Infinite Chains of Vehicles

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(submitted to Automatica)

distributed control
principles?
separation theorem
optimization
anomaly

Introduction

An example

Conclusion

We start with a line of research on vehicle formations:

1. Melzer and Kuo 1971 (-40 years)
2. Bamieh, Paganini, Dahleh 2002
3. D'Andrea and Dullerud 2003
4. Motee and Jadbabaie 2008
5. Curtain 2009
6. Curtain, Iftime, Zwart 2010

Melzer & Kuo, Automatica, 1971

Optimal Regulation of Systems Described by a Countably Infinite Number of Objects

"Consider a system comprised of a large number of indexed identical objects. ... Then, it is of interest to investigate the corresponding case of an infinite number of objects to reveal the behaviour of a typical object."

Central question

N large

model by $N =$ infinity

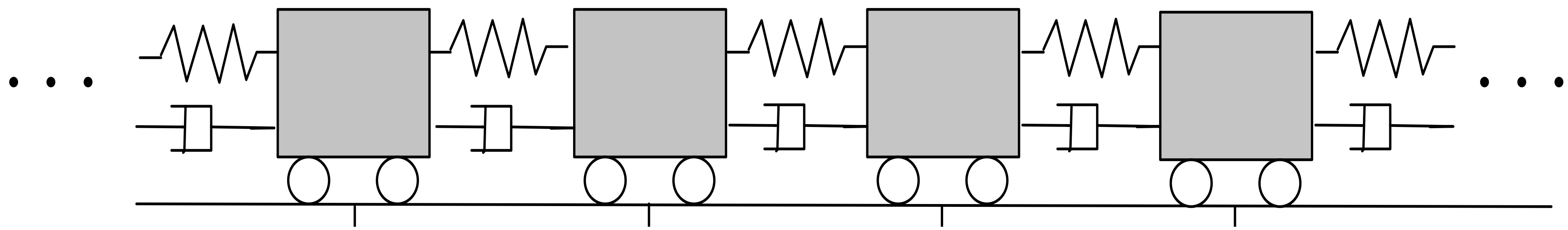
Are the behaviours consistent?

Notice that $N =$ infinity is not necessarily the same as

$$\lim_{N \to \infty}$$

because of boundary conditions.

Infinite chain in a physics context:

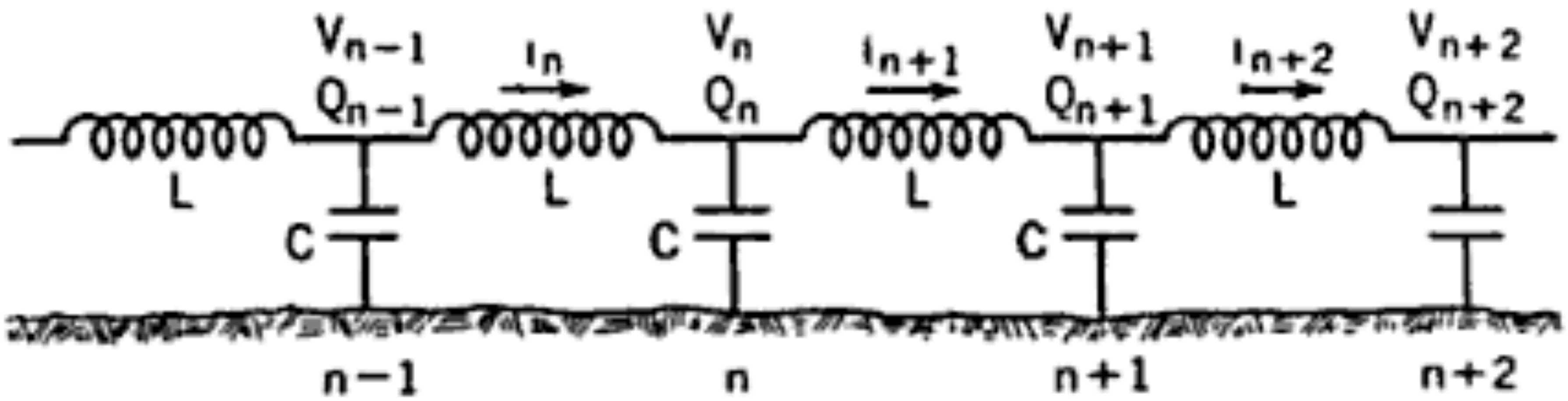


Newton, speed of sound in an elastic medium.

Another physics context:

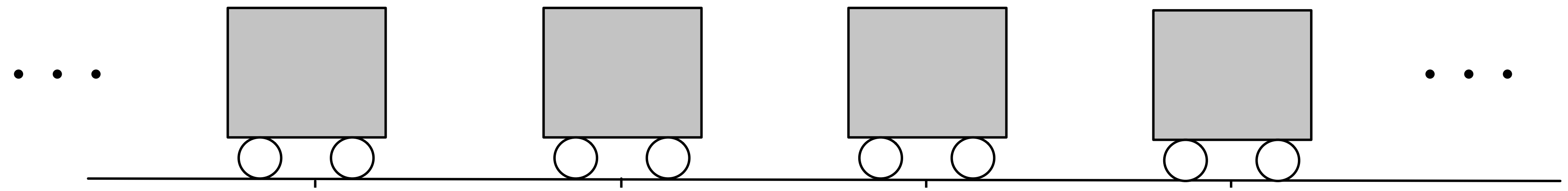
Brillouin: Wave Propagation in Periodic Structures

crystals



In the physics literature, there are derivations (e.g., the velocity of a wave), but nothing is proved.

the Melzer-Kuo problem



n an integer

positions $q_n(t)$

velocities $v_n(t)$

control forces $u_n(t)$

$$\dot{q}_n = v_n$$

$$\dot{v}_n = u_n$$

Vectors

$$q = \left[\begin{array}{c} \vdots \\ q_{-1} \\ \hline q_0 \\ q_1 \\ \vdots \end{array}\right], v, u, \qquad \begin{aligned} \dot{q} &= v \\ \dot{v} &= u \end{aligned}$$

All vectors are functions of t

The control objective

$$q_{n+1} - q_n = h, \qquad v_n = v_{ss}$$

Introduce

$$y = U^{-1}x, \quad y_n = x_{n+1}, \qquad \mathbf{1} = \begin{bmatrix} \vdots \\ 1 \\ \hline 1 \\ 1 \\ \vdots \end{bmatrix}$$

Then

$$U^{-1}q - q = h \cdot \mathbf{1}, \qquad v = v_{ss} \cdot \mathbf{1}$$

The error vectors

$$(U^{-1} - I)q - h \cdot \mathbf{1}$$

$$v - v_{ss} \cdot \mathbf{1}$$

The cost function

$$J = \int_0^\infty \|(U^{-1} - I)q(t) - h \cdot \mathbf{1}\|^2 + \|v(t) - v_{ss} \cdot \mathbf{1}\|^2 dt$$

(+ control penalty)

$$J = \int_0^\infty \|(U^{-1} - I)q(t) - h \cdot \mathbf{1}\|^2 + \|v(t) - v_{ss} \cdot \mathbf{1}\|^2 dt$$

temporal norm $\mathcal{L}^2$

spatial norm $= ?$

What spatial norm do Melzer and Kuo take?

