Distributed constrained optimization under time-varying multi-agent interactions

Sonia Martínez

Mechanical and Aerospace Enginering
University of California, San Diego
soniamd@ucsd.edu

**The 2011 Santa Barbara Control Workshop:
Decision, Dynamics, and Control in Multi-Agent Systems**

Joint work with **Minghui Zhu** (UCSD)

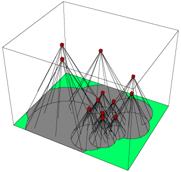
Multi-agent systems and distributed coordination

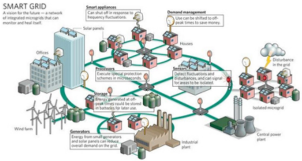
Common features:

- A common global objective
- Lack of a centralized authority
- Time-varying communication network topologies

Desired algorithms:

- Distributed decision-making utilizing local information
- Robust to dynamical changes of network topologies

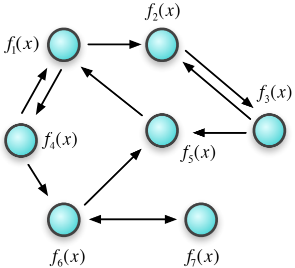




Distributed cooperative optimization

Problem ingredients:

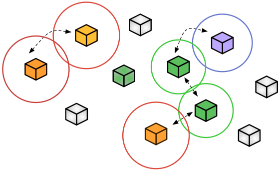
- A group of agents $V := \{1, \cdots, N\}$
- **Local** objective functions $f_i(x, p_i) \equiv f_i(x)$
- **Global** decision vector $x \in \mathbb{R}^n$, $n < dN$
- **Global** constraint functions $g(x)$, $h(x)$
- **Local** constraint sets X_i, $i \in V$



General optimization problem:

$$\min_{x \in \mathbb{R}^n} \sum_{i \in V} f_i(x), \quad \text{s.t. } g(x) \leq 0, \; h(x) = 0, \; x \in X_i, \, i \in V$$

Network model



A **directed weighted** graph $\mathcal{G}(k) := \{V, A(k), E(k)\}$

- **Adjacency matrix:** $A(k) := [a^i_j(k)] \in \mathbb{R}^{N\times N}_{\geq 0}$
- The set of edges with $a^i_j(k) > 0$: $E(k)$

Assumptions:

- **Non-degeneracy:** $a^i_i(k) \geq \alpha > 0$ and $a^i_j(k) \in \{0\} \cup [\alpha, 1]$
- **Balanced communication:** $\sum_{j=1}^N a^i_j(k) = 1$ and $\sum_{j=1}^N a^j_i(k) = 1$
- **Periodic strong connectivity:** $(V, \bigcup_{\tau=0}^B E(k+\tau))$ is strongly connected

Some relevant literature and our contributions

- **Parallel computation and distributed optimization**

  D.P. Bertsekas and J.N. Tsitsiklis, 1997 (book)

  M. Chiang , S.H. Low , A.R. Calderbank , J.C. Doyle, 2007 (survey)

- **Recent references on consensus algorithms**

  A. Jadbabaie, J. Lin and A.S. Morse, 2003

  R. Olfati-Saber and R.M. Murray, 2004

  L. Moreau, 2005

- **Recent "cooperative" convex optimization refs**

  M.G. Rabbat, R.D. Nowak and J.A. Buckley 2005

  A. Nedic and A. Ozdaglar, TAC 2009

  A. Nedic, A. Ozdaglar and P.A. Parrilo, TAC 2010

- **Our contribution**: distributed cooperative **convex** and **non-convex** optimization algorithms under time-varying interactions

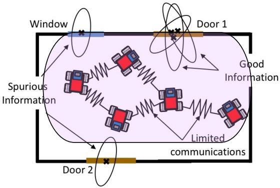
Example problems

Multi-robot/WSN objectives

(1) **Constrained consensus**

(2) **Optimal shape assignment**

(1) **Constrained consensus.** Given robot positions p_i, $i \in V$, and local data z_i, $i \in V$, solve the consensus problem:



$$\min_q \sum_{i \in V} \varphi(q - z_i),$$
$$g(q) \leq 0, \; q \in X$$

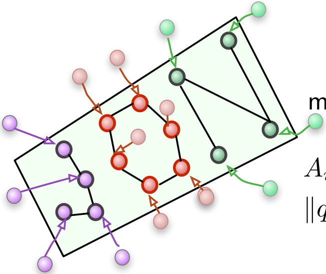
with e.g. $\varphi(e) = \sum_{l=1}^d \rho(e^l)$, $\rho(e^l) = \sigma^2 \left(\frac{|e^l|}{\sigma} - \log\left(1 + \frac{|e^l|}{\sigma}\right)\right)$

