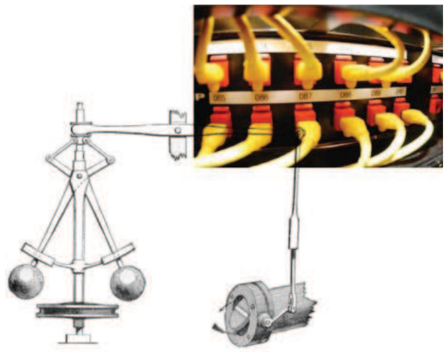
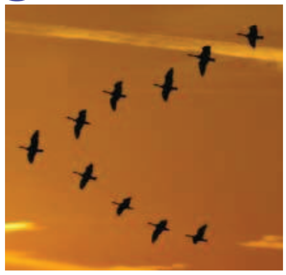
The 2011 Santa Barbara Control Workshop
on Decision, Dynamics and Control in Multi-agent Systems
June 24, 2011



An Internal Model Principle for Consensus in Heterogeneous Multi-Agent Systems



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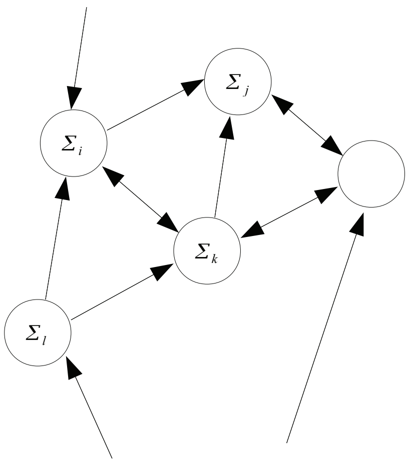
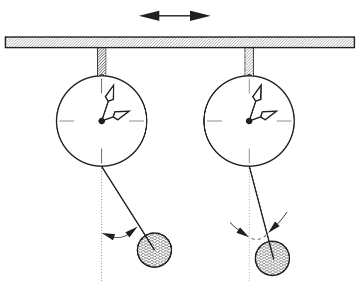
Consensus / Synchronization

Consensus / Synchronization

- deals with agreement about some common behavior in a group
- is relevant for all types of multi-agent systems

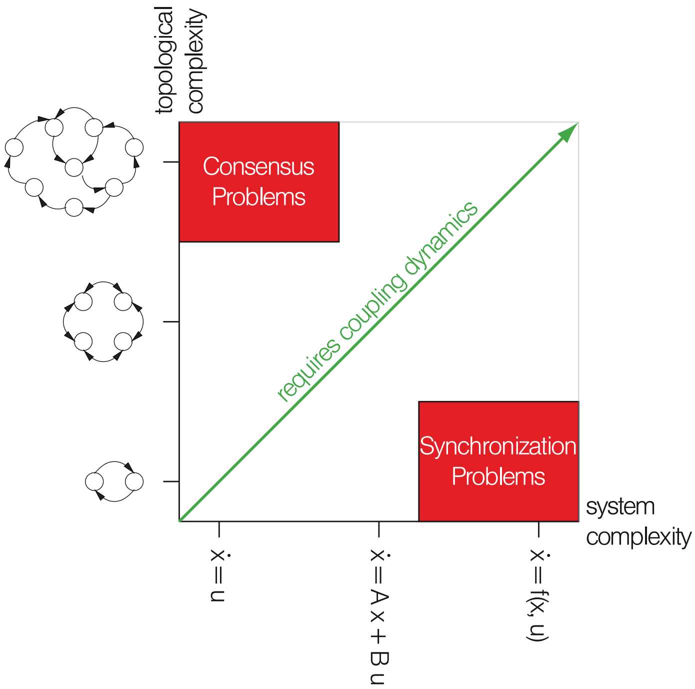




Output consensus/synchronization:

$$\lim_{t\to\infty} \|y_i(t) - y_j(t)\| = 0$$

Consensus / Synchronization

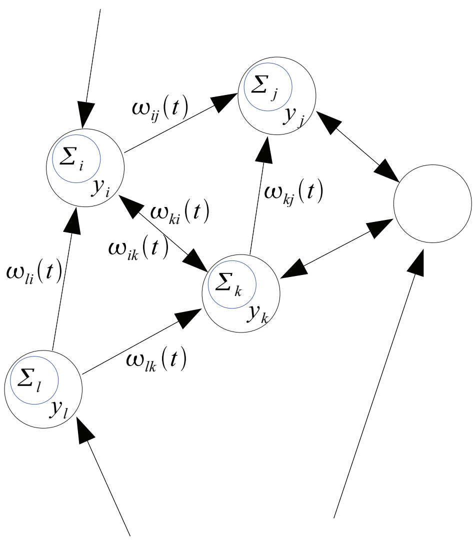


Typical problem setups:

- Consensus: simple systems, complex topologies.
- Synchronization: complex systems, simple topologies.