As well as Bamieh, Paganini, Dahleh, D'Andrea, Dullerud, Motee, Jadbabaie, Curtain, Iftime, and Zwart.

The ℓ^2 norm.

ℓ^2, Hilbert space

$$x, \quad x_n \in \mathbb{R}, \ n \in \mathbb{Z}$$

$$\langle x, y \rangle = \sum x_n y_n$$

$$\|x\|_2 = \left(\sum x_n^2 \right)^{1/2}$$

$$x_n \to 0, \quad n \to \pm\infty$$

If J is finite, then already, without any control, the spacing is correct far away:

$$J = \int_0^\infty \|(U^{-1} - I)q(t) - h \cdot \mathbf{1}\|_2^2 + \ldots dt < \infty$$

$$\implies \quad \lim_{n \to \pm\infty} |q_{n+1}(0) - q_n(0) - h| = 0$$

For every $\varepsilon > 0$, infinitely many vehicles have spacing error $< \varepsilon$.

The formation control problem is surely easier if only a few vehicles are out of whack.

Conclusion: Optimal control for N large is not captured by $N =$ infinity in ℓ^2.

And yet …

Melzer and Kuo claim

“that the infinite object theory accurately describes the properties of the typical vehicle controller in a long finite string.”

They then show by example that vehicle 5 in a string of 9 behaves like the middle of the infinite string.

But 9 is not large, and “most” vehicles are not in the middle.

Our thesis is that, for some problems, if you use ℓ^2 then $N =$ infinity does not approximate the behaviour of large N.

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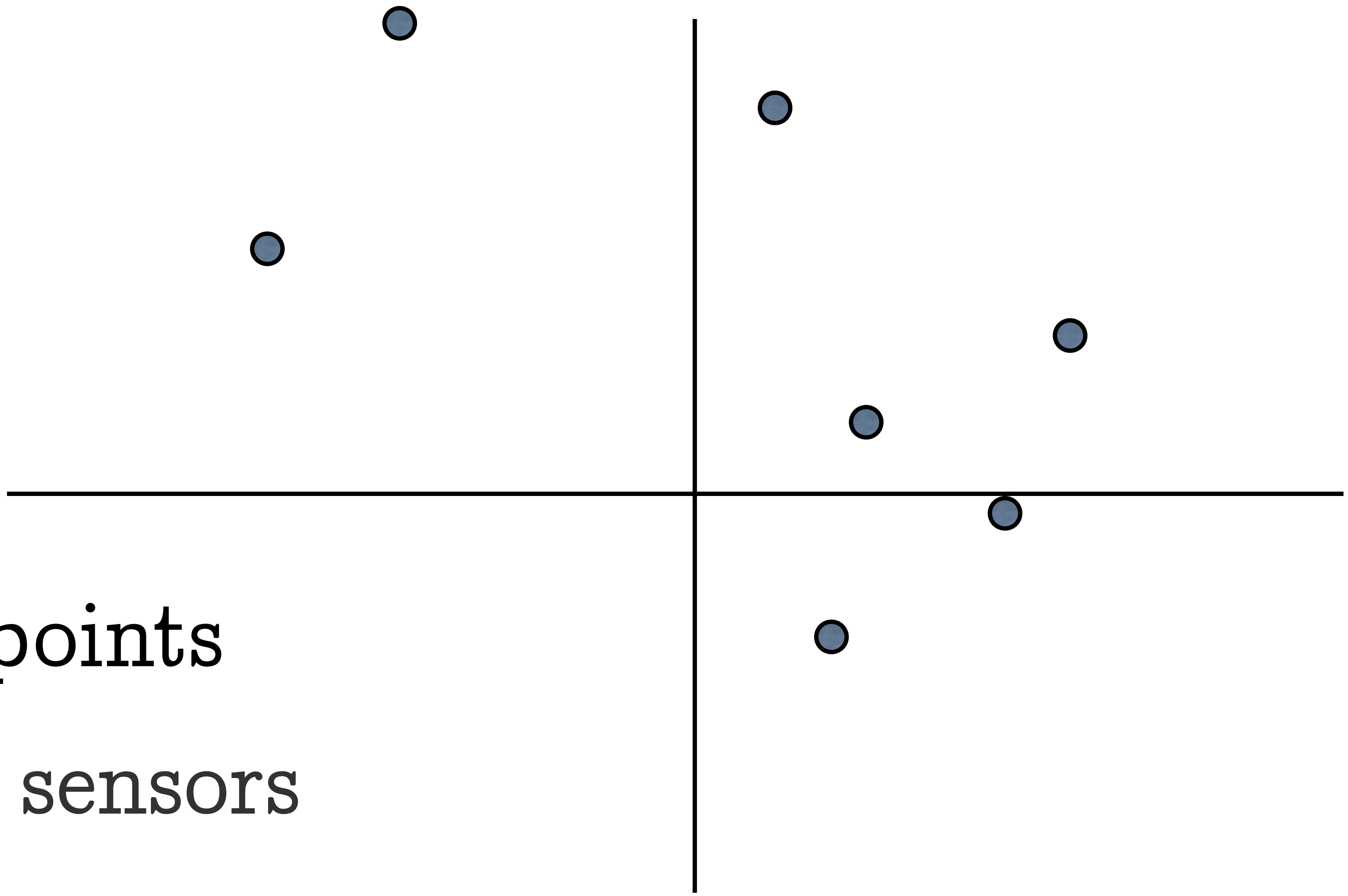
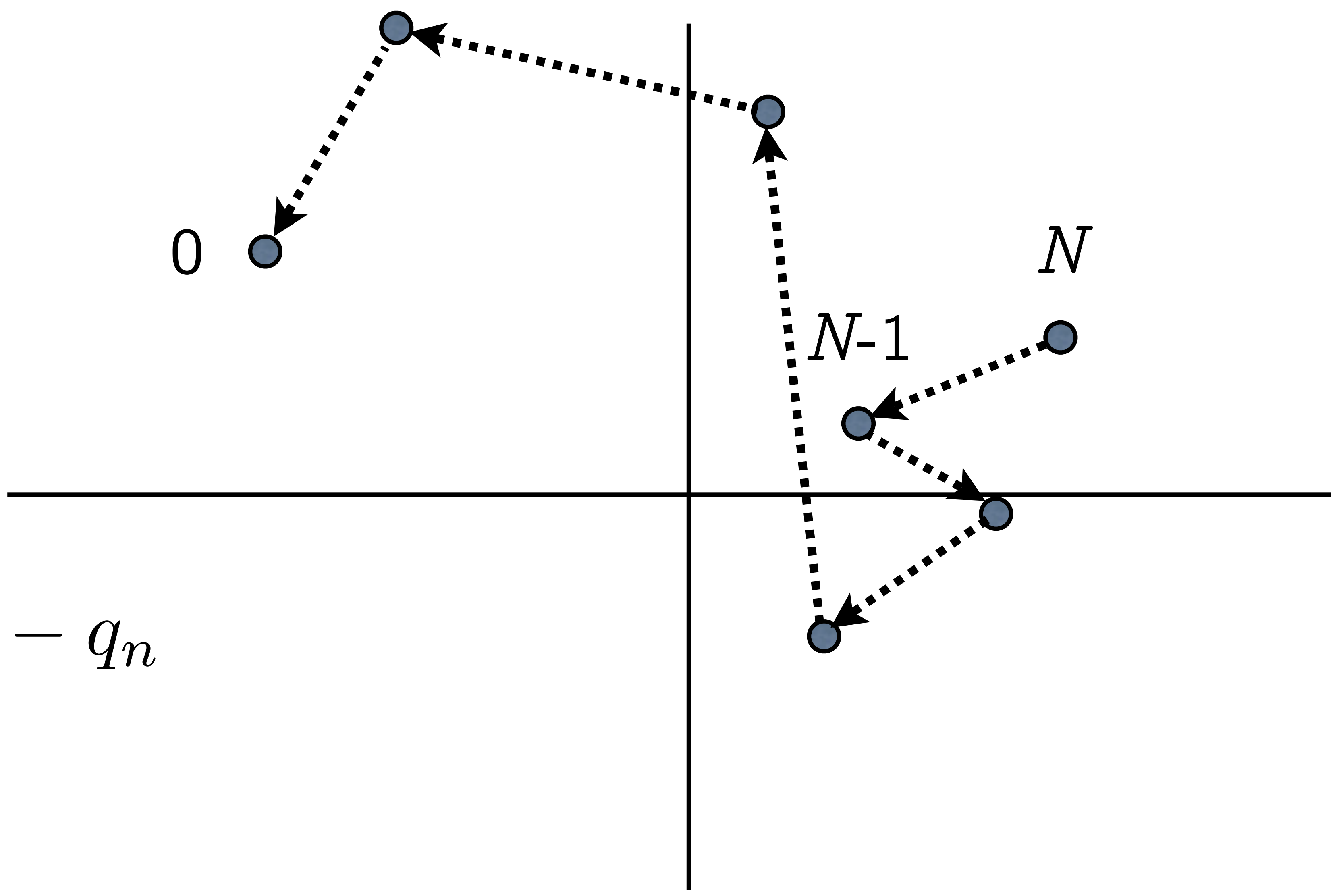
Example of serial pursuit