Example problems

(2) **Optimal shape assignment.** Given a **robotic shape** $S = (s_1, \dots, s_n)$, defining a class

$$[S] = \{\alpha S R + 1_m d^T \mid \alpha \in [0, \alpha_{\max}],\ R \in SO(k),\ d \in \mathbb{R}^k\}$$

and **robot positions** $p_1, \dots, p_n$, find $(q_1, \dots, q_n) \in [S]$:



$$\min \sum_{i=1}^{n} \|q_i - p_i\|^2$$

$$A_{i1}(q_1 - q_2) + A_{i2}(q_i - q_1) = 0,\ i \in \{3, \dots, n\}$$

$$\|q_i - p_i\| \leq r_i$$

Agents agree on $(q_3, \dots, q_n)$, $i \in V$

Outline

Convex Problem (I) – Lagrangian approach

Primal problem

$\min_{x\in\mathbb{R}^n}[f(x) := \sum_{i\in V} f_i(x)]$, s.t. $Ng(x) \leq 0$, $x \in X := \cap_{i\in V} X_i$

assume X_i compact, convexity, Slater condition ($g(z) < 0$)

Global Lagrangian function $\mathcal{L}(x,\mu) = f(x) + N\mu^T g(x)$

Primal problem reduced to $\min_{x\in X}(\sup_{\mu\in\mathbb{R}^m_{\geq 0}} \mathcal{L}(x,\mu))$

Primal solution x^*, Optimal value $p^* = \sup_{\mu\in\mathbb{R}^m_{\geq 0}} \mathcal{L}(x^*,\mu)$

Dual problem $\max_{\mu\in\mathbb{R}^m} q(\mu)$ s.t. $\mu \geq 0$

Dual function $q(\mu) := \inf_{x\in X} \mathcal{L}(x,\mu)$,

Dual solution and optimal value μ^*, $q(\mu^*) = d^*$

Convex Problem (I) – Lagrangian approach

Primal problem

$\min_{x\in\mathbb{R}^n}[f(x) := \sum_{i\in V} f_i(x)]$, s.t. $Ng(x) \leq 0$, $x \in X := \cap_{i\in V} X_i$

assume X_i compact, convexity, Slater condition ($g(z) < 0$)

Global Lagrangian function $\mathcal{L}(x,\mu) = f(x) + N\mu^T g(x)$

Primal problem reduced to $\min_{x\in X}(\sup_{\mu\in\mathbb{R}^m_{\geq 0}} \mathcal{L}(x,\mu))$

Primal solution x^*, Optimal value $p^* = \sup_{\mu\in\mathbb{R}^m_{\geq 0}} \mathcal{L}(x^*,\mu)$

Dual problem $\max_{\mu\in\mathbb{R}^m} q(\mu)$ s.t. $\mu \geq 0$

Dual function $q(\mu) := \inf_{x\in X} \mathcal{L}(x,\mu)$,

Dual solution and optimal value μ^*, $q(\mu^*) = d^*$

Convex Problem (I) – Lagrangian approach

Primal problem

$\min_{x\in\mathbb{R}^n}[f(x) := \sum_{i\in V} f_i(x)]$, s.t. $Ng(x) \leq 0$, $x \in X := \cap_{i\in V} X_i$

assume X_i compact, convexity, Slater condition ($g(z) < 0$)

Global Lagrangian function $\mathcal{L}(x,\mu) = f(x) + N\mu^T g(x)$

Primal problem reduced to $\min_{x\in X}(\sup_{\mu\in\mathbb{R}^m_{\geq 0}} \mathcal{L}(x,\mu))$

Primal solution x^*, Optimal value $p^* = \sup_{\mu\in\mathbb{R}^m_{\geq 0}} \mathcal{L}(x^*,\mu)$

Dual problem $\max_{\mu\in\mathbb{R}^m} q(\mu)$ s.t. $\mu \geq 0$

Dual function $q(\mu) := \inf_{x\in X} \mathcal{L}(x,\mu)$,

Dual solution and optimal value μ^*, $q(\mu^*) = d^*$

Convex Problem (I) – Lagrangian approach

Primal problem

$\min_{x \in \mathbb{R}^n} [f(x) := \sum_{i \in V} f_i(x)]$, s.t. $Ng(x) \leq 0$, $x \in X := \cap_{i \in V} X_i$

assume X_i compact, convexity, Slater condition ($g(z) < 0$)

Global Lagrangian function $\mathcal{L}(x, \mu) = f(x) + N\mu^T g(x)$

Primal problem reduced to $\min_{x \in X} (\sup_{\mu \in \mathbb{R}^m_{\geq 0}} \mathcal{L}(x, \mu))$

Primal solution x^*, Optimal value $p^* = \sup_{\mu \in \mathbb{R}^m_{\geq 0}} \mathcal{L}(x^*, \mu)$