Extend to problems with high topological and system complexity!

Problem Setup



- Individual agents are **nonlinear** dynamical systems

$$\dot{x}_k(t) = f_k(x_k(t), u_k(t))$$
$$y_k(t) = h_k(x_k(t))$$

- Directed, time-varying graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t), W(t)\}$, **uniformly connected** topology
- **nonidentical dynamics** of individual agents $\Rightarrow$ heterogeneous MAS

Very General Problem Setup

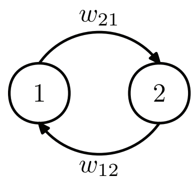
Output synchronization of heterogeneous, nonlinear MAS

Goal of the talk

1. Present necessary conditions for asymptotic synchronization
2. Show a synchronization procedure for classes of problems

Example: Diffusively coupled scalar systems

$$\dot{y}_1(t) = -y_1(t) + u_1(t)$$
$$\dot{y}_2(t) = y_2(t) + u_2(t)$$



- **Diffusive couplings**:

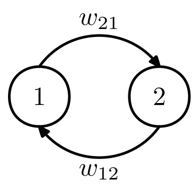
$$u_1(t) = k_1 w_{12}(y_1(t) - y_2(t)),$$
$$u_2(t) = k_2 w_{21}(y_2(t) - y_1(t)).$$

- Find k_1, k_2 such that $(y_1 - y_2) \to 0$ as $t \to \infty$.
- **Observation**: Independently of w_{12}, w_{21}, encoding the interconnection topology, $(y_1 - y_2) \to 0$ as $t \to \infty$ if and only if $y_1 \to 0$ and $y_2 \to 0$ as $t \to \infty$ (e.g., $k_1 = 0$, $k_2 = -2/w_{21}$).

Only trivial synchronization is possible!

Example: Diffusively coupled linear systems

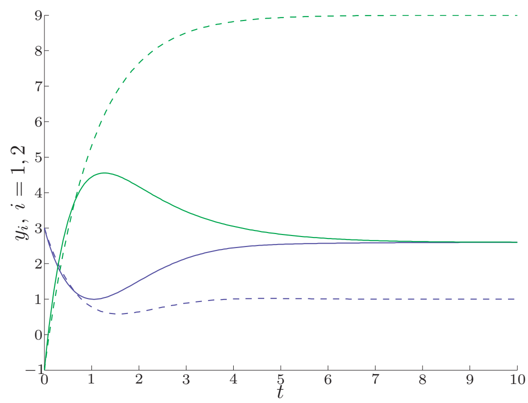
$$\dot{x}_1(t) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & -2 & 0 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} u_1(t),$$
$$y_1(t) = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} x_1(t)$$
$$\dot{x}_2(t) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x_2(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_2(t),$$
$$y_2(t) = \begin{pmatrix} 1 & -1 \end{pmatrix} x_2(t)$$



- **Diffusive couplings**:

$$u_1(t) = -w_{12}(y_1(t) - y_2(t)),$$
$$u_2(t) = -w_{21}(y_2(t) - y_1(t))$$

- **Observation**: Non-trivial synchronization occurs if $w_{12} \geq 0$, $w_{21} \geq 0$, and $\max(w_{12}, w_{21}) > 0$.



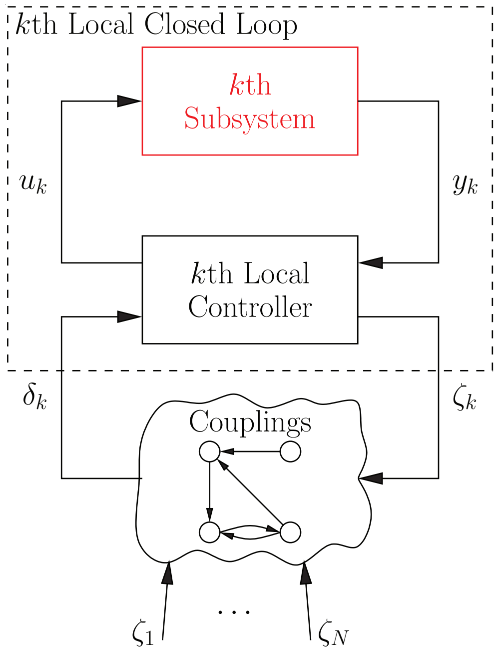
What are structural properties that allow synchronization?