N kinematic points

only onboard sensors

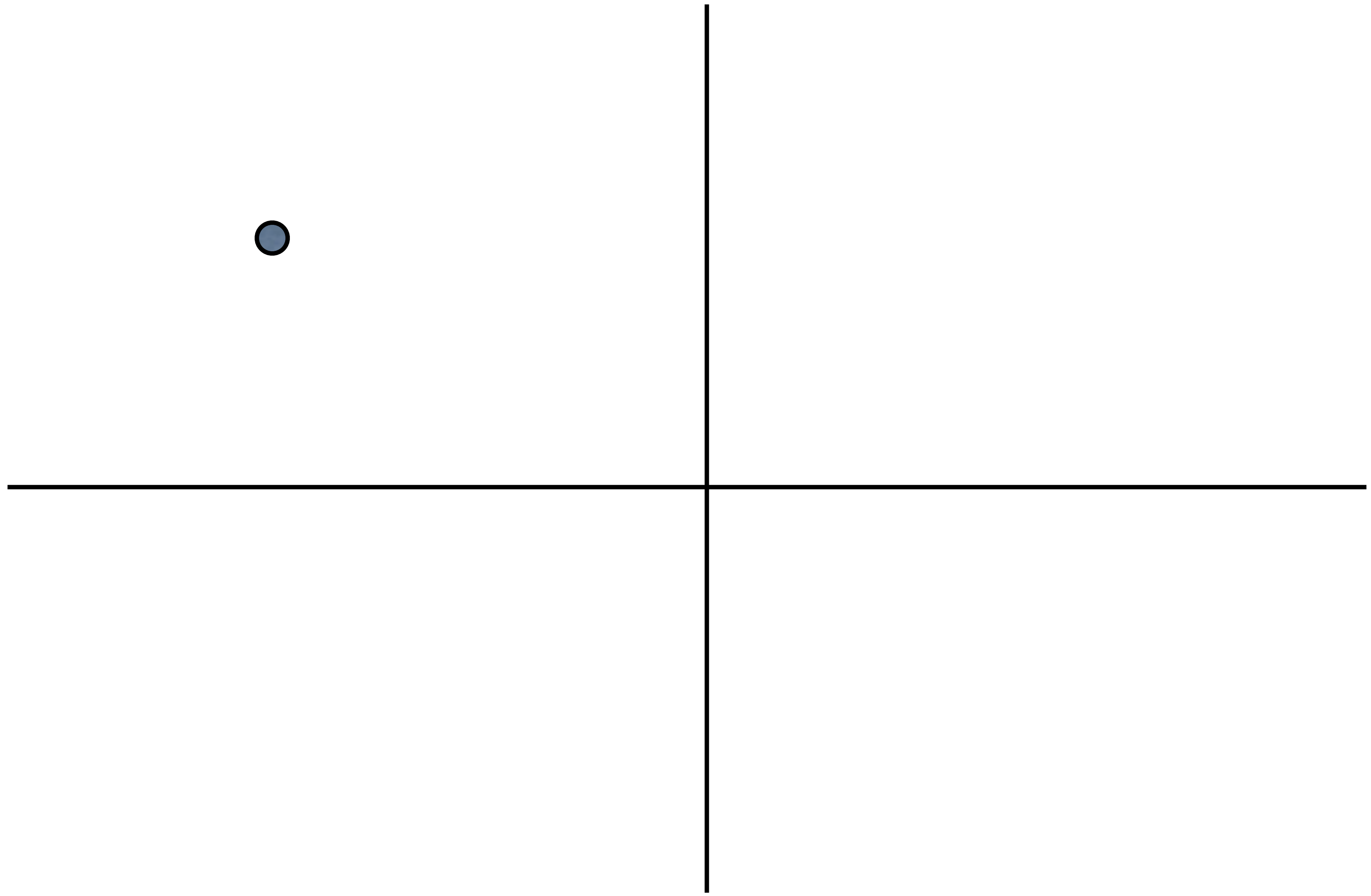
Actually, on the x-axis.



