UC Santa Barbara, Department of Mechanical Engineering

ME103 Dynamical Systems. Chapter Slides.

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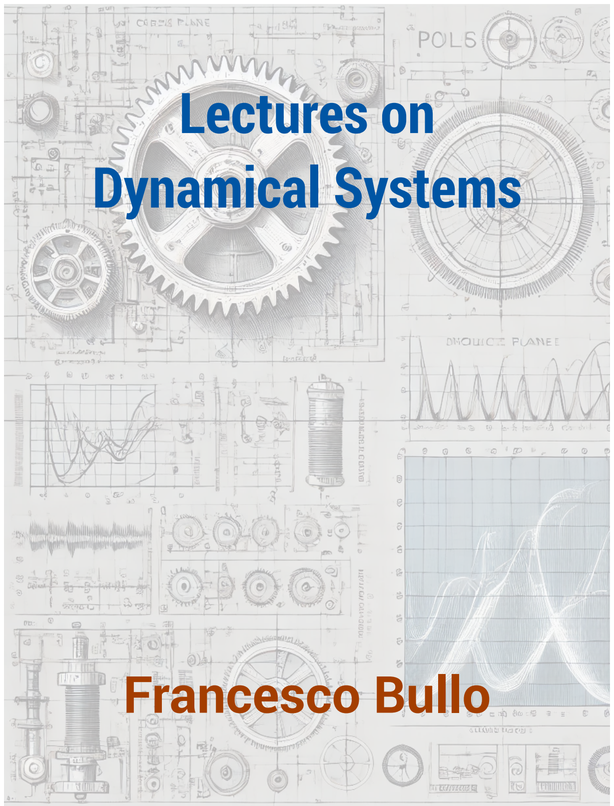
http://motion.me.ucsb.edu/ME103-Fall2025/syllabus.html



Contents

Chapter 6

Frequency Response, Resonances, and Beats in Dynamical Systems

This chapter examines the dynamics of linear time-invariant systems, focusing on their response to sinusoidal inputs, through the concept of *frequency response*, and the phenomena of *resonance* and *beating*.

The frequency response, equal to the transfer function evaluated at $s = \mathrm{i}\omega$, determines how the amplitude and phase of an output sinusoid change relative to the input. In first-order systems (without zeros), low-frequency inputs retain amplitude while high-frequency inputs are attenuated. Second-order systems may experience *resonance*, with large amplification when the input frequency is near the natural frequency and damping is low. Resonance is effectively represented using Bode plots, which show magnitude and phase across frequencies on logarithmic scales.

The chapter also considers *lightly damped systems*, where *beating* arises when two close frequencies interact, producing alternating amplitude patterns due to *constructive* and *destructive interference*. We highlight the role of *proper* and *strictly proper* transfer functions in maintaining causality, particularly under high-frequency inputs. Low-pass filters are introduced to ensure realizability by limiting the effects of improper transfer functions. These analyses establish the steady-state response of stable systems, using tools such as the Laplace transform and phasor representation.

6.1 The frequency response and the resonance phenomenon

We consider the problem illustrated in Figure 6.1 below, where a stable linear time-invariant system with transfer function $G(s)$ is subject to a sinusoidal input with unit magnitude and frequency $\omega > 0$.

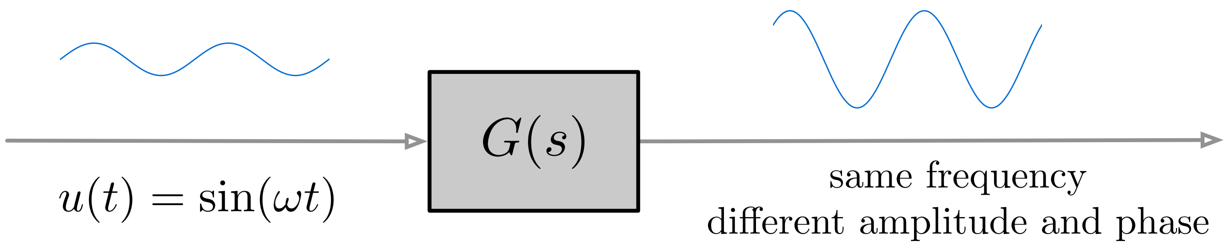


Figure 6.1: A stable system subject to a unit-magnitude sinusoidal input: The main result is that, if the input is a sine wave, so is the output! (after a transient) While the frequency of the output steady state oscillation is the same as the frequency of the input, the magnitude and phase of the output are determined by the frequency response.

6.1.1 The frequency response formula

The main result of this chapter can be stated in one equation.

The steady state response of a stable linear system to a unit-magnitude sinusoidal input satisfies

$$u(t) = \sin(\omega t) \qquad \Longrightarrow \qquad y_{\text{steady-state}}(t) = \left|G(\mathrm{i}\omega)\right| \sin\Big(\omega t + \arg\big(G(\mathrm{i}\omega)\big)\Big) \tag{6.1}$$

where

- given a complex number z, $|z|$ is its magnitude and $\arg(z)$ is its argument or angle,
- $y_{\text{steady-state}}(t)$ is the *steady state solution* of the system, that is, the solution after all exponentially decaying signals have vanished,
- the function $G(\mathrm{i}\omega)$ is called the *frequency response* (also know as the *sinusoidal transfer function*) and is equal to the transfer function $G(s)$ evaluated at $s = \mathrm{i}\omega$.

The correctness of equation (6.1) is studied in Appendix 6.4 via inverse Laplace transforms and partial fraction expansions.

Given a frequency ω, the frequency response of the system $G(\mathrm{i}\omega)$ is a complex number. Given a sinusoidal input at frequency ω,

(i) the *magnitude frequency response* $\left|G(\mathrm{i}\omega)\right|$ determines the amplification (or attenuation) of the output sinusoidal signal as compared with the input; and

(ii) the *angular frequency response* $\arg\bigl(G(\mathrm{i}\omega)\bigr)$ determines the phase shift of the output sinusoidal signal as compared with the input.

6.1.2 First-order systems

Recall from Section 5.2 that the transfer function of a first-order system $\tau\dot{y} + y = u$ is

$$\frac{Y(s)}{U(s)} = G_{\text{first-order}}(s) = \frac{1}{\tau s + 1}. \tag{6.2}$$

Hence, the frequency response function of a first-order system is

$$G_{\text{first-order}}(\mathrm{i}\omega) = \frac{1}{\mathrm{i}\tau\omega + 1}. \tag{6.3}$$

The magnitude frequency response is:

$$\big|G_{\text{first-order}}(\mathrm{i}\omega)\big| = \frac{1}{\sqrt{\tau^2\omega^2 + 1}} \tag{6.4}$$

and the angular frequency response is

$$\arg\big(G_{\text{first-order}}(\mathrm{i}\omega)\big) = -\arctan(\tau\omega) \tag{6.5}$$

In summary, the steady-state response of a first-order system to a unit-magnitude sinusoidal input $u(t) = \sin(\omega t)$ is

$$y_{\text{steady-state}}(t) \;=\; \frac{1}{\sqrt{\tau^2\omega^2 + 1}}\sin\big(\omega t - \arctan(\tau\omega)\big). \tag{6.6}$$

(i) For low frequencies $\omega \ll \frac{1}{\tau}$, we have $\sqrt{\tau^2\omega^2 + 1} \approx 1$ and the output amplitude closely approximates the unit input amplitude in the steady-state response; and

(ii) for high frequencies $\omega \gg \frac{1}{\tau}$, we have $\sqrt{\tau^2\omega^2 + 1} \approx \tau\omega$ and the output amplitude is attenuated, approaching approximately $1/(\tau\omega)$ (meaning that the amplitude decreases as $1/\omega$).

We plot the magnitude frequency response of a first-order system in Figure 6.2.