Output Synchronization among Non-Identical Systems

Subsystems

$$\dot{x}_k(t) = f_k(x_k(t), u_k(t)),$$
$$y_k(t) = h_k(x_k(t)),$$

with state $x_k(t) \in \mathbb{R}^{n_k}$, input $u_k(t) \in \mathbb{R}^{p_k}$, and output $y_k(t) \in \mathbb{R}^q$.



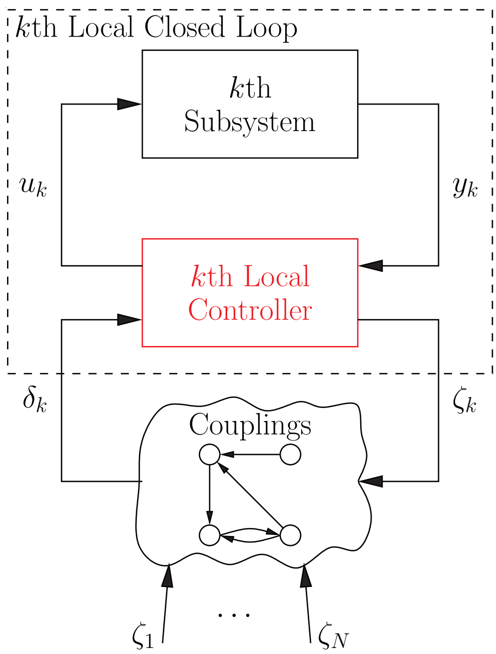
Output Synchronization among Non-Identical Systems

Subsystems

Local Controllers for Subsystems

$$\dot{z}_k(t) = \phi_k(z_k(t), y_k(t), \delta_k(t)),$$
$$u_k(t) = \alpha_k(z_k(t), y_k(t), \delta_k(t)),$$
$$\zeta_k(t) = \beta_k(z_k(t), y_k(t)),$$

with state $z_k(t) \in \mathbb{R}^{m_k}$, inputs $y_k(t) \in \mathbb{R}^q$ and $\delta_k(t) \in \mathbb{R}^r$, and outputs $\zeta_k(t) \in \mathbb{R}^r$ and $u_k(t) \in \mathbb{R}^{p_k}$.



Output Synchronization among Non-Identical Systems

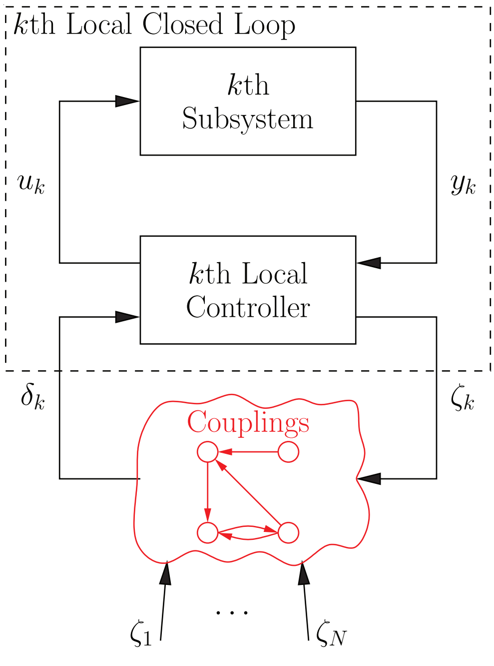
Subsystems

Local Controllers for Subsystems

Couplings

Diffusive couplings ($\simeq$ exchange of relative information)