$$\dot{q}_n = q_{n-1} - q_n$$

say $\dot{q}_0 = 0$

formation in the limit

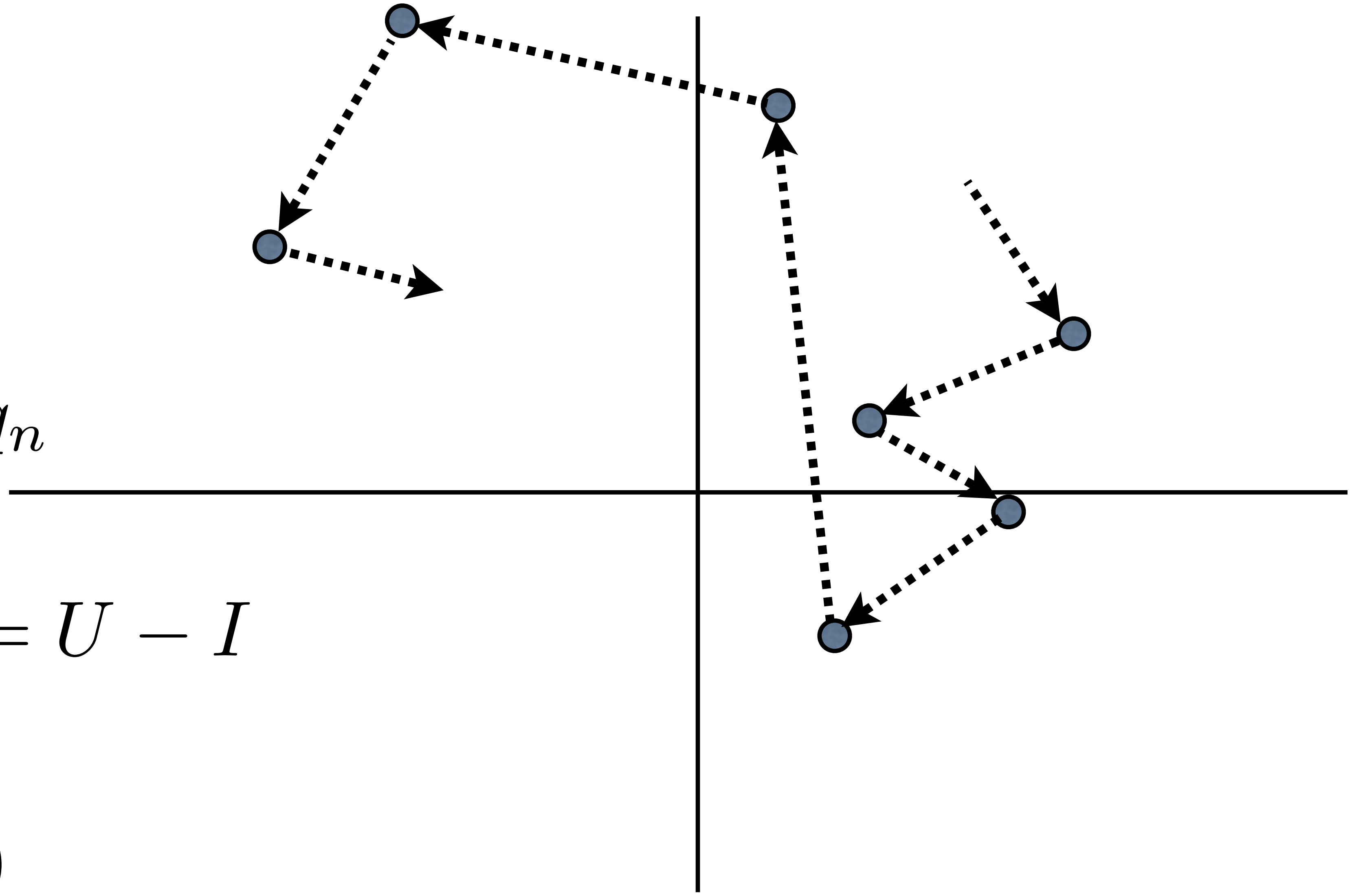


$$N = \infty$$

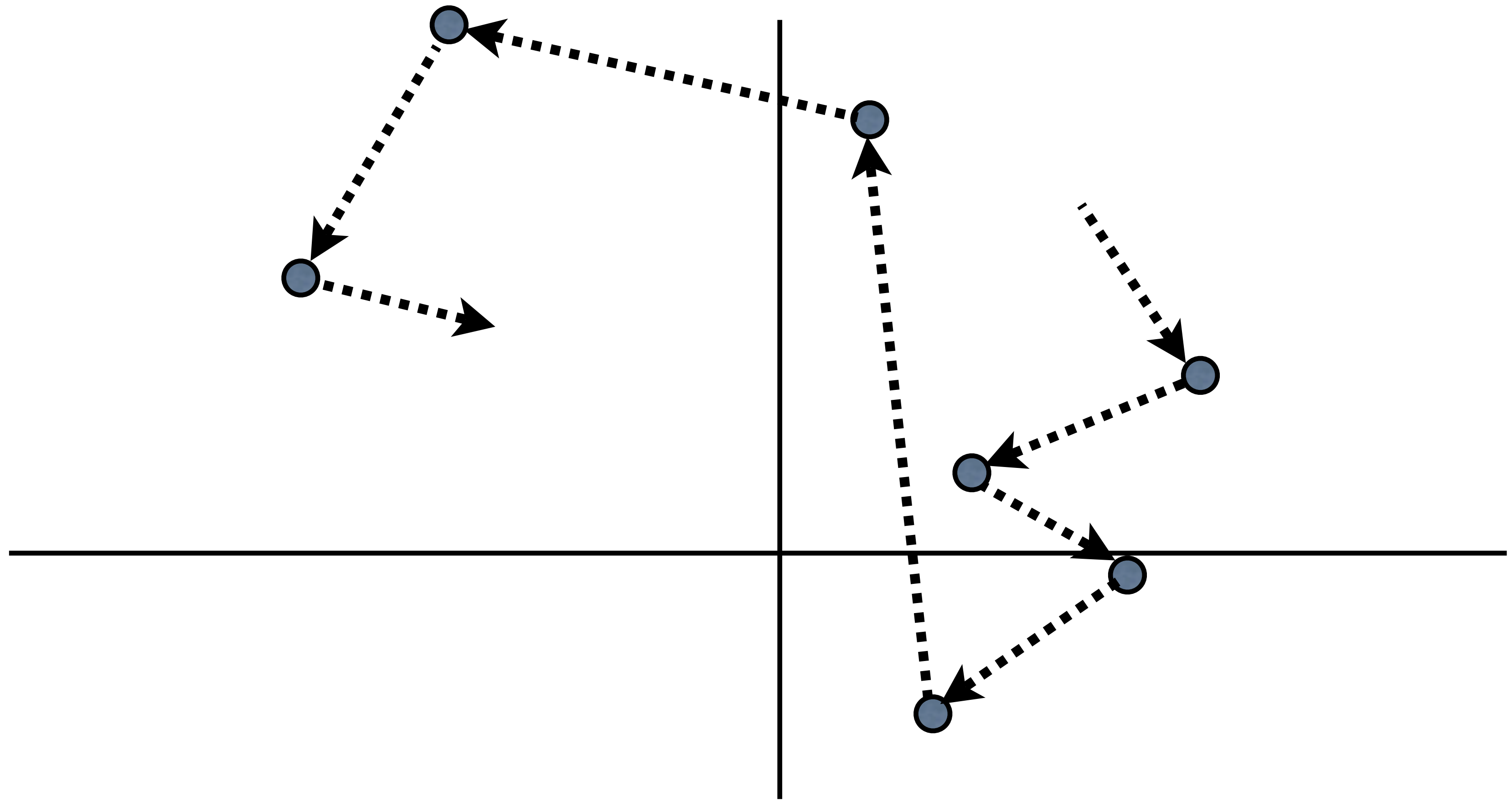
$$\dot{q}_n = q_{n-1} - q_n$$

$$\dot{q} = Aq, \quad A = U - I$$

$$q(t) = \mathrm{e}^{At} q(0)$$



What happens?



If the state space is ℓ^2 , the vehicles rendezvous at the origin.