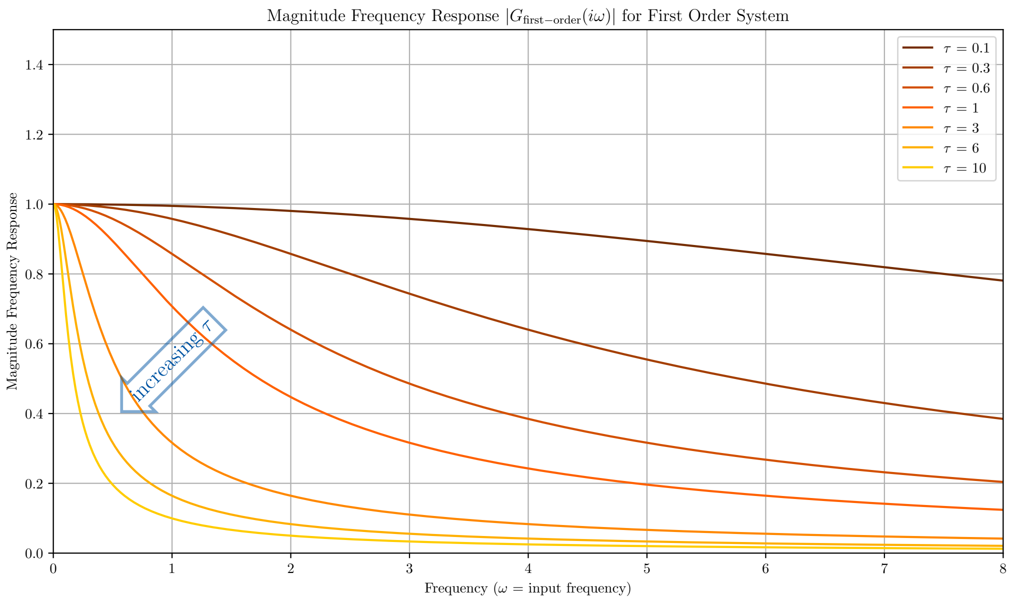


Figure 6.2: The magnitude frequency response $\left|G_{\text{first-order}}(\mathrm{i}\omega)\right|$ of a first-order system, as in equation (6.4).
On the horizontal axis, the variable is the frequency ω of the sinusoidal input.
Python code available at `frequencyresponse-firstorder.py`

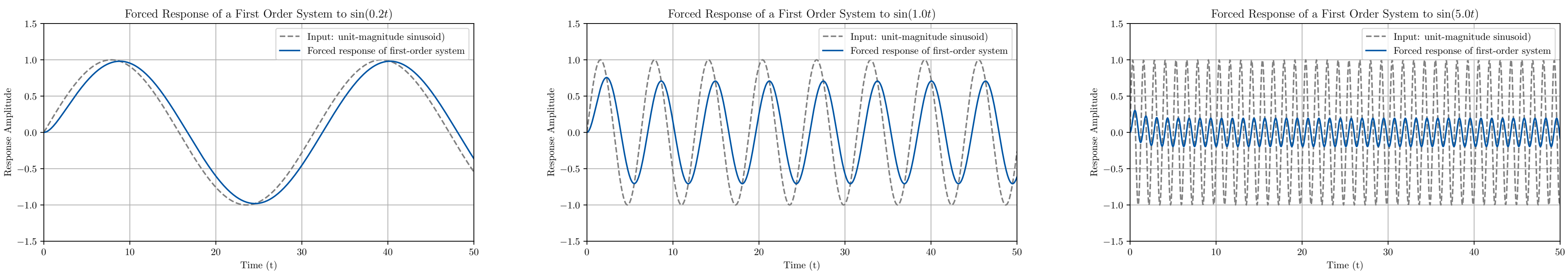


Figure 6.3: The forced response (blue solid line) of a first order system subject to a unit-magnitude sinusoidal forcing (gray dashed lines) for three values of the frequency $\omega = 0.2, 1.0, 5.0$ and $\tau = 1$.
As ω increases, the magnitude of the response decreases and the phase delay increases.

6.1.3 Second-order systems and the resonance phenomenon

Recall from Section 5.3 that the transfer function of a second-order system $\ddot{y}(t) + 2\zeta\omega_{\mathrm{n}}\dot{y}(t) + \omega_{\mathrm{n}}^2 y(t) = \omega_{\mathrm{n}}^2 u(t)$ is

$$G_{\text{second-order}}(s) = \frac{Y(s)}{U(s)} = \frac{\omega_{\mathrm{n}}^2}{s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2}. \tag{6.7}$$

We now introduce the *input frequency* ω (also called *driving frequency*). Note: In the frequency response of second order systems (especially, underdamped systems) there are two relevant frequencies: ω is the frequency of the input sinusoid and ω_{n} is the natural frequency of the system. We now compute the frequency response:

$$G_{\text{second-order}}(\mathrm{i}\omega) = \frac{\omega_{\mathrm{n}}^2}{(\omega_{\mathrm{n}}^2 - \omega^2) + \mathrm{i}(2\zeta\omega_{\mathrm{n}}\omega)} = \frac{1}{(1 - \omega^2/\omega_{\mathrm{n}}^2) + \mathrm{i}(2\zeta\omega/\omega_{\mathrm{n}})} \tag{6.8}$$

and the magnitude frequency response:

$$\left|G_{\text{second-order}}(\mathrm{i}\omega)\right| = \frac{1}{\sqrt{(1 - (\omega/\omega_{\mathrm{n}})^2)^2 + (2\zeta\omega/\omega_{\mathrm{n}})^2}} \tag{6.9}$$

We plot this magnitude frequency response in Figure 6.4.

(i) For low frequencies $\omega \ll \omega_{\mathrm{n}}$, we have $\sqrt{(1 - (\omega/\omega_{\mathrm{n}})^2)^2 + (2\zeta\omega/\omega_{\mathrm{n}})^2} \approx 1$ and the output amplitude closely approximates the unit input amplitude in the steady-state response;

(ii) for high frequencies $\omega \gg \omega_{\mathrm{n}}$, we have $\sqrt{(1 - (\omega/\omega_{\mathrm{n}})^2)^2 + (2\zeta\omega/\omega_{\mathrm{n}})^2} \approx (\omega/\omega_{\mathrm{n}})^2$ and the output amplitude is attenuated, approaching approximately $1/(\omega/\omega_{\mathrm{n}})^2$ (meaning that the amplitude decreases as $1/\omega^2$); and

(iii) for $\omega = \omega_{\mathrm{n}}$, we have $\sqrt{(1 - (\omega/\omega_{\mathrm{n}})^2)^2 + (2\zeta\omega/\omega_{\mathrm{n}})^2} = 2\zeta$ and the output amplitude is $1/(2\zeta)$.

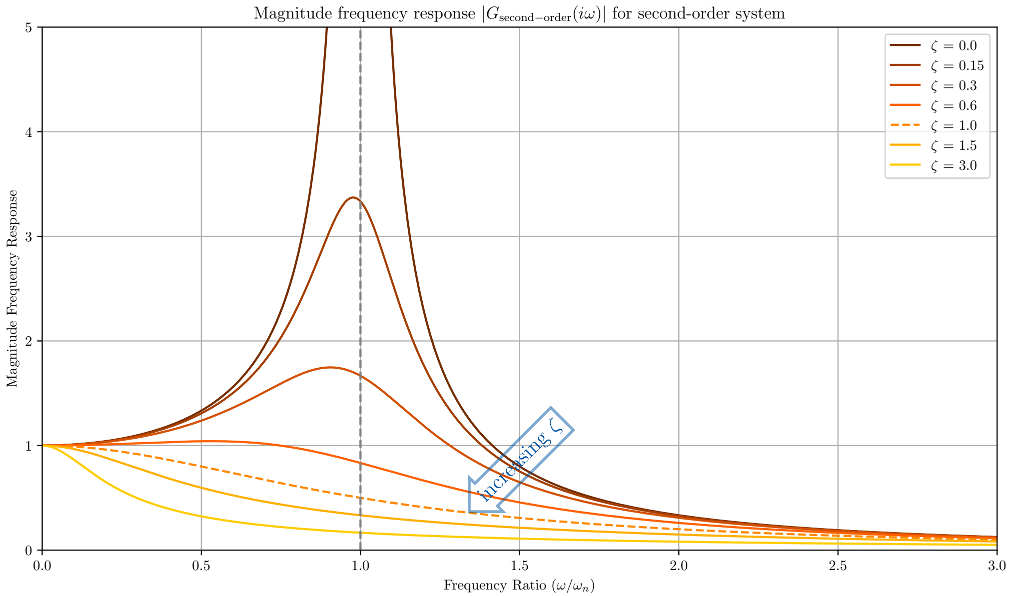


Figure 6.4: The magnitude frequency response $|G_{\text{second-order}}(\mathrm{i}\omega)|$ of a second-order system, as given in equation (6.9).
On the horizontal axis, the variable is the frequency ratio $\omega/\omega_{\mathrm{n}}$, where ω is the frequency of the sinusoidal input.
Image generated by `frequencyresponse-secondorder.py`

From equation (6.9) and Figure 6.4, we learn that

- when $\zeta \geq 1$ (critically damped and overdamped system), the input magnitude is always attenuated,
- when $0 < \zeta < 1$ (underdamper system), there is a range of input frequencies for which the input magnitude is amplified, and
- when $0 < \zeta \ll 1$ (the *lightly damped regime*), the amplification can be very large. We call this amplification *resonance*.

We say that *resonance* happens when

(i) the input frequency is very close to the natural frequency $\omega/\omega_{\mathrm{n}} \approx 1$, and

(ii) the second-order system is lightly damped, meaning that the damping ratio ζ is much smaller than 1.

