A(1)R(2)A(1) + \cdots \\ &+ (A(1)\cdots A(k-1))R(k)(A(k-1)\cdots A(1)) \neq 0\}\end{aligned}$$

Controlled Diffusion over Random Networks

The controlled diffusion is weakly observable if for some $k \geq 1$,

$$\mathbf{P}\{\Omega_k\} \neq 0$$

Then

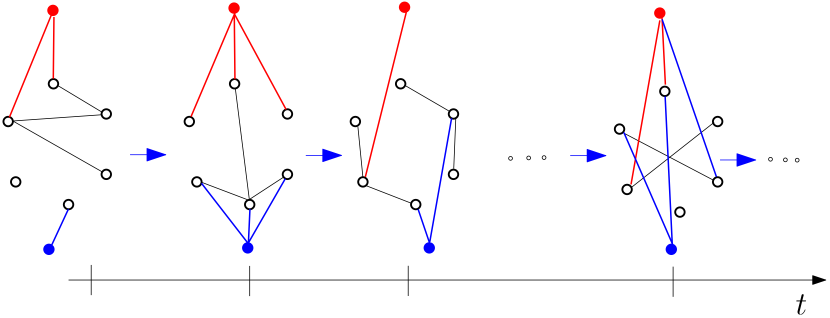
(**Bougerol**) If the system is **weakly observable**, it admits an **observer** such that by observing $y(k)$, one can estimate $\widehat{x}(k)$ for which

$$\limsup_{k \to \infty} \frac{1}{k} \log \|x(k) - \widehat{x}(k)\| < -\gamma$$

for some $\gamma > 0$ almost surely

pictorially ...

Let blue edges denote influence from external blue nodes and red edges denote observations on the node states by the external red nodes



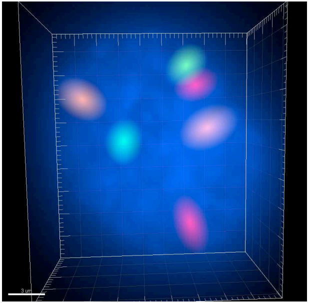
- diffusion process on a random network is weakly observable and weakly controllable

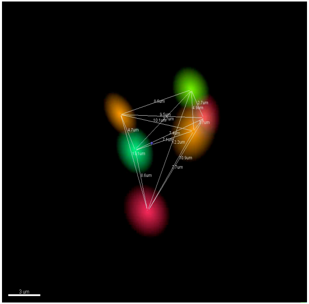
let us now revisit some of our earlier examples ...

- chromosomal interaction networks
- quantum networks

- if time permits we will discuss another facet of the input-output point of view for graph isomorphism problem

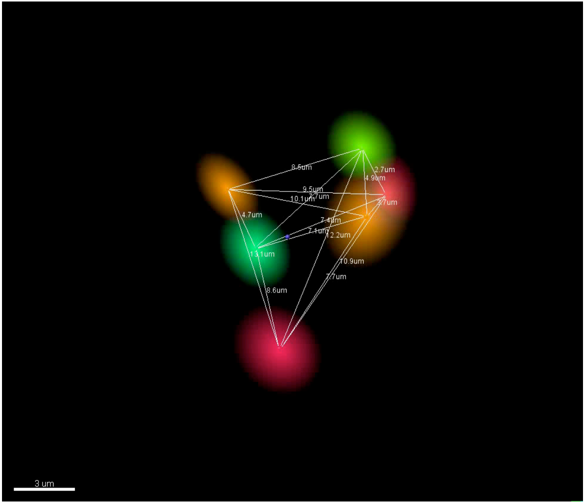
Genomic organization





Courtesy of I. Rajapakse et al. (2011)

Genomic organization ...



Courtesy of I. Rajapakse et al. (2011)

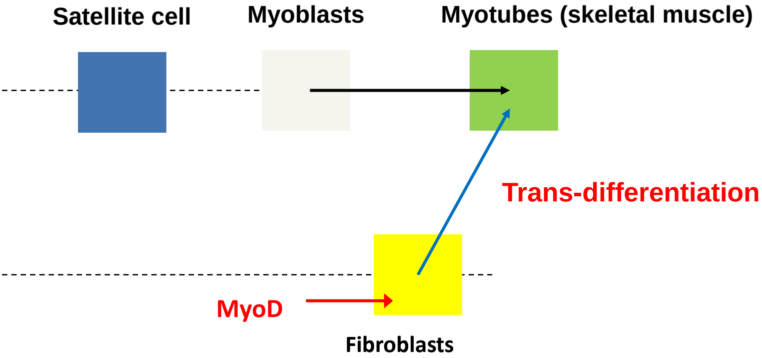
networks in genomic biology

- genome is non-randomly located in the cell
- cell function (gene expression) and cell geometry/form (chromosome distribution), are highly linked after an initial alignment
- cell form and function: reorganization of stem cell

One may ask whether cell reorganizes itself to better accomodate being efficiently steered in a given direction.

disclaimer: all biologically significant statements are due to my FHCRC colleagues **Indika Rajapakse and Mark Groudine***.*

Master Switch for Muscle



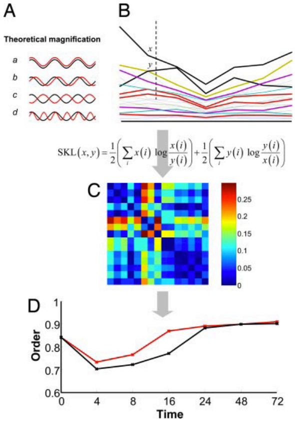
Davis, R. L., H. Weintraub, et al. (1987). Expression of a single transfected cDNA converts fibroblasts to myoblasts. Cell 51(6): 987-1000.

controllability and cell differentiation?!