As if they had GPS! But they don't.

In fact, the only equilibrium in ℓ^2 is the origin.

Let's try for a better model of N large.

ℓ^∞, Banach space

$x, \quad x_n \in \mathbb{R}, \; n \in \mathbb{Z}$

$\langle x, y \rangle$, none

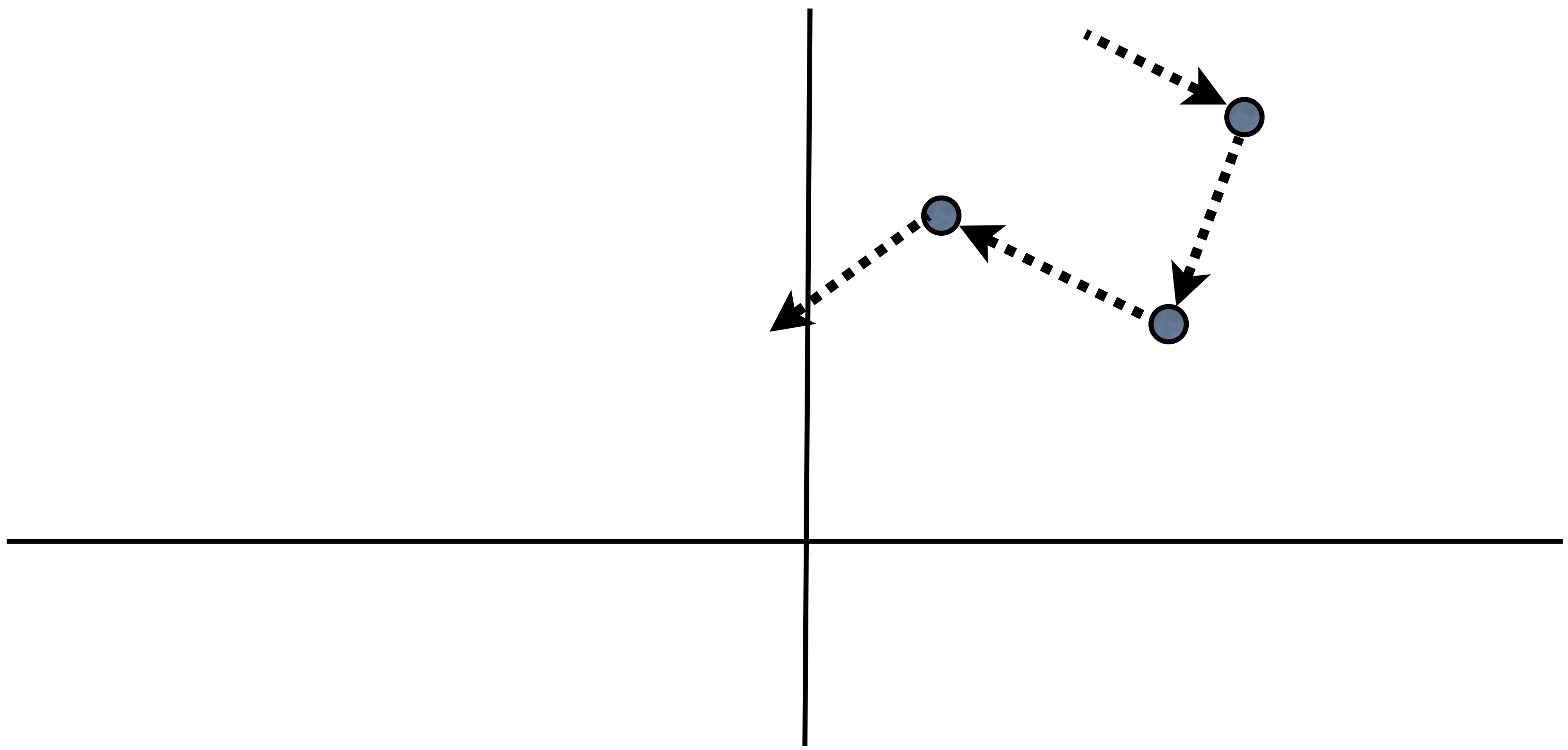
$\|x\|_\infty = \sup_n |x_n|$

The key ℓ^2 tools are not available.