Under these two conditions, the magnitude frequency response is very large: the input sinusoidal signal is *efficiently amplified* to a potentially destructive effect. The physical reason for this efficient amplification is that the input signal adds energy to the system during each oscillation cycle. Even a small periodic driving force can produce large amplitude oscillations due to the constructive interference between external force and natural frequency.

6.1.4 Bode plots

In engineering practice, it is convenient to draw the frequency response in logarithmic coordinates. Specifically, the *Bode magnitude plot* of the frequency response adopts:

(i) the ω horizontal axis is logarithmic, and

(ii) the magnitude $|G(\mathrm{i}\omega)|$ is plotted in *decibels*, that is, a value $|G(\mathrm{i}\omega)|$ is plotted at $20\log_{10}|G(\mathrm{i}\omega)|$.

The *Bode phase plot* uses a logarithmic scale for ω and a linear scale for the angle $\arg\big(G(\mathrm{i}\omega)\big)$. Note that, for all frequencies where $|G(\mathrm{i}\omega)| = 1$, we have $20\log_{10}|G(\mathrm{i}\omega)| = 20\log_{10}(1) = 0$.

For a first-order factor of the form $(1+\tau s)^{-1}$, the magnitude is approximately constant (0 dB) for $\omega \ll 1/\tau$ and decays at -20 dB/decade for $\omega \gg 1/\tau$. For a first-order factor of the form $(1+\tau s)$, the magnitude grows at $+20$ dB/decade for $\omega \gg 1/\tau$. In general, each real pole contributes a -20 dB/decade slope beyond its break frequency, and each real zero contributes a $+20$ dB/decade slope. The break frequency is $\omega_b = 1/\tau$ for these examples.

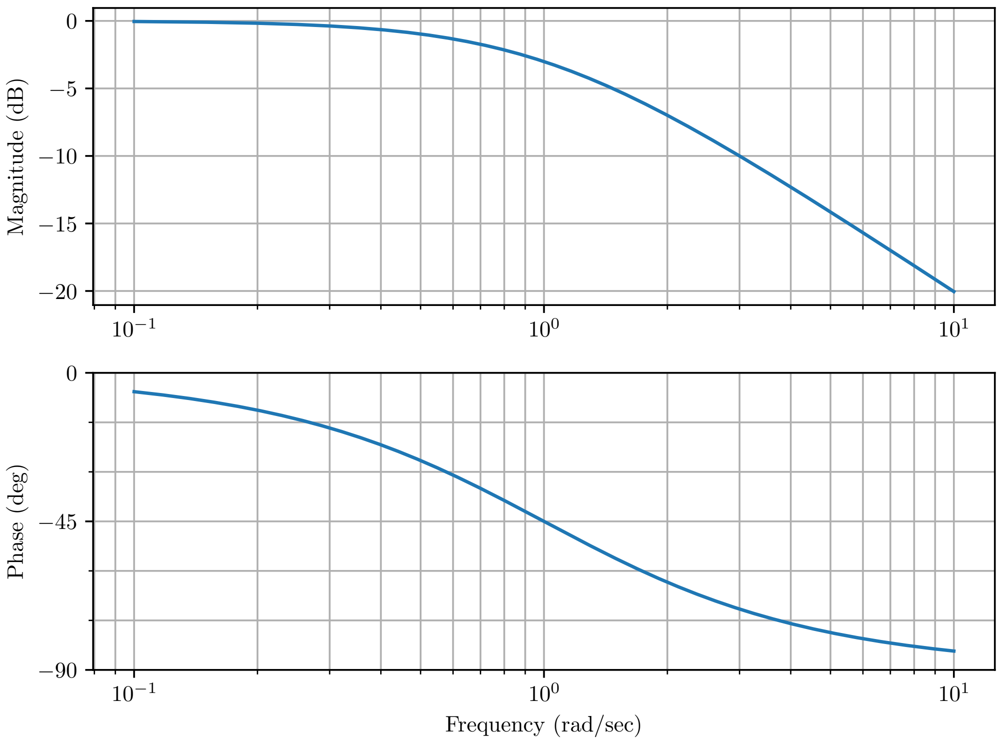


Figure 6.5: Bode magnitude and phase plots for a first-order system as in equation (6.4), with time constant $\tau = 1$. The slope change of -20 dB/decade begins at the break frequency $\omega_b = 1$.

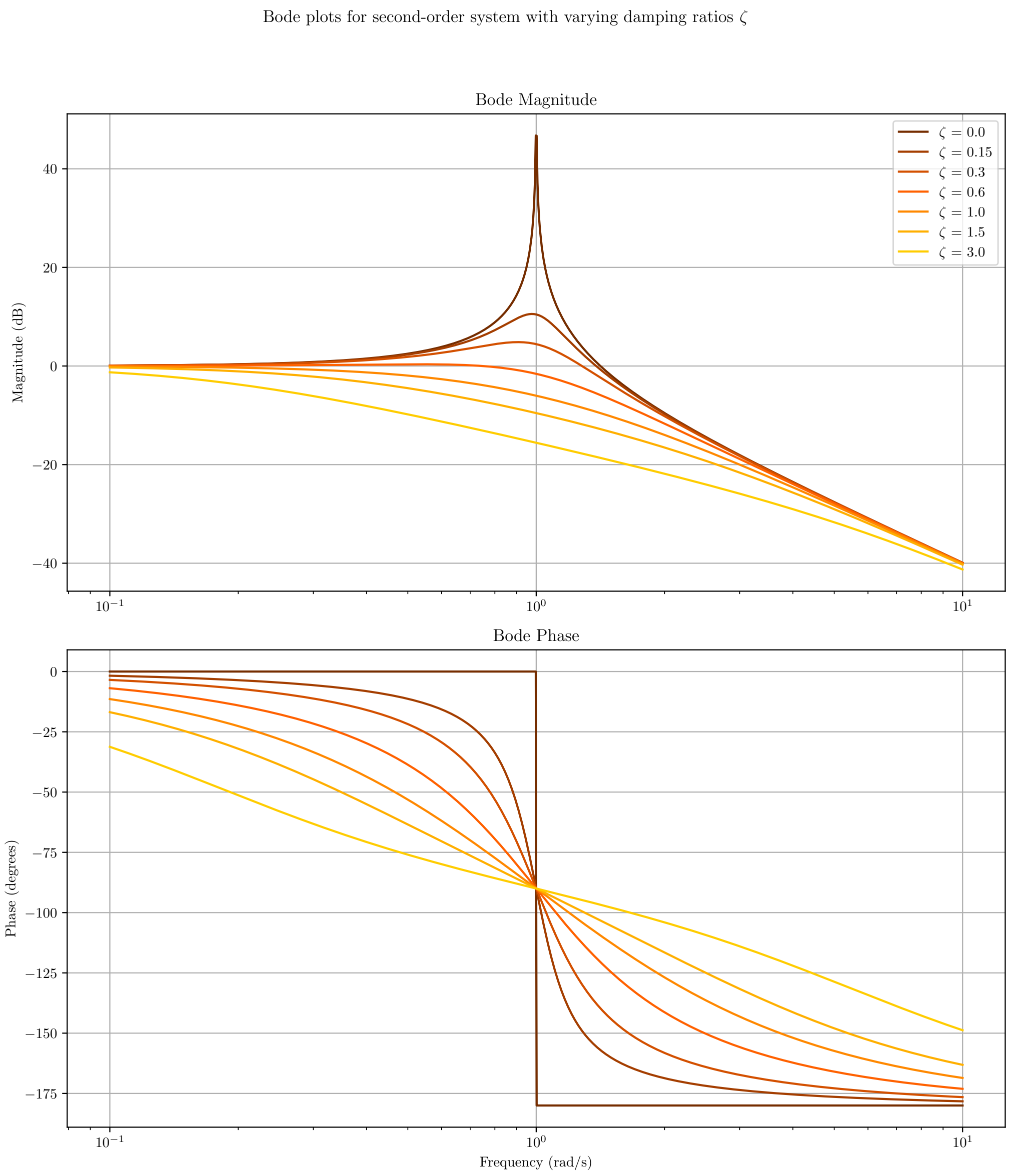


Figure 6.6: Bode magnitude and phase plots for a second-order system with unit natural frequency $\omega_n = 1$ and multiple values of the damping ratio ζ. For low damping, the magnitude plot shows a pronounced resonance near ω_n.

6.1.5 Control System Design Software: The Python Control Systems Library

Programming notes

It is useful to compare popular tools for control system design, including both commercial and open-source domains.

- Leading commercial tools include Matlab's Control System Toolbox and Simulink, which are widely used for their comprehensive features in control design, system simulation, and robust analysis capabilities.
- The Python Control Systems Library (`python-control`) is an open-source library designed for analyzing and designing feedback control systems, see (Fuller et al., 2021). It offers functionality for modeling linear time-invariant (LTI) systems, computing step responses, performing stability and frequency response / Bode analysis, and designing controllers using techniques such as root locus and frequency-domain methods.