Dual problem $\max_{\mu \in \mathbb{R}^m} q(\mu)$ s.t. $\mu \geq 0$

Dual function $q(\mu) := \inf_{x \in X} \mathcal{L}(x, \mu)$,

Dual solution and optimal value μ^*, $q(\mu^*) = d^*$

(x^*, μ^*) **saddle point** of $\mathcal{L} \iff (x^*, \mu^*)$ is a **primal-dual solution** and

$$\sup_{\mu \in \mathbb{R}^m_{\geq 0}} \inf_{x \in X} \mathcal{L}(x, \mu) = \mathcal{L}(x^*, \mu^*) = \inf_{x \in X} \sup_{\mu \in \mathbb{R}^m_{\geq 0}} \mathcal{L}(x, \mu)$$

Primal-dual algorithm

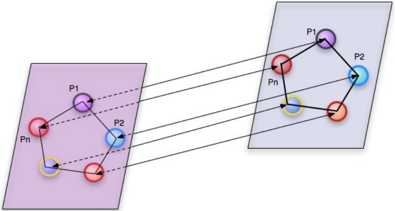
Centralized optimization

Player 1 : $U(x) = \mathsf{sup}_\mu \mathcal{L}(x, \mu)$
Player 2 : $V(\mu) = \mathsf{inf}_x \mathcal{L}(x, \mu)$

gradient descent/ascent $\Longrightarrow$ convergence to saddle point

Decentralized optimization

The network has to align the corresponding primal-dual actions



Network Lagrangian decomposition

Local primal problems

$$\mathcal{L}(x,\mu) = \sum_{i\in V} \mathcal{L}_i(x,\mu),\ \mathcal{L}_i(x,\mu) = f_i(x) + \mu^T g(x)$$

Local dual problem functions satisfy

$$q(\mu) \geq \sum_{i\in V} \inf_{x\in X_i}(f_i(x) + \mu^T g(x)) = \sum_{i\in V} q_i(\mu)$$

There exist M_i, compact and convex, such that $D^* \subseteq M_i$ for all i. The M_i depend on a Slater vector

Let $D^* \subset M = \cap_i M_i$ be a compact superset of the set of dual solutions

- If (x^*, μ^*) **saddle point** of $\mathcal{L}$ over $X \times \mathbb{R}^m_{\geq 0}$, then (x^*, μ^*) **saddle point** of $\mathcal{L}$ over $X \times M$
- If $(\tilde{x}, \tilde{\mu})$ **saddle point** of $\mathcal{L}$ over $X \times M$, then $\mathcal{L}(\tilde{x}, \tilde{\mu}) = p^*$ and $\tilde{\mu}$ is Lagragian dual optimal

Network Lagrangian decomposition

Local primal problems

$$\mathcal{L}(x,\mu) = \sum_{i\in V} \mathcal{L}_i(x,\mu),\ \mathcal{L}_i(x,\mu) = f_i(x) + \mu^T g(x)$$

Local dual problem functions satisfy

$$q(\mu) \geq \sum_{i\in V} \inf_{x\in X_i}(f_i(x) + \mu^T g(x)) = \sum_{i\in V} q_i(\mu)$$

There exist M_i, compact and convex, such that $D^* \subseteq M_i$ for all i. The M_i depend on a Slater vector

Let $D^* \subset M = \cap_i M_i$ be a compact superset of the set of dual solutions

- If (x^*,μ^*) **saddle point** of $\mathcal{L}$ over $X \times \mathbb{R}^m_{\geq 0}$, then (x^*,μ^*) **saddle point** of $\mathcal{L}$ over $X \times M$
- If $(\tilde{x},\tilde{\mu})$ **saddle point** of $\mathcal{L}$ over $X \times M$, then $\mathcal{L}(\tilde{x},\tilde{\mu}) = p^*$ and $\tilde{\mu}$ is Lagragian dual optimal

Network Lagrangian decomposition

Local primal problems

$$\mathcal{L}(x,\mu) = \sum_{i \in V} \mathcal{L}_i(x,\mu),\ \mathcal{L}_i(x,\mu) = f_i(x) + \mu^T g(x)$$

Local dual problem functions satisfy

$$q(\mu) \geq \sum_{i \in V} \inf_{x \in X_i}(f_i(x) + \mu^T g(x)) = \sum_{i \in V} q_i(\mu)$$

There exist M_i, compact and convex, such that $D^* \subseteq M_i$ for all i. The M_i depend on a Slater vector

Let $D^* \subset M = \cap_i M_i$ be a compact superset of the set of dual solutions

- If (x^*, μ^*) **saddle point** of $\mathcal{L}$ over $X \times \mathbb{R}^m_{\geq 0}$, then (x^*, μ^*) **saddle point** of $\mathcal{L}$ over $X \times M$
- If $(\tilde{x}, \tilde{\mu})$ **saddle point** of $\mathcal{L}$ over $X \times M$, then $\mathcal{L}(\tilde{x}, \tilde{\mu}) = p^*$ and $\tilde{\mu}$ is Lagragian dual optimal