No inner product means no Cauchy-Schwarz.

No Fourier transform. (there is in a distributional sense)

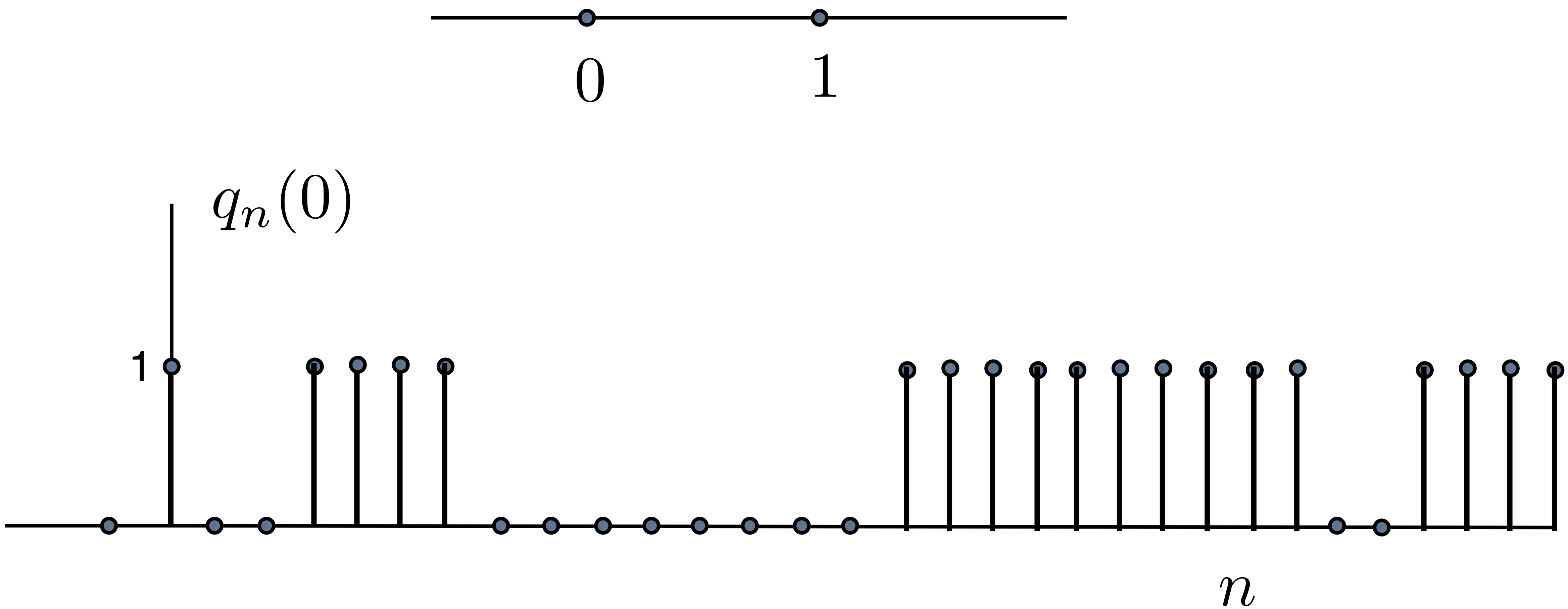
Serial pursuit in ℓ^∞



Every point is a possible rendezvous point.

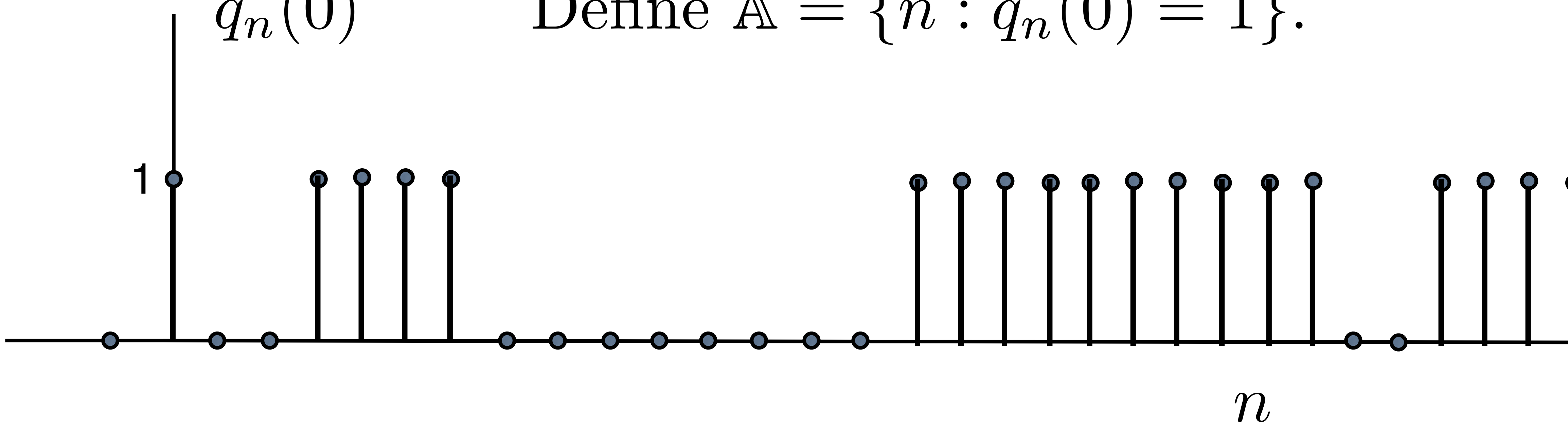
But, as it turns out, the points don't necessarily converge.

An example ...



0
1
$q_n(0)$
1
n

Define $\mathbb{A} = \{n : q_n(0) = 1\}$.