  Its tight integration with Python ensures compatibility with other scientific libraries like NumPy and SciPy, making it a strong choice for users who prefer open-source, Python-based workflows, although commercial tools might provide more specialized features and enhanced usability for large-scale projects.

FBMaybe: in chapter 5 we already use the control toolbox to (1) define a transfer function object, (2) compute step responses.

6.2 Lightly damped systems and the beating phenomenon

By *lightly damped system* we mean a second-order system with very small damping ratio $\zeta \ll 1$.

In this section, we study *beating*, a phenomenon that arises when the driving frequency is close to the natural frequency of the lightly damped system, often observed as the system approaches resonance.

To understand the behavior of a lightly-damped second order system, we study the forced response of the *undamped second-order system with* $\zeta = 0$ subject to a unit-magnitude sinusoidal input.

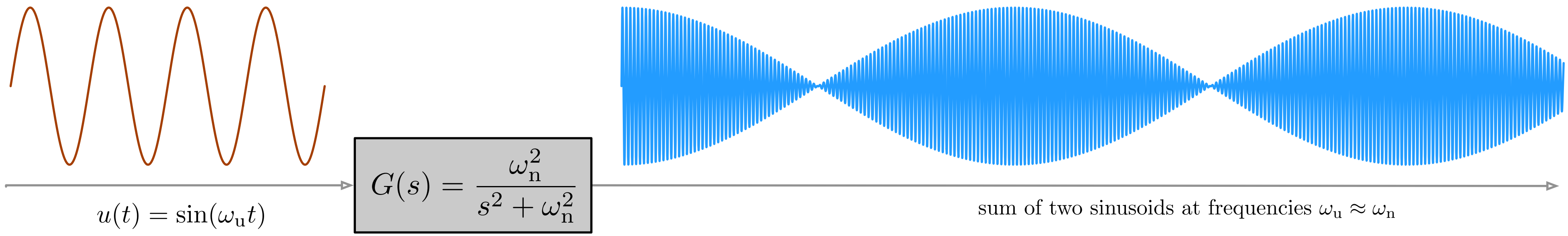


Figure 6.7: A sinusoidal input forcing an undamped second-order system, that is, an harmonic oscillator. (For illustration purposes, the time duration of the output signal is larger than that of the input signal.)

6.2.1 Background: Constructive and destructive interference of sinusoidal waves

We start by considering the sum of two sinusoidal waves with frequencies ω_1 and ω_2 such that $\omega_1 \approx \omega_2$.

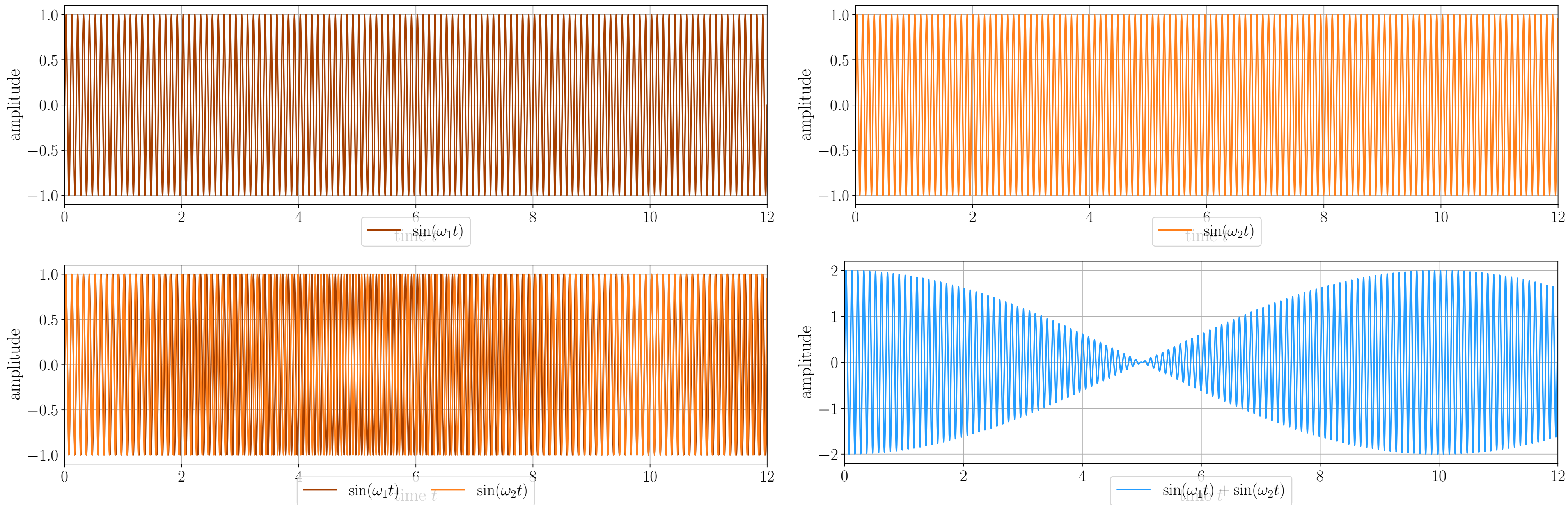


Figure 6.8: These plot illustrates the phenomenon of *constructive* and *destructive interference* between two sine waves, $\sin(\omega_1 t)$ and $\sin(\omega_2 t)$, with close but distinct frequencies ($\omega_1 = 20\pi$ rad/s and $\omega_2 = 20.2\pi$ rad/s). The sum of the two waves exhibits a *beating pattern*, where the amplitude alternates between high (constructive interference) and low (destructive interference), creating the characteristic modulation observed in the combined signal.

To better understand the beating phenomenon for two sinusoidal waves, we use the *sum-to-product* trigonometric formula:[1]

$$\sin\alpha + \sin\beta = 2\sin\Big(\frac{\alpha+\beta}{2}\Big)\cos\Big(\frac{\alpha-\beta}{2}\Big) \qquad \text{(valid for each pair of angles } \alpha, \beta\text{)}$$

and show that the sum (the result of the interference) of two sinusoidal waves satisfies the following equality:

$$\sin(\omega_1 t) + \sin(\omega_2 t) \;=\; \underbrace{2\cos\Big(\frac{\omega_1-\omega_2}{2}\,t\Big)}_{\text{slowly-varying beating amplitude}} \cdot \underbrace{\sin\Big(\frac{\omega_1+\omega_2}{2}\,t\Big)}_{\frac{\omega_1+\omega_2}{2}\approx\omega_1\approx\omega_2} \tag{6.10}$$

When the two frequencies ω_1 and ω_2 are approximately equal, this expression shows the resulting wave (see the blue curve in Figure 6.8) has

(i) amplitude that slowly varies with the *beat frequency* $\frac{1}{2}|\omega_1-\omega_2| \ll (\omega_1+\omega_2)/2$, and

(ii) frequency equal to the average of the similar frequencies $(\omega_1+\omega_2)/2 \approx \omega_1 \approx \omega_2$.

[1] More sum-to-product formulas are reviewed in Appendix 6.5

6.2.2 The response of an harmonic oscillator to a sinusoidal input

At zero damping $\zeta = 0$ and *natural frequency* $\omega_{\mathsf{n}} > 0$, the transfer function (in canonical form) is

$$G_{\mathsf{undamped}}(s) = \frac{Y(s)}{U(s)} = \frac{\omega_{\mathsf{n}}^2}{s^2 + \omega_{\mathsf{n}}^2} \tag{6.11}$$

and its two poles are purely imaginary so that the system is *marginally stable* (but not stable). Therefore, even when we study the forced response from zero initial conditions, there will be non-vanishing terms due to the marginally stable system dynamics.

We now take $u(t) = \sin(\omega_{\mathsf{u}} t)$ at some *input frequency* ω_{u}, and compute the response in the Laplace domain:

$$Y(s) = G_{\mathsf{undamped}}(s) \cdot \mathcal{L}[\sin(\omega_{\mathsf{u}} t)] = \frac{\omega_{\mathsf{n}}^2}{s^2 + \omega_{\mathsf{n}}^2} \cdot \frac{\omega_{\mathsf{u}}}{s^2 + \omega_{\mathsf{u}}^2} \tag{6.12}$$

Using the inverse Laplace transform (see Exercise E6.5), one can verify that, for $\omega \neq \omega_{\mathsf{n}}$, the forced response is the weighted sum of two sinusoidal signals:

$$y_{\mathsf{forced}}(t) = \mathcal{L}^{-1}\left[\frac{\omega_{\mathsf{u}}\omega_{\mathsf{n}}^2}{(\omega_{\mathsf{u}}^2 + s^2)(\omega_{\mathsf{n}}^2 + s^2)}\right] = \frac{\omega_{\mathsf{n}}}{\omega_{\mathsf{u}}^2 - \omega_{\mathsf{n}}^2}\big(\omega_{\mathsf{u}} \sin(\omega_{\mathsf{n}} t) - \omega_{\mathsf{n}} \sin(\omega_{\mathsf{u}} t)\big) \tag{6.13}$$

In class assignment

Why are there two sinusoids in $y_{\mathsf{forced}}(t)$?