$$\delta_k(t) = \sum_{j=1}^{N} w_{kj}(t)\big(\zeta_k(t) - \zeta_j(t)\big)$$



Output Synchronization among Non-Identical Systems

Subsystems

Local Controllers for Subsystems

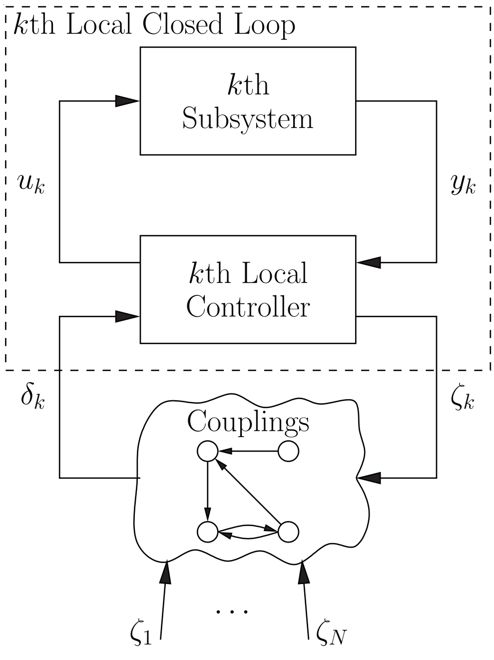
Couplings

Control Objective

$$\lim_{t\to\infty} \|y_i(t) - y_j(t)\| = 0,$$

$$\lim_{t\to\infty} \|\zeta_i(t) - \zeta_j(t)\| = 0$$

for all i, j.



1. When do local controllers exist?
2. What are structural properties of the local controllers?
3. What are the dynamics of the synchronous outputs?

Assumptions for Global Coupled System

Global System

$$\left.\begin{aligned}\dot{x}^*(t) &= f^*(x^*(t), \delta(t))\\ y(t) &= h^*(x^*(t))\\ \zeta(t) &= \beta^*(x^*(t))\end{aligned}\right\}\quad \text{stacked subsystems + local controllers}$$

$$\delta(t) = \Delta(t, \zeta(t)) \;\}\quad \text{couplings } (= \text{feedback})$$

Assumptions

- Solutions $x^*(t)$ exist for all positive times in some closed set $\mathcal{X}^*$.
- Solutions $x^*(t)$ eventually enter and remain in a bounded set $\mathcal{B}^* \subset \mathcal{X}^*$.

Steady State of Global Coupled System

The set $\Omega^* \triangleq \omega(\mathbb{R} \times \mathcal{B}^*)$ uniquely defines the **steady state locus** of the global coupled system.

How does the steady state locus relate to synchronization?

Synchronous Steady State

Theorem (Wieland and Allgöwer, 2010b)

Synchronization occurs **asymptotically** if and only if

$$\Omega^* \subset \{x^* \in \mathcal{X}^* | y_1 = y_2 = \cdots \wedge \zeta_1 = \zeta_2 = \cdots\}$$

Synchronization occurs **asymptotically** if and only if synchronization occurs **identically** in **steady state**.

Couplings use relative information, thus

$$\Delta(t, \beta^*(x^*)) \equiv 0 \text{ in steady state}$$

Synchronization occurs **asymptotically** if and only if the subsystems are **decoupled** and **identically synchronous** in **steady state**.

Results characterize steady state **locus** of synchronizing systems.
What can be deduced about steady state **dynamics**?

Implicit Internal Model Principle

Theorem (Wieland and Allgöwer, 2010b)

If the global coupled system is detectable and synchronization occurs asymptotically, then there exists a **virtual exosystem**

$$\dot{\xi}(t) = s(\xi(t)), \qquad \eta(t) = \hat{h}(\xi(t)) \tag{VEx}$$

with state $\xi(t) \in \hat{\mathcal{X}}$ and output $\eta(t) \in \mathbb{R}^q$ characterizing the steady state dynamics, and there exist maps $\pi_k \colon \hat{\mathcal{X}} \to \mathbb{R}^{n_k}$, $\sigma_k \colon \hat{\mathcal{X}} \to \mathbb{R}^{m_k}$ such that

$$\frac{\partial \pi_k(\xi)}{\partial \xi} s(\xi) = f_k\Big(\pi_k(\xi), \alpha_k\big(\sigma_k(\xi), h_k(\pi_k(\xi)), 0\big)\Big), \tag{Impl/a}$$

$$\frac{\partial \sigma_k(\xi)}{\partial \xi} s(\xi) = \phi_k\big(\sigma_k(\xi), h_k(\pi_k(\xi)), 0\big), \tag{Impl/b}$$

$$\hat{h}(\xi) = h_k(\pi_k(\xi)) \tag{Impl/c}$$

If in addition Ω^* has the asymptotic phase property, then

$$\lim_{t\to\infty} (y_k(t) - \eta(t)) = 0$$

along some solution of (VEx).

Implicit Internal Model Principle

- PDEs (Impl/a), (Impl/b) are an infinitesimal characterization of the statement

  "The graph of the map $(x_k, z_k) = \big(\pi_k(\xi), \sigma_k(\xi)\big)$ is an invariant set for the local closed loop + (VEx); the dynamics restricted to this set is given by (VEx)."