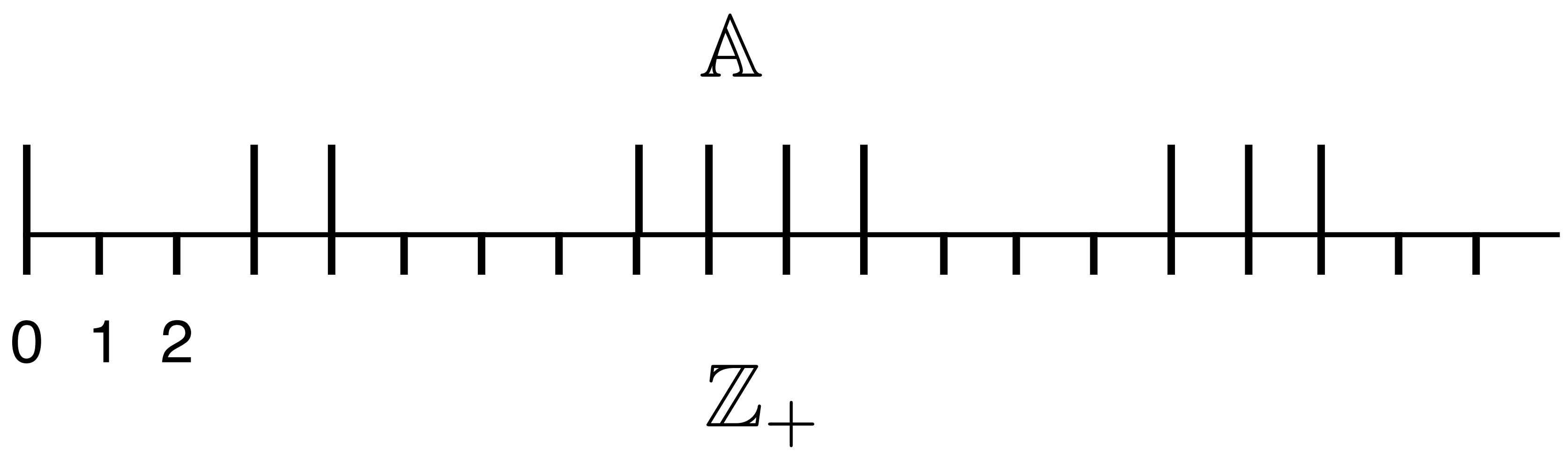


When does $q_n(t)$ converge as $t \to \infty$?

It depends on what kind of set $\mathbb{A}$ is.

Diaconis and Stein, 1978

$\mathbb{A} \subset \mathbb{Z}_+$



toss a coin n times

S_n is the number of heads

$\{S_n \in \mathbb{A}\}$ is an event

When does $\Pr(S_n \in \mathbb{A})$ converge as $n \to \infty$?

It depends on what kind of set $\mathbb{A}$ is.

Tauberian theory

Surprise!

$q_n(t)$ converges as $t \to \infty$

iff

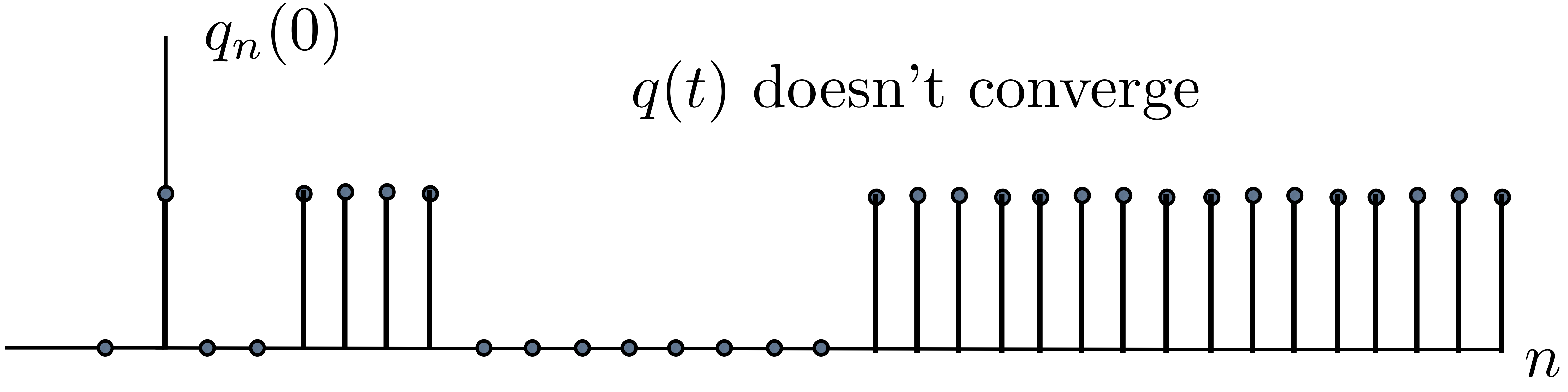
$Pr(S_n \in \mathbb{A})$ converges as $n \to \infty$