6.2.3 Approximating the solution at approximately-equal frequencies

Next, it is important to study the case when input and natural frequencies are approximately equal:

$$\omega_u \approx \omega_n \qquad \Longrightarrow \qquad \omega_u + \omega_n \approx 2\omega_n \quad \text{and} \quad |\omega_u - \omega_n| \ll \omega_n.$$

Using the trigonometric analysis in Appendix 6.5, the forced response $y_{forced}(t)$ in equation (6.13) can be approximated as:

$$y_{forced}(t) = \frac{\omega_n}{\omega_u^2 - \omega_n^2}\big(\omega_u \sin(\omega_n t) - \omega_n \sin(\omega_u t)\big) \quad \approx \quad \underbrace{\frac{\omega_n}{\omega_u - \omega_n} \cdot \sin\Big(\frac{\omega_n - \omega_u}{2} t\Big)}_{\text{large slowly-varying beating amplitude}} \quad \cdot \quad \cos\Big(\frac{\omega_n + \omega_u}{2} t\Big) \tag{6.14}$$

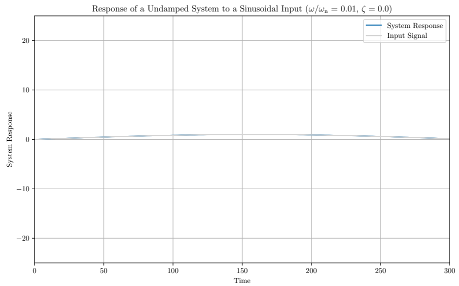
where

- $\dfrac{\omega_n}{\omega_u - \omega_n}$ is a *large amplitude* proportional to $1/|\omega_u - \omega_n|$,
- $\sin\Big(\dfrac{\omega_n - \omega_u}{2} t\Big)$ is a *slow oscillatory enclosing envelope* with *beat frequency* $|\omega_u - \omega_n|/2 \ll \omega_n$,
- $\cos\Big(\dfrac{\omega_n + \omega_u}{2} t\Big)$ is a cosine wave at high frequency $(\omega_u + \omega_n)/2 \approx \omega_n$.

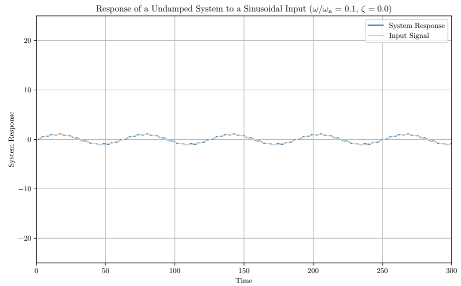
The response alternates between constructive and destructive interference, causing the amplitude of the oscillation to slowly rise and fall periodically. In other words,

> When input and natural frequency are approximately equal $\omega_u \approx \omega_n$, the response is an oscillation at the natural frequency ω_n with a large amplitude that rises and falls periodically with a slow beat frequency. This is called the *beating phenomenon*.

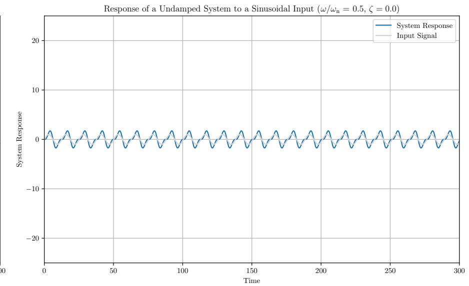
Response of an undamped system to a sinusoidal forcing ($\zeta = 0$)



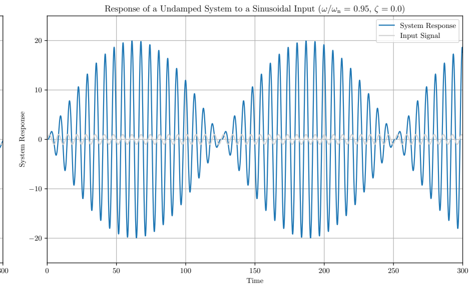
(a) $\omega_u/\omega_n = 0.01$



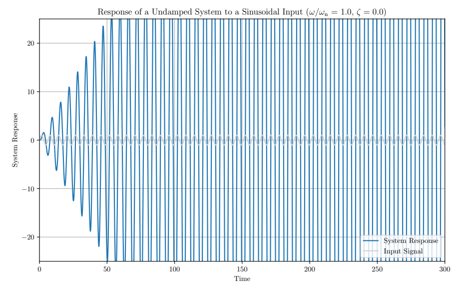
(b) $\omega_u/\omega_n = 0.1$



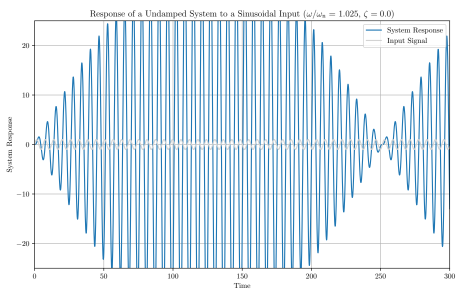
(c) $\omega_u/\omega_n = 0.5$



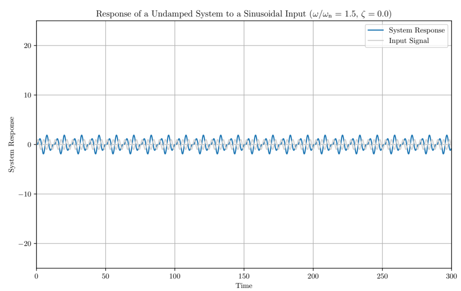
(d) $\omega_u/\omega_n = 0.95$ – *beats*



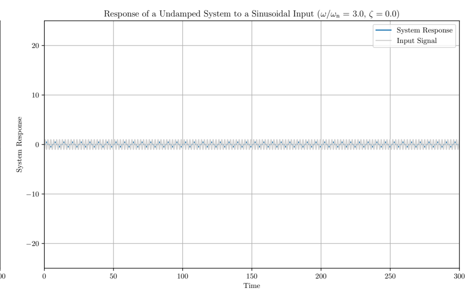
(e) $\omega_u/\omega_n = 1.0$ – *resonance*



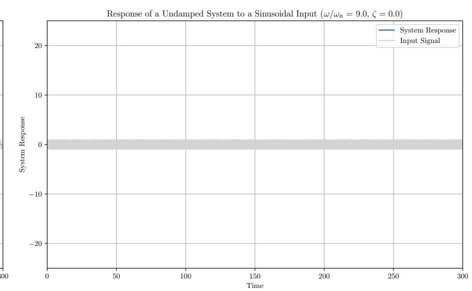
(f) $\omega_u/\omega_n = 1.025$ – *beats*