- Condition (Impl/c) states

  "When restricted to this invariant set, $y_k(t) = \eta(t)$."

Synchronization implies that all local closed loops contain an **internal model** of a **common virtual exosystem**

Invariance conditions are implicit in the sense that they depend on the local controllers, i.e., the solution to the problem.

Can we get rid of dependency on controllers?

Explicit Internal Model Principle

Theorem (Wieland and Allgöwer, 2010b)

If the global coupled system is detectable and synchronization occurs asymptotically, then there exists a virtual exosystem (VEx) as before, and there exist maps $\pi_k : \hat{\mathcal{X}} \to \mathbb{R}^{n_k}$, $\lambda_k : \hat{\mathcal{X}} \to \mathbb{R}^{p_k}$ such that

$$\frac{\partial \pi_k(\xi)}{\partial \xi} s(\xi) = f_k\big(\pi_k(\xi), \lambda_k(\xi)\big) \qquad \text{(Expl/a)}$$

$$\hat{h}(\xi) = h_k(\pi_k(\xi)) \qquad \text{(Expl/b)}$$

- PDE (Expl/a) $\Rightarrow$ The graph of the map $x_k = \pi_k(\xi)$ is a controlled invariant set for the subsystem + (VEx),
  rendered invariant with the **feedforward** control $u_k(t) = \lambda_k(\xi(t))$
- Condition (Expl/b) is identical to condition (Impl/c)).
- Solvability of (Expl/a), (Expl/b) is equivalent to existence of a local controller that admits a solution of (Impl/a)–(Impl/c).

Implicit Internal Model Principle

Local Closed Loop Systems

$$\left.\begin{aligned} \dot{x}_k^*(t) &= A^* x_k^*(t) + B_k^* \delta_k(t) \\ y_k(t) &= C_k^* x_k^*(t) \\ \zeta_k(t) &= P_k^* x_k^*(t) \end{aligned}\right\} \quad \text{subsystems + local controllers}$$

Theorem

If synchronization occurs asymptotically, then there exists a **virtual exosystem**

$$\dot{\xi}(t) = S\xi(t), \quad \eta(t) = R\xi(t) \qquad \text{(VEx)}$$

with state $\xi(t) \in \mathbb{R}^{\nu}$ and output $\eta(t) \in \mathbb{R}^{q}$, and there exist matrices $\Psi_k \in \mathbb{R}^{(n_k+m_k)\times\nu}$ such that

$$\Psi_k S = A_k^* \Psi_k, \qquad \text{(Impl/a)}$$

$$R = C_k^* \Psi_k. \qquad \text{(Impl/b)}$$

In addition

$$\lim_{t\to\infty} (y_k(t) - \eta(t)) = 0$$

along some solution of (VEx).

Explicit Internal Model Principle

Theorem

If synchronization occurs asymptotically, then there exists a virtual exosystem (VEx) as before, and there exist matrices $\Pi_k \in \mathbb{R}^{n_k \times \nu}$, $\Lambda_k \in \mathbb{R}^{p_k \times \nu}$ such that

$$\Pi_k S = A_k \Pi_k + B_k \Lambda_k, \qquad \text{(Expl/a)}$$

$$R = C_k \Pi_k. \qquad \text{(Expl/b)}$$

- Condition (Expl/a) $\Rightarrow$ The subspace of $\mathbb{R}^\nu \times \mathbb{R}^{n_k}$ spanned by the columns of $\left(I_\nu, \Pi_k^T\right)^T$ is a controlled invariant subspace for (VEx) + subsystem,
  rendered invariant with the **feedforward** control $u_k(t) = \Lambda_k \xi(t)$
- Condition (Expl/b) is identical to condition (Impl/b)).
- Solvability of (Expl/a), (Expl/b) is equivalent to existence of a local controller that admits a solution of (Impl/a), (Impl/b).

Discussion of Internal Model Principle

Synchronization vs. Output Regulation

- Conditions (Expl/a), (Expl/b) correspond to the F.B.I.-Equations, that are solvability conditions for the output regulation problem.
- Synchronization is not Output Regulation!
  - Output Regulation: the exosystem is an autonomous system external to the system to be controlled;
  - Synchronization: no autonomous exosystem exists, the virtual exosystem only exists internal to the network.

Synchronization of non-identical systems requires

- **Feedforward control** that ensures existence of an invariant set on which the network is identically synchronous
- **Feedback control** that renders this set attractive

The internal model conditions are **existence conditions** for the **feedforward** part of the **control**.