Algorithm sketch

Init phase. Find $D^* \subseteq M_i$, compact superset of dual solutions
Common Slater vector computation through max consensus

Let $x^i(k) \approx x^*$, $\mu^i(k) \approx \mu^*$

Main algorithm. At each $k \geq 0$, agents apply:

Average computation:

$$[v_x^i(k), v_\mu^i(k)]^T = \sum_{i=1}^N a_j^i(k)[x_j^i(k), \mu_j^i(k)]^T$$

Primal-dual step:

$$x^i(k+1) = P_{X_i}[v_x^i(k) - \alpha(k)\mathcal{D}_x^i(k)]$$
$$\mu^i(k+1) = P_{M_i}[v_\mu^i(k) + \alpha(k)\mathcal{D}_\mu^i(k)]$$

$\mathcal{D}_x^i(k)$ **subgradient** of $\mathcal{L}_i(v_x^i(k), v_\mu^i(k))$ at $x = v_x^i(k)$
$\mathcal{D}_\mu^i(k)$ **supgradient** of $\mathcal{L}_i(v_x^i(k), v_\mu^i(k))$ at $\mu = v_\mu^i(k)$

Algorithm sketch

Init phase. Find $D^* \subseteq M_i$, compact superset of dual solutions
Common Slater vector computation through max consensus

Let $x^i(k) \approx x^*$, $\mu^i(k) \approx \mu^*$

Main algorithm. At each $k \geq 0$, agents apply:

Average computation:

$$[v_x^i(k), v_\mu^i(k)]^T = \sum_{i=1}^N a_j^i(k)[x_j^i(k), \mu_j^i(k)]^T$$

Primal-dual step:

$$x^i(k+1) = P_{X_i}[v_x^i(k) - \alpha(k)\mathcal{D}_x^i(k)]$$
$$\mu^i(k+1) = P_{M_i}[v_\mu^i(k) + \alpha(k)\mathcal{D}_\mu^i(k)]$$

$\mathcal{D}_x^i(k)$ **subgradient** of $\mathcal{L}_i(v_x^i(k), v_\mu^i(k))$ at $x = v_x^i(k)$
$\mathcal{D}_\mu^i(k)$ **supgradient** of $\mathcal{L}_i(v_x^i(k), v_\mu^i(k))$ at $\mu = v_\mu^i(k)$

Algorithm sketch

Init phase. Find $D^* \subseteq M_i$, compact superset of dual solutions
Common Slater vector computation through max consensus

Let $x^i(k) \approx x^*$, $\mu^i(k) \approx \mu^*$

Main algorithm. At each $k \geq 0$, agents apply:

Average computation:

$$[v_x^i(k), v_\mu^i(k)]^T = \sum_{i=1}^N a_j^i(k)[x_j^i(k), \mu_j^i(k)]^T$$

Primal-dual step:

$$x^i(k+1) = P_{X_i}[v_x^i(k) - \alpha(k)\mathcal{D}_x^i(k)]$$
$$\mu^i(k+1) = P_{M_i}[v_\mu^i(k) + \alpha(k)\mathcal{D}_\mu^i(k)]$$

$\mathcal{D}_x^i(k)$ **subgradient** of $\mathcal{L}_i(v_x^i(k), v_\mu^i(k))$ at $x = v_x^i(k)$
$\mathcal{D}_\mu^i(k)$ **supgradient** of $\mathcal{L}_i(v_x^i(k), v_\mu^i(k))$ at $\mu = v_\mu^i(k)$

Algorithm sketch

Init phase. Find $D^* \subseteq M_i$, compact superset of dual solutions
Common Slater vector computation through max consensus

Let $x^i(k) \approx x^*$, $\mu^i(k) \approx \mu^*$

Main algorithm. At each $k \geq 0$, agents apply:

Average computation:

$$[v_x^i(k), v_\mu^i(k)]^T = \sum_{i=1}^N a_j^i(k)[x_j^i(k), \mu_j^i(k)]^T$$

Primal-dual step:

$$x^i(k+1) = P_{X_i}[v_x^i(k) - \alpha(k)\mathcal{D}_x^i(k)]$$

$$\mu^i(k+1) = P_{M_i}[v_\mu^i(k) + \alpha(k)\mathcal{D}_\mu^i(k)]$$

$\mathcal{D}_x^i(k)$ **subgradient** of $\mathcal{L}_i(v_x^i(k), v_\mu^i(k))$ at $x = v_x^i(k)$
$\mathcal{D}_\mu^i(k)$ **supgradient** of $\mathcal{L}_i(v_x^i(k), v_\mu^i(k))$ at $\mu = v_\mu^i(k)$