(g) $\omega_u/\omega_n = 1.5$



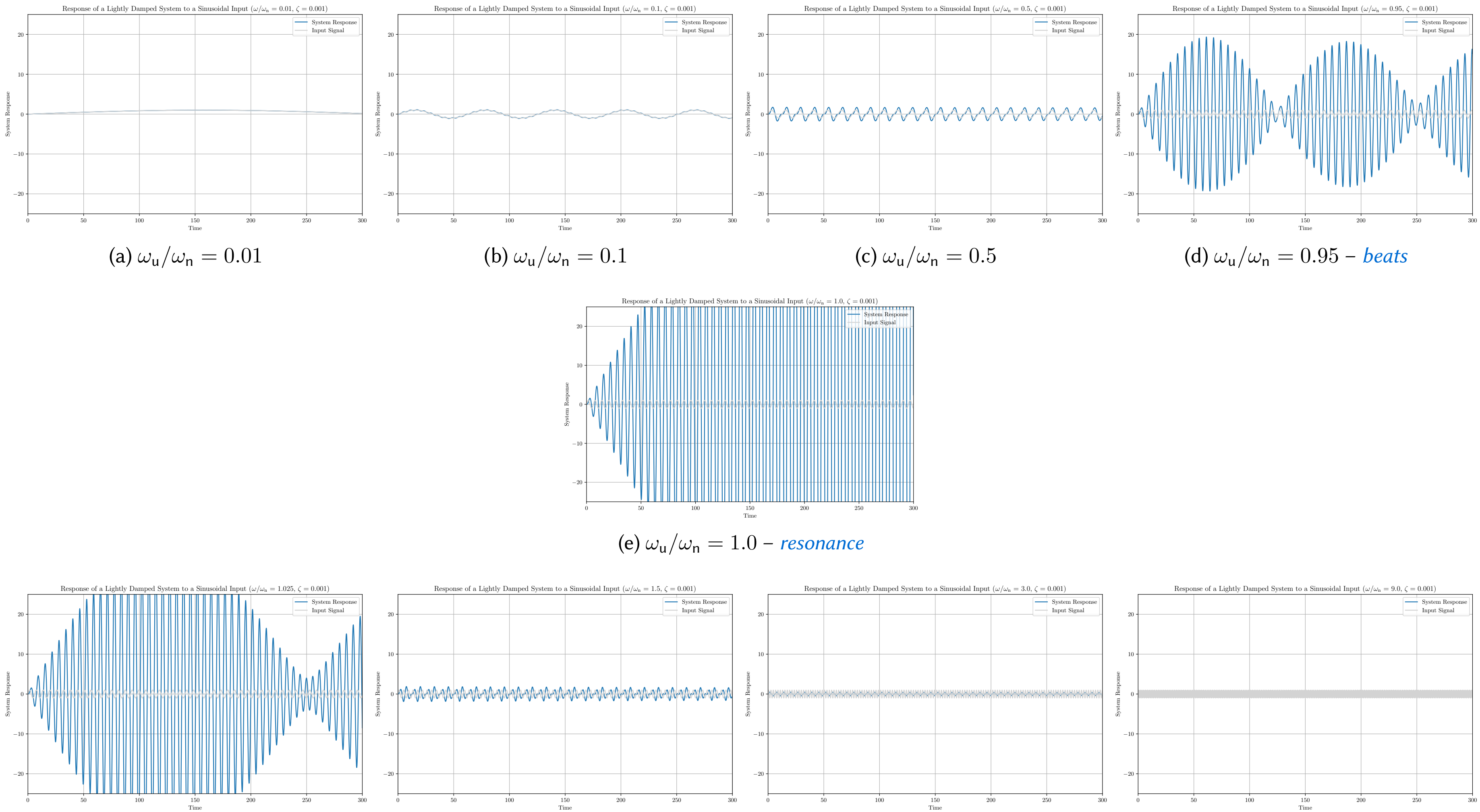
(h) $\omega_u/\omega_n = 3.0$



(i) $\omega_u/\omega_n = 9.0$

Figure 6.9: Forced response of an *undamped second-order system* ($\zeta = 0$ and $\omega_n = 1$) to a unit-magnitude sinusoidal input, plotted for different values of the ratio ω_u/ω_n. Python code available at `frequencyresponse-undamped.py`
For $\omega_u \approx \omega_n$, the forced response consists of an oscillation at nearly the driving frequency ω_u, whose amplitude rises and falls periodically with a *beat frequency* $(1/2)|\omega_u - \omega_n|$.

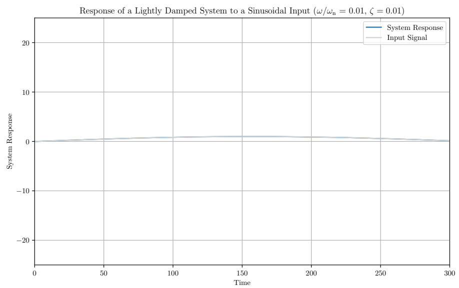
Response of a lightly damped system to a sinusoidal forcing ($\zeta = 0.001$)



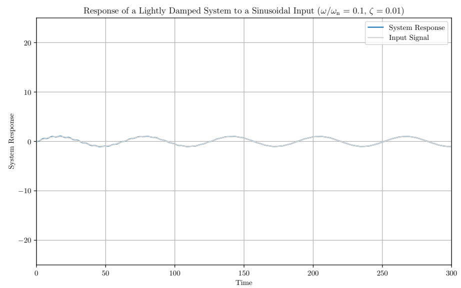
(a) $\omega_u/\omega_n = 0.01$
(b) $\omega_u/\omega_n = 0.1$
(c) $\omega_u/\omega_n = 0.5$
(d) $\omega_u/\omega_n = 0.95$ – *beats*
(e) $\omega_u/\omega_n = 1.0$ – *resonance*
(f) $\omega_u/\omega_n = 1.025$ – *beats*
(g) $\omega_u/\omega_n = 1.5$
(h) $\omega_u/\omega_n = 3.0$
(i) $\omega_u/\omega_n = 9.0$

Figure 6.10: Forced response of an *lightly-damped second-order system* ($\zeta = 0.001$ and $\omega_n = 1$) to a unit-magnitude sinusoidal input, plotted for different values of the ratio ω_u/ω_n.
The lesson is that: for very small damping and short duration of time, the solution is very similar to that for the undamped solution.

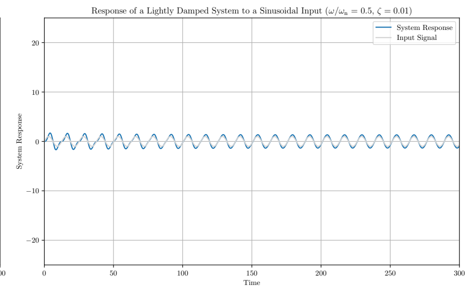
Response of a damped system to a sinusoidal forcing ($\zeta = 0.01$)



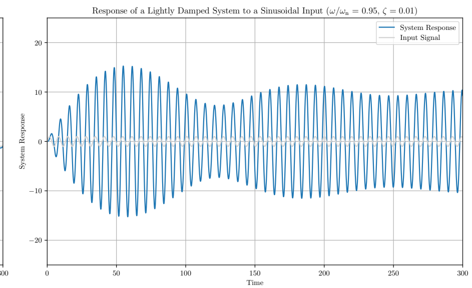
(a) $\omega_u/\omega_n = 0.01$



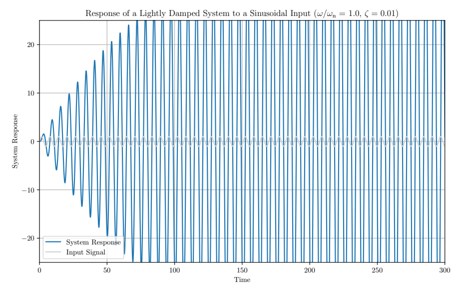
(b) $\omega_u/\omega_n = 0.1$



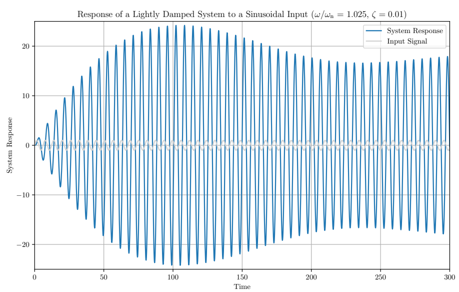
(c) $\omega_u/\omega_n = 0.5$



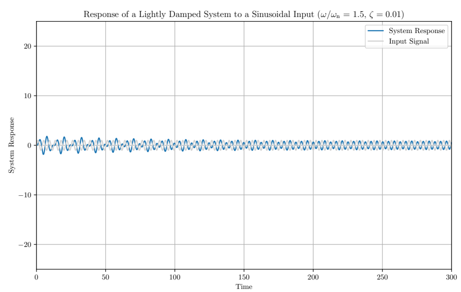
(d) $\omega_u/\omega_n = 0.95$ – *beats*



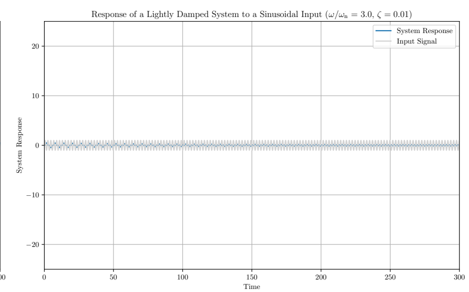
(e) $\omega_u/\omega_n = 1.0$ – *resonance*



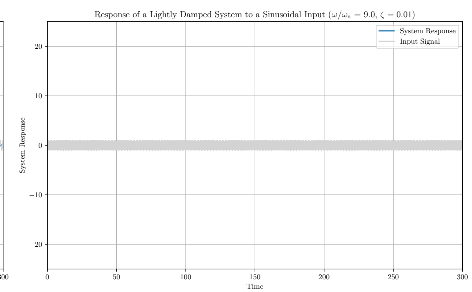
(f) $\omega_u/\omega_n = 1.025$ – *beats*



(g) $\omega_u/\omega_n = 1.5$



(h) $\omega_u/\omega_n = 3.0$



(i) $\omega_u/\omega_n = 9.0$

Figure 6.11: Forced response of an *lightly-damped second-order system* ($\zeta = 0.01$ and $\omega_n = 1$) to a unit-magnitude sinusoidal input, plotted for different values of the ratio ω_u/ω_n.
The lesson is that: as ζ increases, the beating phenomenon disappears.