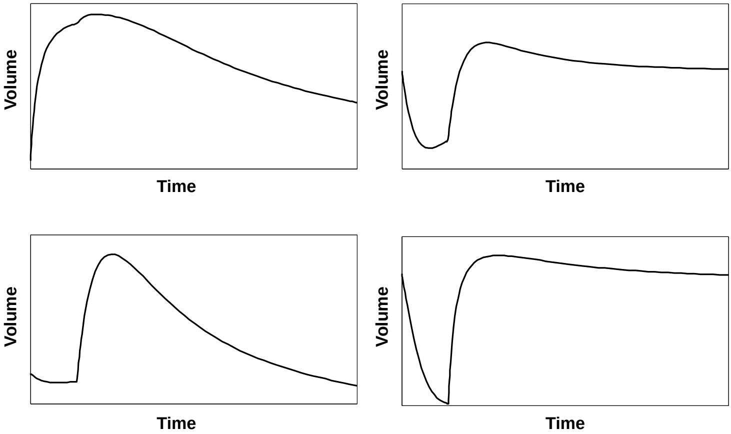
Possible translation?

biological	*control theoretic*
entropy / disorder	*asymmetry*

If this is valid, it implies that the cancer causes the cell to reorganize itself so that it can be efficiently steered!



a model



Dynamics and Control of State-dependent Networks for Understanding Genomic Organization, PNAS, Rajapakse, Mark Groudine, & M (submitted)

quantum tunneling

quantum networks

- motion of a quantum particle on a manifold is governed by the Schrödinger equation

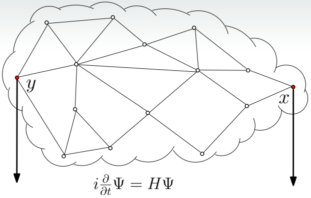
$$i\frac{\partial}{\partial t}\Psi = H\Psi$$

  where Ψ is quantum state of the particle, and H is the Hamiltonian $H = L - K$; L is the Laplace operator and K is the potential
- Spatial discretization/normalization on the graph $G = (V, E)$:

$$i\frac{\partial}{\partial t}\Psi = (\mathcal{L}(\mathcal{G}) - K)\Psi$$

  where $\mathcal{L}$ is the normalized Laplacian. For simplicity we let $K = qB$, $q \in \Re$, for some fixed B and vary q only
- $|\Psi_t(x)|^2$ can be interpreted as the probability that the particle is in vertex x at time t

tunneling ...



- Let the particle be initially in pure x state
- Define the tunneling coefficient between two nodes x and y as

$$\tau(x, y) = \lim_{q \to \infty} \limsup_{t \to \infty} |\Psi_t(y)|^2$$

- When $\tau(x, y) = 1$ perfect (asymptotic) tunneling has occurred.

Theorem

(Lin, Lippner, Yau) There is a perfect tunneling between two vertices if there is an involution of $\mathcal{G}$, taking x to y, such that B is preserved under the action of the involution.

graph isomorphism

controllability and graph isomorphism problem

- there has been a surge of interest by algebraic graph theorists/quantum network researchers in understanding the controllability of the pair $(A(\mathcal{G}), \delta(\mathcal{G}))$
- let

$$G(\mathcal{G}, \delta(\mathcal{G}) = \delta(\mathcal{G})^T (sI - A(\mathcal{G}))^{-1} \delta(\mathcal{G})$$

  be the input-output map for the graph
- recent work by Godsil points out a connection between graph controllability and graph isomorphism problem

Definition: The pairs $(A(\mathcal{G}), g)$ and $(A(\mathcal{H}), h)$ are isomorphic if there exists an orthogonal matrix U such that

$$UA(\mathcal{G})U^T = A(\mathcal{H}) \quad \text{and} \quad Ug = h$$

an observation

Godsil (2010) If $(A(\mathcal{G}), g)$ and $(A(\mathcal{H}), h)$ are controllable and

$$G(\mathcal{G}, g) = G(\mathcal{H}, h),$$

the pairs $(A(\mathcal{G}), g)$, $(A(\mathcal{H}), h)$ are isomorphic

- note that if the isomorphism was done with respect to the $GL(n)$ then the above statement follows directly from linear systems theory
- (Zavlanos & Pappas- 2010):

$$\text{permutation matrices} = (\perp \text{ matrices }) \cap (\geq 0 \text{ matrices})$$

- the graph isomorphism problem can be thought of a particular instance of system realization problem!

recap and ongoing work

- system-theoretic perspective on network security and semi-autonomoy
  - security, controllability, and observability over networks
  - performance; network formation games
  - random networks
  - controllability outside of "control"?
- some of the ongoing work
  - threshold phenomena in system theoretic properties of random networks; applications in social networks
  - state-dependent graphs and biological networks; graph sequences
  - human swarm interaction
  - social networks
  - advection and advection-diffusion over graphs and their system-theoretic properties
  - games on graphs and how the structure influences equilibria