Convergence properties

Assume that:

- The time-varying network topologies are **non degenerate**, **balanced**, and **periodically strongly connected**
- The **step-sizes** $\{\alpha(k)\}$ satisfy (C1):

$$\lim_{k \to +\infty} \alpha(k) = 0, \sum_{k=0}^{+\infty} \alpha(k) = +\infty, \text{ and } \sum_{k=0}^{+\infty} \alpha(k)^2 < +\infty$$

Then, each agent estimates $x^i(k)$, $\mu^i(k)$ converge:

$$\lim_{k \to +\infty} x^i(k) = x^*, \lim_{k \to +\infty} \mu^i(k) = \mu^*,$$

to a pair (x^*, μ^*) of **primal-dual optimal solutions**

Convergence properties

Assume that:

- The time-varying network topologies are **non degenerate**, **balanced**, and **periodically strongly connected**
- The **step-sizes** $\{\alpha(k)\}$ satisfy (C1):

$$\lim_{k \to +\infty} \alpha(k) = 0, \sum_{k=0}^{+\infty} \alpha(k) = +\infty, \text{ and } \sum_{k=0}^{+\infty} \alpha(k)^2 < +\infty$$

Then, each agent estimates $x^i(k)$, $\mu^i(k)$ converge:

$$\lim_{k \to +\infty} x^i(k) = x^*, \lim_{k \to +\infty} \mu^i(k) = \mu^*,$$

to a pair (x^*, μ^*) of **primal-dual optimal solutions**

Convergence properties

Assume that:

- The time-varying network topologies are **non degenerate**, **balanced**, and **periodically strongly connected**
- The **step-sizes** $\{\alpha(k)\}$ satisfy (C1):

$$\lim_{k\to+\infty} \alpha(k) = 0, \sum_{k=0}^{+\infty} \alpha(k) = +\infty, \text{ and } \sum_{k=0}^{+\infty} \alpha(k)^2 < +\infty$$

Then, each agent estimates $x^i(k)$, $\mu^i(k)$ converge:

$$\lim_{k\to+\infty} x^i(k) = x^*, \quad \lim_{k\to+\infty} \mu^i(k) = \mu^*,$$

to a pair (x^*, μ^*) of **primal-dual optimal solutions**

Convergence properties

Assume that:

- The time-varying network topologies are **non degenerate**, **balanced**, and **periodically strongly connected**
- The **step-sizes** $\{\alpha(k)\}$ satisfy (C1):

$$\lim_{k\to+\infty} \alpha(k) = 0, \sum_{k=0}^{+\infty} \alpha(k) = +\infty, \text{ and } \sum_{k=0}^{+\infty} \alpha(k)^2 < +\infty$$

Then, each agent estimates $x^i(k)$, $\mu^i(k)$ converge:

$$\lim_{k\to+\infty} x^i(k) = x^*, \lim_{k\to+\infty} \mu^i(k) = \mu^*,$$

to a pair (x^*, μ^*) of **primal-dual optimal solutions**

Main idea of the analysis

Decomposition:

$$x^i(k+1) = v_x^i(k) + e_x^i(k)$$
$$\mu^i(k+1) = v_\mu^i(k) + e_\mu^i(k)$$

Projection errors:

$$e_x^i(k) := P_{X_i}[v_x^i(k) - \alpha(k)\mathcal{D}_x^i(k)] - v_x^i(k)$$
$$e_\mu^i(k) := P_{M_i}[v_\mu^i(k) + \alpha(k)\mathcal{D}_\mu^i(k)] - v_\mu^i(k)$$

Main idea:

Main idea of the analysis

Decomposition:

$$x^i(k+1) = v_x^i(k) + e_x^i(k)$$
$$\mu^i(k+1) = v_\mu^i(k) + e_\mu^i(k)$$

Projection errors:

$$e_x^i(k) := P_{X_i}[v_x^i(k) - \alpha(k)\mathcal{D}_x^i(k)] - v_x^i(k)$$
$$e_\mu^i(k) := P_{M_i}[v_\mu^i(k) + \alpha(k)\mathcal{D}_\mu^i(k)] - v_\mu^i(k)$$

Main idea:

Main idea of the analysis

Decomposition:

$$x^i(k+1) = v_x^i(k) + e_x^i(k)$$
$$\mu^i(k+1) = v_\mu^i(k) + e_\mu^i(k)$$

Projection errors:

$$e_x^i(k) := P_{X_i}[v_x^i(k) - \alpha(k)\mathcal{D}_x^i(k)] - v_x^i(k)$$
$$e_\mu^i(k) := P_{M_i}[v_\mu^i(k) + \alpha(k)\mathcal{D}_\mu^i(k)] - v_\mu^i(k)$$