Comments on the beats phenomenon According to wikipedia:beats: "In acoustics, a *beat* is an interference pattern between two sounds of slightly different frequencies, perceived as a periodic variation in volume." Beats are often used to tune musical instruments to the correct pitch by comparing the instrument's frequency with a reference frequency.

In *mechanical systems*, beats can be observed when two oscillating components (like springs or pendulums) with similar but not identical natural frequencies are coupled. Beats can be an indicator of potential resonance problems. For instance, in a machine with rotating parts, the presence of a beat frequency might indicate a close match in the rotational frequencies, potentially leading to resonance.

6.3 Appendix: Proper transfer functions, causality, and low-pass filters

A rational transfer function $G(s)$ is classified to be:

- *strictly proper* if the degree of the numerator polynomial is strictly less than the degree of the denominator polynomial,
- *proper* if the degree of the numerator polynomial is less than or equal to the degree of the denominator polynomial, and
- *improper* if the degree of the numerator polynomial is greater than the degree of the denominator polynomial.

For example, letting $G(s) = \mathrm{Num}(s)/\mathrm{Den}(s)$,

Strictly proper: $$G_1(s) = \frac{1}{s+1}, \qquad \deg c\,\mathrm{Num}_1(s) = 0 \;<\; \deg \mathrm{Den}_1(s) = 1,$$

Proper (not strictly): $$G_2(s) = \frac{s+1}{s+2}, \qquad \deg \mathrm{Num}_2(s) = 1 \;=\; \deg \mathrm{Den}_2(s) = 1,$$

Improper: $$G_3(s) = \frac{s^2+1}{s+3}, \qquad \deg \mathrm{Num}_3(s) = 2 \;>\; \deg \mathrm{Den}_3(s) = 1.$$

In physical terms, strict properness ensures that the output does not react instantaneously to infinitely fast variations in the input, because the gain at high frequency tends to zero.

Causality. For continuous-time linear systems, *causality* means that the output at time t depends only on the values of the input for times $\tau \leq t$. Equivalently, the impulse response $g(t)$ must satisfy

$$g(t) = 0 \quad \text{for all } t < 0.$$

If $G(s)$ is *improper*, its inverse Laplace transform contains derivatives of the Dirac delta $\delta(t)$ at $t = 0$. Such terms represent instantaneous dependence on time derivatives of the input and this behavior is incompatible with any finite-speed physical system. Thus, causal finite-dimensional linear systems must have *proper* transfer functions.

Remark 6.1 (Frequency-domain insight). *An improper transfer function grows without bound as $|\omega| \to \infty$, implying that it would amplify arbitrarily fast input variations without limit. Such infinite-bandwidth behavior is not physically realizable because real systems have finite propagation speeds and finite energy storage.* •

Examples of proper transfer functions.

- *Spring–mass–damper:* $G(s) = \frac{1}{ms^2+bs+k}$ is strictly proper ($\deg \mathrm{Num}(s) < \deg \mathrm{Den}(s)$). Its impulse response is a decaying oscillation for $t \geq 0$ and zero for $t < 0$, meaning the output reacts only after the impulse is applied.
- *Static gain:* $G(s) = 10$ is proper ($\deg \mathrm{Num}(s) = \deg \mathrm{Den}(s) = 0$) and causal. Its impulse response $g(t) = 10\,\delta(t)$ represents instantaneous proportional scaling of the input.

An example of an improper transfer function: the derivative action. The pure derivative operator

$$G(s) = s$$

is *improper* and cannot be implemented as a causal finite-dimensional linear system: this transfer function has unbounded high-frequency gain and would require instantaneous response.

Derivative action is important in control engineering, particularly in PID controllers (which we will review in later chapters). In practice, the derivative action is implemented with high-frequency roll-off:

$$G_D(s) = \frac{s}{\tau_{\text{filter}} s + 1}, \qquad \tau_{\text{filter}} > 0,$$

which is *proper*. The low-pass transfer function $(\tau_{\text{filter}} s + 1)^{-1}$ limits high-frequency gain, attenuates noise, and makes the derivative action physically realizable. Such a transfer function is called *low-pass filter*. This is the standard form used in PID controllers.

6.4 Appendix: Steady-state response of stable systems to sinusoidal inputs

As for the step response in Section 5.4, we consider only *stable* dynamical systems, that is, all poles of the transfer function are in the (strict) left half plane. The formula (6.1) can be then derived in three steps.

Step 1: Computing the Laplace transform. Assuming $G(s)$ has distinct stable poles $-p_1, \dots, -p_n$, since $s^2+\omega^2 = (s-\mathrm{i}\omega)(s+\mathrm{i}\omega)$

$$Y(s) \overset{u(t)=\sin(\omega t)}{=} G(s)\frac{\omega}{s^2+\omega^2} \overset{\text{partial fraction expansion}}{=} \frac{r_-}{s-\mathrm{i}\omega} + \frac{r_+}{s+\mathrm{i}\omega} + \sum_{i=1}^{n} \frac{r_i}{s+p_i} \tag{6.15}$$

where the residues r_- and r_+ may be complex. Since the poles $-p_i$ are stable, each term $\frac{r_i}{s+p_i}$ gives rise to an exponentially decaying term. Therefore, the steady state response is

$$y_{\text{steady-state}}(t) = r_- \,\mathrm{e}^{+\mathrm{i}\omega t} + r_+ \,\mathrm{e}^{-\mathrm{i}\omega t} \tag{6.16}$$

Step 2: Using the single-pole residue formula on both complex poles. From the residue formula (4.37), we compute

$$\begin{aligned} r_- &= (s-\mathrm{i}\omega)G(s)\frac{\omega}{s^2+\omega^2}\Big|_{s=+\mathrm{i}\omega} = G(s)\frac{\omega}{s+\omega}\Big|_{s=\mathrm{i}\omega} = \frac{1}{2\mathrm{i}}G(\mathrm{i}\omega) &&= \frac{1}{2\mathrm{i}}|G(\mathrm{i}\omega)|\,\mathrm{e}^{\mathrm{i}\phi} \qquad (\phi = \arg\big(G(\mathrm{i}\omega)\big)) \\ r_+ &= (s+\mathrm{i}\omega)G(s)\frac{\omega}{s^2+\omega^2}\Big|_{s=-\mathrm{i}\omega} = G(s)\frac{\omega}{s-\omega}\Big|_{s=-\mathrm{i}\omega} = -\frac{1}{2\mathrm{i}}G(-\mathrm{i}\omega) &&\overset{(*)}{=} -\frac{1}{2\mathrm{i}}|G(\mathrm{i}\omega)|\,\mathrm{e}^{-\mathrm{i}\phi}, \end{aligned} \tag{6.17}$$

where the equality $(*)$ follows from the property $\overline{G(s)} = G(\bar{s})$ for any complex number s and rational function G.

Step 3: Using the inverse Euler formula. Plugging the expressions for r_- and r_+ into formula (6.16), we obtain

$$y_{\text{steady-state}}(t) = |G(\mathrm{i}\omega)|\frac{\mathrm{e}^{\mathrm{i}(\omega t+\phi)} - \mathrm{e}^{-\mathrm{i}(\omega t+\phi)}}{2\mathrm{i}} = |G(\mathrm{i}\omega)|\sin(\omega t+\phi), \tag{6.18}$$

where we used the inverse Euler formula $\sin(\theta) = \frac{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}}{2\mathrm{i}}$ from Figure 4.7.

6.5 Appendix: Trigonometric explanation of the beating phenomenon

The sum-to-product formulas in trigonometry The *sum-to-product formulas* are trigonometric identities that transform the sum or difference of trigonometric functions into a product of trigonometric functions. The four main formulas are:

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{6.19}$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right), \tag{6.20}$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right), \tag{6.21}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right). \tag{6.22}$$

But the general concept holds even more generally.