Questions Answered

1. *When do local controllers exist?*

   A **necessary condition** is solvability of the **explicit internal model equations** for some virtual exosystem.

2. *What are structural properties of the local controllers?*

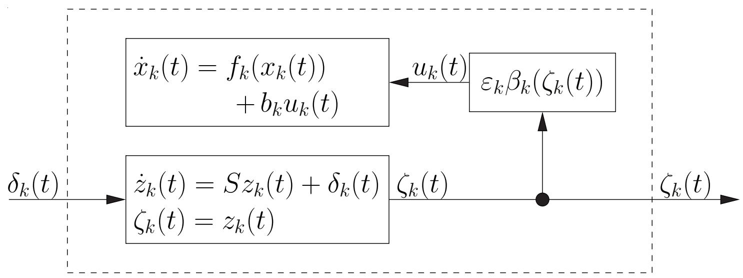
   They solve the **implicit internal model equations**. Thus they contain a feedforward part that renders appropriate sets invariant with dynamics corresponding to the virtual exosystem dynamics.

3. *What are the dynamics of the synchronous outputs?*

   All possible synchronous outputs are given by outputs generated by the **virtual exosystem**

How can we use this for
synchronization of non-identical exponentially stable oscillators?

Synchronization of Non-Identical Oscillators



- Basic idea: synchronize copies of virtual exosystem ($\simeq$ coupling dynamics) and use synchronized signals to entrain oscillators.
- Coupling dynamics used to compensate for non-identical dynamics and to compensate for high topological complexity

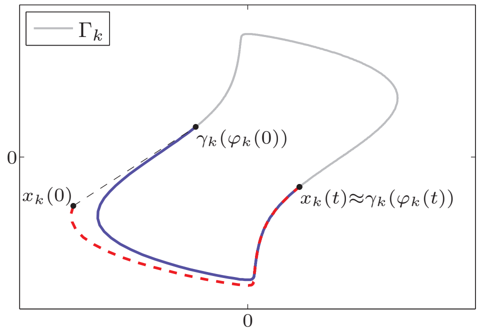
Generic method to synchronize non-identical oscillators with
weak assumptions on subsystems and couplings
($\simeq$ high system and topological complexity).

Exponentially Stable Oscillators

Oscillators

$$\dot{x}_k(t) = f(x_k(t)) + b_k u_k(t)$$

with state $x_k(t) \in \mathbb{R}^{n_k}$ and input $u_k(t) \in \mathbb{R}$.



Properties of unforced systems

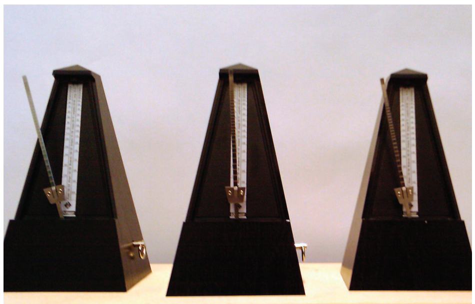
- Periodic solution $x_k(t) = \gamma_k(\omega_k t)$ with frequency ω_k and periodic orbit $\Gamma_k \triangleq \gamma_k(\mathbb{R})$.
- Solutions $x_k(t)$ starting close to Γ_k satisfy
$$\left\| x_k(t) - \gamma_k\Big(\omega_k t + \theta_k\big(x_k(0)\big)\Big) \right\| \leq M e^{-\mu t}$$
- $\varphi_k(t) = \theta_k(x_k(t)) \in \mathbb{S}^1$ is the **asymptotic phase** of $x_k(t)$ with
$$\dot{\varphi}_k(t) = \omega_k$$
(= phase of limiting solution $\gamma_k\big(\omega_k t + \varphi_k(0)\big) = \gamma_k\big(\varphi_k(t)\big)$)

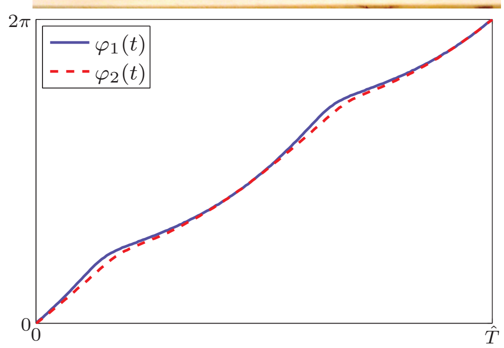
What is phase synchronization?