Main idea:

- Errors are diminishing
- Reach consensus values
- Verify that consensus values coincide with a pair of primal-dual optimal solutions (saddle point)

Outline

Convex Problem (II) – Penalty approach

Primal problem

$$\min_x \sum_{i=1}^{N} f_i(x),\ g(x) \leq 0,\ h(x) = 0,\ x \in X_i = X,\ i \in V$$

Penalty function $\mathcal{H}(x, \mu, \lambda) = f(x) + N\mu^T[g(x)]^+ + N\lambda^T|h(x)|$

Primal problem reduced to $\min_{x \in X}(\sup_{\mu \geq 0, \lambda \geq 0} \mathcal{H}(x, \mu, \lambda))$

Dual problem: $\max_{\mu,\lambda} q(\mu, \lambda)$ s.t. $\mu \geq 0, \quad \lambda \geq 0,$

Dual function $q(\mu, \lambda) := \inf_{x \in X} \mathcal{H}(x, \mu, \lambda)$

(x^*, μ^*, λ^*) **saddle point** of $\mathcal{H} \iff$
(x^*, μ^*, λ^*) **primal-dual solution** and

$$\sup_{\mu \geq 0, \lambda \geq 0} \inf_{x \in X} \mathcal{H}(x, \mu, \lambda) = \inf_{x \in X} \sup_{\mu \geq 0, \lambda \geq 0} \mathcal{H}(x, \mu, \lambda)$$

Network penalty decomposition:

$$\mathcal{H}(x, \mu, \lambda) = \sum_{i \in V} \mathcal{H}_i(x, \mu, \lambda)$$
$$\mathcal{H}_i(x, \mu, \lambda) = f_i(x) + \mu^T[g(x)]^+ + \lambda^T|h(x)|$$ **convex-concave**

Convex Problem (II) – Penalty approach

Primal problem

$$\min_x \sum_{i=1}^N f_i(x),\ g(x) \leq 0,\ h(x) = 0,\ x \in X_i = X,\ i \in V$$

Penalty function $\mathcal{H}(x, \mu, \lambda) = f(x) + N\mu^T[g(x)]^+ + N\lambda^T|h(x)|$

Primal problem reduced to $\min_{x\in X}(\sup_{\mu\geq 0, \lambda\geq 0} \mathcal{H}(x, \mu, \lambda))$

Dual problem: $\max_{\mu,\lambda} q(\mu, \lambda)$ s.t. $\mu \geq 0,\ \ \lambda \geq 0,$

Dual function $q(\mu, \lambda) := \inf_{x\in X} \mathcal{H}(x, \mu, \lambda)$

(x^*, μ^*, λ^*) **saddle point** of $\mathcal{H} \iff$
(x^*, μ^*, λ^*) **primal-dual solution** and

$$\sup_{\mu\geq 0, \lambda\geq 0} \inf_{x\in X} \mathcal{H}(x, \mu, \lambda) = \inf_{x\in X} \sup_{\mu\geq 0, \lambda\geq 0} \mathcal{H}(x, \mu, \lambda)$$

Network penalty decomposition:

$$\mathcal{H}(x, \mu, \lambda) = \sum_{i\in V} \mathcal{H}_i(x, \mu, \lambda)$$
$$\mathcal{H}_i(x, \mu, \lambda) = f_i(x) + \mu^T[g(x)]^+ + \lambda^T|h(x)|$$ **convex-concave**

Convex Problem (II) – Penalty approach

Primal problem

$$\min_x \sum_{i=1}^N f_i(x),\ g(x) \leq 0,\ h(x) = 0,\ x \in X_i = X,\ i \in V$$

Penalty function $\mathcal{H}(x, \mu, \lambda) = f(x) + N\mu^T[g(x)]^+ + N\lambda^T|h(x)|$

Primal problem reduced to $\min_{x\in X}(\sup_{\mu\geq 0,\lambda\geq 0} \mathcal{H}(x, \mu, \lambda))$

Dual problem: $\max_{\mu,\lambda} q(\mu, \lambda)$ s.t. $\mu \geq 0$, $\lambda \geq 0$,

Dual function $q(\mu, \lambda) := \inf_{x\in X} \mathcal{H}(x, \mu, \lambda)$

(x^*, μ^*, λ^*) **saddle point** of $\mathcal{H} \iff$
(x^*, μ^*, λ^*) **primal-dual solution** and

$$\sup_{\mu\geq 0,\lambda\geq 0} \inf_{x\in X} \mathcal{H}(x, \mu, \lambda) = \inf_{x\in X} \sup_{\mu\geq 0,\lambda\geq 0} \mathcal{H}(x, \mu, \lambda)$$