Thanks to Ronen Peretz

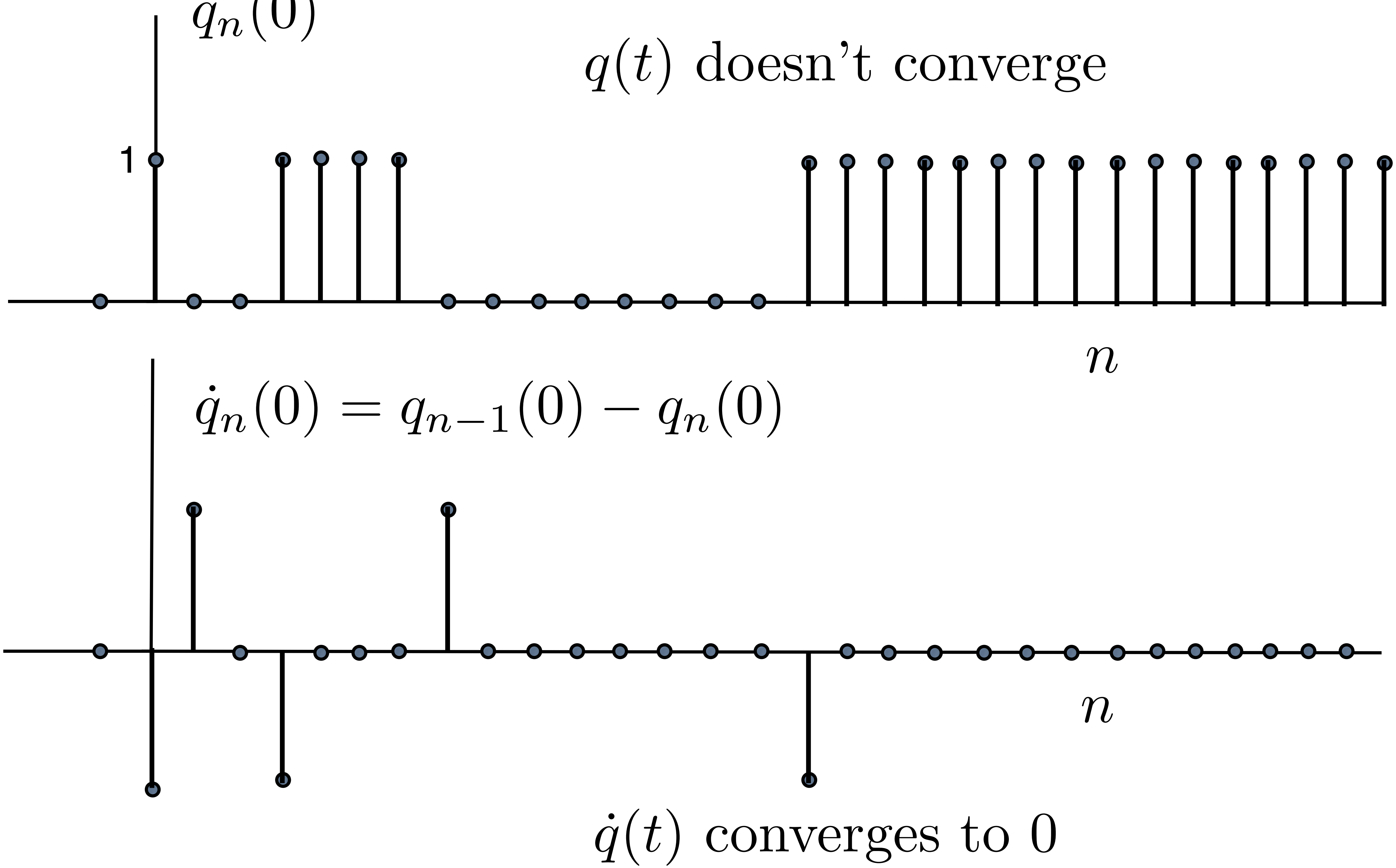
From this theory, we can get a $q(0) \in \ell^\infty$

for which $q(t)$ does not converge.

something like this



But ...



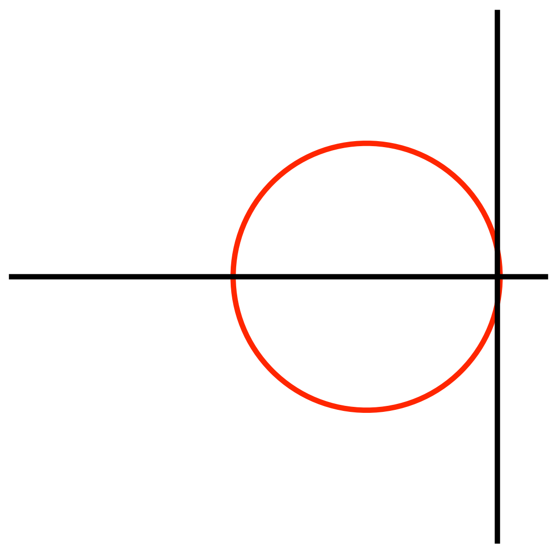
$q_n(0)$
$q(t)$ doesn't converge
1
n
$\dot{q}_n(0) = q_{n-1}(0) - q_n(0)$
n
$\dot{q}(t)$ converges to 0

Results

$$\dot{q} = Aq \qquad A = U - I$$

$$q(t) = \mathrm{e}^{At} q(0)$$

spectrum of A



$$\|\mathrm{e}^{At}\| = 1$$

$$A\mathrm{e}^{At} \to 0 \quad \text{in } \mathcal{B}(\ell^\infty)$$

Therefore

$$Aq(t) \to 0$$

Daleckii and Krein

Stability of Solutions of Differential Equations in Banach Space

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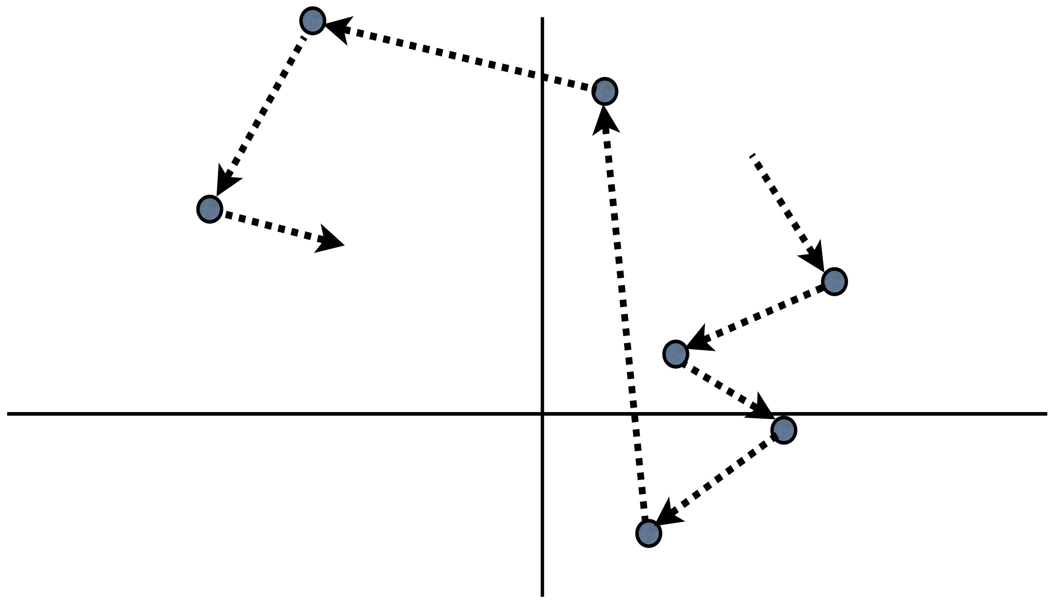
Summary



If the state space is ℓ^2 , the vehicles rendezvous at the origin.
If the state space is ℓ^∞, the vehicles may not converge.

Extension in the paper to

$$\dot{q}_n = (q_{n+1} - q_n) + (q_{n-1} - q_n)$$

$$\dot{q} = (U + U^{-1} - 2I)q$$

Partial results for masses

$$\ddot{q} = \dots$$

Finally

We had hoped for a more definitive treatment.

There are lots of open questions in ℓ^∞ .

Thanks for your attention.