Phase Synchronization of Non-Identical Oscillators

Phase Synchronization

- Synchronization of zero phase times:
$$\lim_{l \to \infty} \|\varphi_k(t_l)\| = 0$$
for all k, with time instants t_l such that $\exists j : \varphi_j(t_l) = 0$.





- Coupled oscillators admit perturbed periodic solutions $\hat{\gamma}_k(\hat{\omega} t)$ with some common frequency $\hat{\omega}$.
- Asymptotic phase relative to $\hat{\gamma}_k(\hat{\omega} t)$: $\hat{\varphi}_k(t) = \hat{\theta}_k(x_k(t))$ with
$$\dot{\hat{\varphi}}_k(t) = \hat{\omega}.$$

Find solution to F.B.I. equations with virtual exosystem $\dot{\xi}(t) = \hat{\omega}$, $\xi(t) \in \mathbb{S}^1$ and **unknown** system output $\hat{\varphi}_k(t) = \hat{\theta}_k(x_k(t))$!

Solution to the F.B.I. Equations

Entrainment by Small Harmonic Forcing

For $\hat{\omega} - \omega_k$ small enough, there exist ε_k (small enough) and $\hat{\xi}_k \in \mathbb{S}^1$ such that the control

$$u_k(t) = \varepsilon_k \cos(\hat{\omega} t + \hat{\xi}_k)$$

ensures convergence of $x_k(t)$ to a unique periodic solution $\hat{\gamma}_k(\hat{\omega} t)$ with frequency $\hat{\omega}$ for almost all $x_k(0) \in \mathbb{R}^{n_k}$ close enough to Γ_k.

F.B.I. equations are satisfied with virtual exosystem $\dot{\xi}(t) = \hat{\omega}$, $\xi(t) \in \mathbb{S}^1$, and maps

$$\pi_k(\xi) \triangleq \hat{\gamma}_k(\xi),$$
$$\lambda_k(\xi) \triangleq \varepsilon_k \cos(\xi + \hat{\xi}_k).$$

Find local controllers that asymptotically synchronize and generate the appropriate feedforward controls!

Phase Synchronization by Entrainment and Consensus

Theorem (Wieland, 2010)

If $\hat{\omega} - \omega_k$ small enough, there exist ε_k (small enough) and $\hat{\xi}_k \in \mathbb{S}^1$ such that the local controllers

$$\dot{z}_k(t) = \begin{pmatrix} 0 & -\hat{\omega} \\ \hat{\omega} & 0 \end{pmatrix} z_k(t) + \delta_k(t)$$
$$u_k(t) = \frac{\varepsilon_k}{\|z_k(t)\|} \Big(\cos(\hat{\xi}_k) z_{k,1}(t) + \sin(\hat{\xi}_k) z_{k,2}(t) \Big)$$
$$\zeta_k(t) = z_k(t)$$

with $z_k(t), \zeta_k(t) \in \mathbb{R}^2$ yield asymptotic synchronization over uniformly connected topologies ($\simeq$ high topological complexity) for almost all initial conditions $x_k(0) \in \mathbb{R}^{n_k}$, $z_k(0) \in \mathbb{R}^2$.

- Consensus type results can be used to show synchronization of harmonic oscillators, i.e., $(z_i(t) - z_j(t)) \to 0$.
- Feedforward controls $u_k(t) \Rightarrow$ existence of invariant synchronous set.
- Robustness of exp. stable limit cycles $\Rightarrow$ attractivity of this set.

Example

Subsystems

Different Van der Pol oscillators (varying in parameter μ_k):

$$\dot{x}_k(t) = \begin{pmatrix} x_{k,2}(t) + \mu_k \left(x_{k,1}(t) - \frac{1}{3} x_{k,1}^3(t) \right) \\ -x_{k,1}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_k(t)$$

k	1	2	3	4	5
μ_k	3.0	3.5	4.0	4.5	5.0
ω_k	0.7092	6.599	0.6158	0.5764	0.5411
T_k	8.859	9.521	10.20	10.90	11.61
ε_k	0.5	0.5	0.5	0.5	0.5
$\hat{\xi}_k$	4.065	4.578	4.960	5.310	5.704