Network penalty decomposition:

$$\mathcal{H}(x, \mu, \lambda) = \sum_{i\in V} \mathcal{H}_i(x, \mu, \lambda)$$
$$\mathcal{H}_i(x, \mu, \lambda) = f_i(x) + \mu^T[g(x)]^+ + \lambda^T|h(x)|$$ **convex-concave**

Algorithm sketch

Main algorithm. At each $k \geq 0$, agents apply:

Average computation:

$$v_x^i(k) = \sum_{i=1}^{N} a_j^i(k) x^j(k)$$

$$[v_\mu^i(k), v_\lambda^i(k)]^T = \sum_{i=1}^{N} a_j^i(k)[\mu^j(k), \lambda^j(k)]^T$$

Primal-dual update:

$$x^i(k+1) = P_X[v^i x(k) - \alpha(k)\mathcal{S}_x^i(k)]$$

$$\mu^i(k+1) = v_\mu^i(k) + \alpha(k)[g(v_x^i(k))]^+$$

$$\lambda^i(k+1) = v_\lambda^i(k) + \alpha(k)|h(v_x^i(k))|$$

$\mathcal{S}_x^i(k)$ **subgradient** of $\mathcal{H}_i(\cdot, v_\mu^i(k), v_\lambda^i(k))$ at $x = v_x^i(k)$
$([g(v_x^i(k))]^+, |h(v_x^i(k))|)$ **supgradient** of $\mathcal{H}_i(v_x^i(k), \cdot, \cdot)$
at $(\mu, \lambda) = (v_\mu^i(k), v_\lambda^i(k))$

Algorithm sketch

Main algorithm. At each $k \geq 0$, agents apply:

Average computation:

$$v_x^i(k) = \sum_{i=1}^N a_j^i(k) x^j(k)$$

$$[v_\mu^i(k), v_\lambda^i(k)]^T = \sum_{i=1}^N a_j^i(k)[\mu^j(k), \lambda^j(k)]^T$$

Primal-dual update:

$$x^i(k+1) = P_X[v^i x(k) - \alpha(k)\mathcal{S}_x^i(k)]$$

$$\mu^i(k+1) = v_\mu^i(k) + \alpha(k)[g(v_x^i(k))]^+$$

$$\lambda^i(k+1) = v_\lambda^i(k) + \alpha(k)|h(v_x^i(k))|$$

$\mathcal{S}_x^i(k)$ **subgradient** of $\mathcal{H}_i(\cdot, v_\mu^i(k), v_\lambda^i(k))$ at $x = v_x^i(k)$
$([g(v_x^i(k))]^+, |h(v_x^i(k))|)$ **supgradient** of $\mathcal{H}_i(v_x^i(k), \cdot, \cdot)$ at $(\mu, \lambda) = (v_\mu^i(k), v_\lambda^i(k))$

Algorithm sketch

Main algorithm. At each $k \geq 0$, agents apply:

Average computation:

$$v_x^i(k) = \sum_{i=1}^{N} a_j^i(k) x^j(k)$$
$$[v_\mu^i(k), v_\lambda^i(k)]^T = \sum_{i=1}^{N} a_j^i(k)[\mu^j(k), \lambda^j(k)]^T$$

Primal-dual update:

$$x^i(k+1) = P_X[v^i x(k) - \alpha(k)\mathcal{S}_x^i(k)]$$
$$\mu^i(k+1) = v_\mu^i(k) + \alpha(k)[g(v_x^i(k))]^+$$
$$\lambda^i(k+1) = v_\lambda^i(k) + \alpha(k)|h(v_x^i(k))|$$

$\mathcal{S}_x^i(k)$ **subgradient** of $\mathcal{H}_i(\cdot, v_\mu^i(k), v_\lambda^i(k))$ at $x = v_x^i(k)$
$([g(v_x^i(k))]^+, |h(v_x^i(k))|)$ **supgradient** of $\mathcal{H}_i(v_x^i(k), \cdot, \cdot)$ at $(\mu, \lambda) = (v_\mu^i(k), v_\lambda^i(k))$

Convergence properties

Assume that:

- the time-varying network topologies are **non degenerate**, **balanced**, and **periodically strongly connected**, and
- the **step sizes** $\{\alpha(k)\}$ satisfy (C1) and

$$\lim_{k\to+\infty}\alpha(k+1)\sum_{\ell=0}^{k}\alpha(\ell)=0,\quad \sum_{k=0}^{+\infty}\alpha(k+1)^2(\sum_{\ell=0}^{k}\alpha(\ell))<+\infty,$$

$$\sum_{k=0}^{+\infty}\alpha(k+1)^2(\sum_{\ell=0}^{k}\alpha(\ell))^2<+\infty$$

Then, each agent primal estimates $x^i(k)$ converge, $\lim_{k\to+\infty}x^i(k)=x^*$, to a **primal optimal solution** x^*

Convergence properties

Assume that:

- the time-varying network topologies are **non degenerate**, **balanced**, and **periodically strongly connected**, and
- the **step sizes** $\{\alpha(k)\}$ satisfy (C1) and

$$\lim_{k\to+\infty} \alpha(k+1)\sum_{\ell=0}^{k}\alpha(\ell) = 0, \quad \sum_{k=0}^{+\infty}\alpha(k+1)^2\Big(\sum_{\ell=0}^{k}\alpha(\ell)\Big) < +\infty,$$

$$\sum_{k=0}^{+\infty}\alpha(k+1)^2\Big(\sum_{\ell=0}^{k}\alpha(\ell)\Big)^2 < +\infty$$

Then, each agent primal estimates $x^i(k)$ converge, $\lim_{k\to+\infty} x^i(k) = x^*$, to a **primal optimal solution** x^*

Convergence properties

Assume that:

- the time-varying network topologies are **non degenerate**, **balanced**, and **periodically strongly connected**, and
- the **step sizes** $\{\alpha(k)\}$ satisfy (C1) and

$$\lim_{k\to+\infty} \alpha(k+1)\sum_{\ell=0}^{k}\alpha(\ell)=0, \quad \sum_{k=0}^{+\infty}\alpha(k+1)^2\Big(\sum_{\ell=0}^{k}\alpha(\ell)\Big)<+\infty,$$

$$\sum_{k=0}^{+\infty}\alpha(k+1)^2\Big(\sum_{\ell=0}^{k}\alpha(\ell)\Big)^2<+\infty$$

Then, each agent primal estimates $x^i(k)$ converge, $\lim_{k\to+\infty} x^i(k)=x^*$, to a **primal optimal solution** x^*

Convergence properties

Assume that:

- the time-varying network topologies are **non degenerate**, **balanced**, and **periodically strongly connected**, and
- the **step sizes** $\{\alpha(k)\}$ satisfy (C1) and

$$\lim_{k \to +\infty} \alpha(k+1) \sum_{\ell=0}^{k} \alpha(\ell) = 0, \quad \sum_{k=0}^{+\infty} \alpha(k+1)^2 (\sum_{\ell=0}^{k} \alpha(\ell)) < +\infty,$$

$$\sum_{k=0}^{+\infty} \alpha(k+1)^2 (\sum_{\ell=0}^{k} \alpha(\ell))^2 < +\infty$$

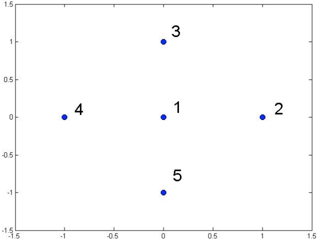
Then, each agent primal estimates $x^i(k)$ converge, $\lim_{k \to +\infty} x^i(k) = x^*$, to a **primal optimal solution** x^*

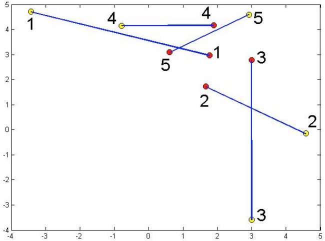
Outline

Example simulation

Optimal shape assignment problem:

$$\min_{q\in\mathbb{R}^{10}} \sum_{i=1}^{5} |q_{2i-1} - z_{2i-1}| + |q_{2i} - z_{2i}|,$$
$$Aq = 0,\ q \in X = [-5,5]^{10}$$





Desired Shape: $S = \{[0,0],[1,0],[0,1],[-1,0],[0,-1]\}$
Initial positions: $\mathbf{z}_i = [z_{2i}, z_{2i+1}]$, **Final positions:** $\mathbf{q}_i = [q_{2i}, q_{2i+1}]$
Agents agree on: $\mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5$

Example simulation

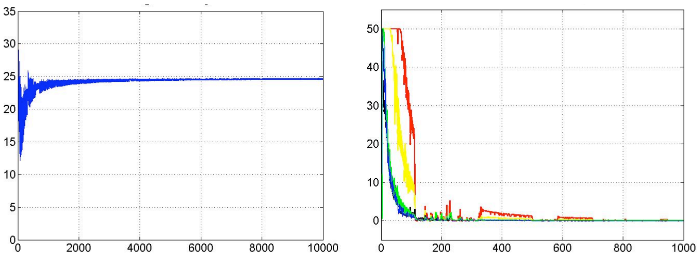


Figure: Objective function evolution and disagreement evolution

Outline

Conclusions

- Presented distributed algorithms to solve a class of cooperative convex programs
- Guarantee the convergence to primal-dual solutions
- Future and current work: convergence-time study, uncertainty effect

Conclusions

- Presented distributed algorithms to solve a class of cooperative convex programs
- Guarantee the convergence to primal-dual solutions
- Future and current work: convergence-time study, uncertainty effect

Thank you!