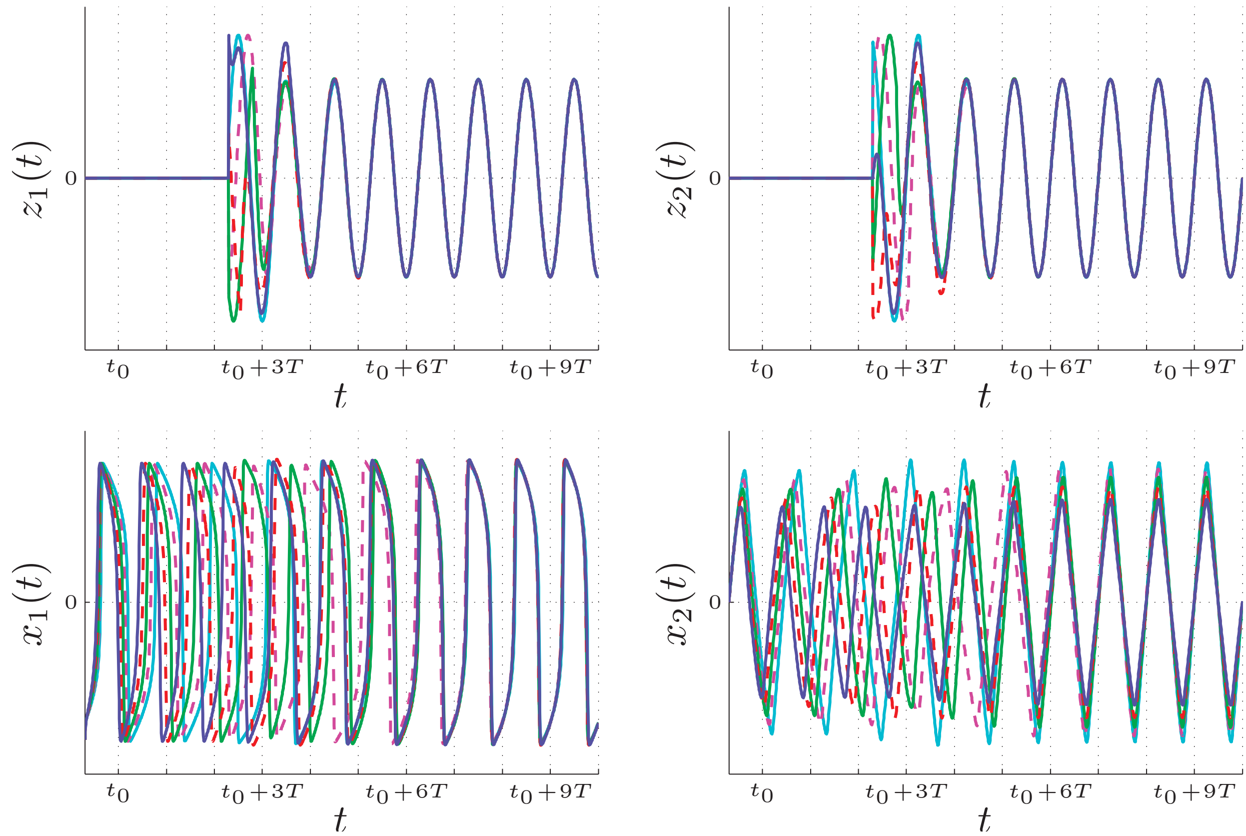
Parameter values for simulation.

Couplings

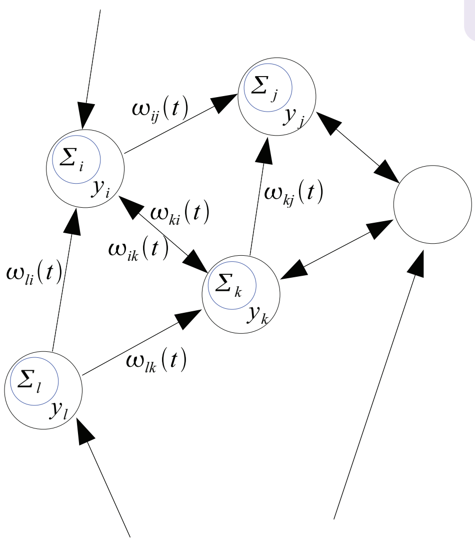
Graph contains exactly one link at each time instant and switches every $T = 2.5$ units of time (seconds).

Example

figs/vdposcsync.avi



Summary



Key result: **Internal Model Principle for Synchronization**

- Presents a necessary condition for output synchronization.
- Links synchronization problems to output regulation problems.
- Suggests a control paradigm for output synchronization of heterogenous MAS using dynamic couplings.
- Presented a new result for synchronization of nonlinear oscillators over uniformly connected communication graphs.

Acknowledgements

Peter Wieland

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