Approximate frequency response of an undamped system In this appendix, we study the expression $\frac{1}{\omega_u^2-\omega_n^2}\big(\omega_u \sin(\omega_n t) - \omega_n \sin(\omega_u t)\big)$ and obtain an approximate equality when $\omega_u \approx \omega_n$. Summing and subtracting the sum-to-product formulas (6.19) and (6.20), we obtain:

$$\sin\alpha = \cos\Big(\frac{\alpha+\beta}{2}\Big)\sin\Big(\frac{\alpha-\beta}{2}\Big) + \sin\Big(\frac{\alpha+\beta}{2}\Big)\cos\Big(\frac{\alpha-\beta}{2}\Big) \tag{6.23}$$

$$\sin\beta = \sin\Big(\frac{\alpha+\beta}{2}\Big)\cos\Big(\frac{\alpha-\beta}{2}\Big) - \cos\Big(\frac{\alpha+\beta}{2}\Big)\sin\Big(\frac{\alpha-\beta}{2}\Big) \tag{6.24}$$

We let $\alpha = \omega_n t$, $\beta = \omega_u t$ and scale the two equations with appropriate coefficients to obtain:

$$\omega_u \sin(\omega_n t) = \omega_u \cos\big(\frac{\omega_n+\omega_u}{2}t\big)\sin\big(\frac{\omega_n-\omega_u}{2}t\big) + \omega_u \sin\big(\frac{\omega_n+\omega_u}{2}t\big)\cos\big(\frac{\omega_n-\omega_u}{2}t\big), \tag{6.25}$$

$$\omega_n \sin(\omega_u t) = \omega_n \sin\big(\frac{\omega_n+\omega_u}{2}t\big)\cos\big(\frac{\omega_n-\omega_u}{2}t\big) - \omega_n \cos\big(\frac{\omega_n+\omega_u}{2}t\big)\sin\big(\frac{\omega_n-\omega_u}{2}t\big). \tag{6.26}$$

Next, we subtract the second equation from the first

$$\omega_u \sin(\omega_n t) - \omega_n \sin(\omega_u t) = (\omega_u+\omega_n)\cos\Big(\frac{\omega_n+\omega_u}{2}t\Big)\sin\Big(\frac{\omega_n-\omega_u}{2}t\Big) + (\omega_u-\omega_n)\sin\Big(\frac{\omega_n+\omega_u}{2}t\Big)\cos\Big(\frac{\omega_n-\omega_u}{2}t\Big) \tag{6.27}$$

and we scale the result to obtain

$$\frac{1}{\omega_u^2-\omega_n^2}\Big(\omega_u \sin(\omega_n t) - \omega_n \sin(\omega_u t)\Big) = \frac{1}{\omega_u-\omega_n}\cos\Big(\frac{\omega_n+\omega_u}{2}t\Big)\sin\Big(\frac{\omega_n-\omega_u}{2}t\Big) + \frac{1}{\omega_u+\omega_n}\sin\Big(\frac{\omega_n+\omega_u}{2}t\Big)\cos\Big(\frac{\omega_n-\omega_u}{2}t\Big). \tag{6.28}$$

Finally, when $\omega_u \approx \omega_n$, we note $\left|\frac{1}{\omega_u-\omega_n}\right| \gg \frac{1}{\omega_u+\omega_n}$ so that

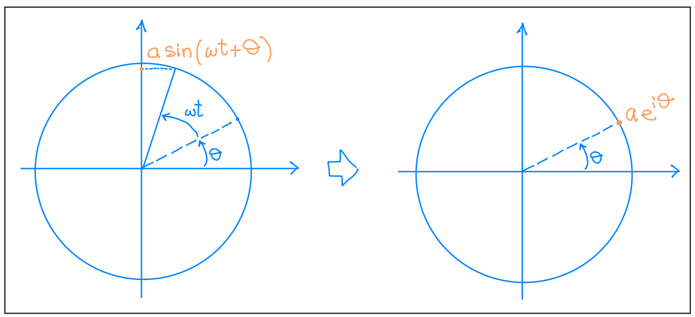
$$\frac{1}{\omega_u^2-\omega_n^2}\Big(\omega_u \sin(\omega_n t) - \omega_n \sin(\omega_u t)\Big) \approx \frac{1}{\omega_u-\omega_n}\sin\Big(\frac{\omega_n-\omega_u}{2}t\Big)\cos\Big(\frac{\omega_n+\omega_u}{2}t\Big) \tag{6.29}$$

Physical interpretation of beating via phasors The *phasor representation* of a sinusoidal signal is a complex number that encodes the amplitude and phase of the sinusoid. Specifically,

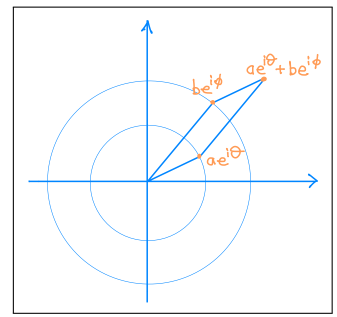
$$a\sin(\omega t+\theta) \qquad \mapsto \qquad \underbrace{a\,\mathrm{e}^{\mathrm{i}\theta}}_{\text{phasor representation}} \tag{6.30}$$

Phasors simplify the analysis of sinusoidal signals by expressing them as a constant magnitude and phase angle, ignoring the explicit time dependence. A key property is the *sum property* for sinusoidal signals with *equal frequency*:

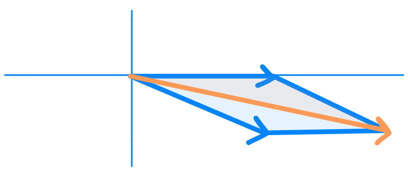
$$a\sin(\omega t+\theta)+b\sin(\omega t+\phi) \qquad \mapsto \qquad \underbrace{a\,\mathrm{e}^{\mathrm{i}\theta}+b\,\mathrm{e}^{\mathrm{i}\phi}}_{\text{sum of the phasors in complex plane}} \tag{6.31}$$



(a) definition of phasor



(b) sum of two phasors (with equal frequency)


(c) constructive

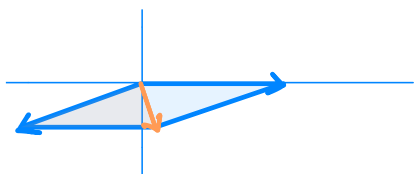

(d) destructive

Figure 6.12: Sum of phasors: constructive and destructive interference

6.6 Historical notes and further resources

A classic reference on vibrations is the famous textbook by Den Hartog (1956).

These following videos are recommended:

- resonance in tuning forks,
- beating in tuning forks,
- explanations and useful animations on understanding vibration and resonance (20 min), including a discussion of vibrations in multiple degree of freedom mechanical systems,
- more information about interference beats, for music and guitar lovers.

For further references, see

- the role of aerodynamic flutter in the 1940 Tacoma bridge collapse (less than 1 minute); read about it at wikipedia:Tacoma Narrows Bridge, and
- synchronizing oscillators and metronomes.

6.7 Exercises

Section 6.1: The frequency response and the resonance phenomenon

E6.1 **Low-pass and high-pass filters.** Consider the first-order system with transfer function

$$G(s) = \frac{\tau s}{\tau s + 1}, \tag{6.32}$$

where the time constant $\tau > 0$.

(i) Derive expressions for the magnitude frequency response $|G(\mathrm{i}\omega)|$ and the phase $\arg\big(G(\mathrm{i}\omega)\big)$.

(ii) Using your result from part (i), write the steady-state response $y_{\mathrm{ss}}(t)$ of the system to a unit-magnitude sinusoidal input $u(t) = \sin(\omega t)$.

(iii) Using your answer from part (ii), determine:

(a) the approximate steady-state response for a low-frequency input $\omega \ll 1/\tau$,

(b) the approximate steady-state response for a high-frequency input $\omega \gg 1/\tau$.

(iv) A *low-pass filter* passes low-frequency sinusoids and attenuates high-frequency sinusoids. A *high-pass filter* passes high-frequency sinusoids and attenuates low-frequency sinusoids. Comparing your results here with those from Section 6.1.2, identify which transfer function

$$G(s) = \frac{1}{\tau s + 1} \quad \text{and} \quad G(s) = \frac{\tau s}{\tau s + 1}$$

is a low-pass filter and which is a high-pass filter.

Answer:

(i) Substituting $s = \mathrm{i}\omega$ gives

$$G(\mathrm{i}\omega) = \frac{\mathrm{i}\tau\omega}{1 + \mathrm{i}\tau\omega}. \tag{6.33}$$

Hence,

$$|G(\mathrm{i}\omega)| = \frac{\tau\omega}{\sqrt{1+\tau^2\omega^2}}, \quad \arg\big(G(\mathrm{i}\omega)\big) = \frac{\pi}{2} - \arctan(\tau\omega). \tag{6.34}$$

(ii) By the steady-state sinusoidal response formula,

$$y_{\mathrm{ss}}(t) = \frac{\tau\omega}{\sqrt{1+\tau^2\omega^2}} \sin\Big(\omega t + \frac{\pi}{2} - \arctan(\tau\omega)\Big). \tag{6.35}$$

(iii) For $\omega \ll 1/\tau$,

$$y_{\mathrm{ss}}(t) \approx 0. \tag{6.36}$$

For $\omega \gg 1/\tau$,

$$y_{\mathrm{ss}}(t) \approx \sin(\omega t). \tag{6.37}$$

(iv) The transfer function $\dfrac{1}{\tau s + 1}$ is a low-pass filter. The transfer function $\dfrac{\tau s}{\tau s + 1}$ is a high-pass filter. We show the magnitude Bode plots for these two filters in Figure 6.13.

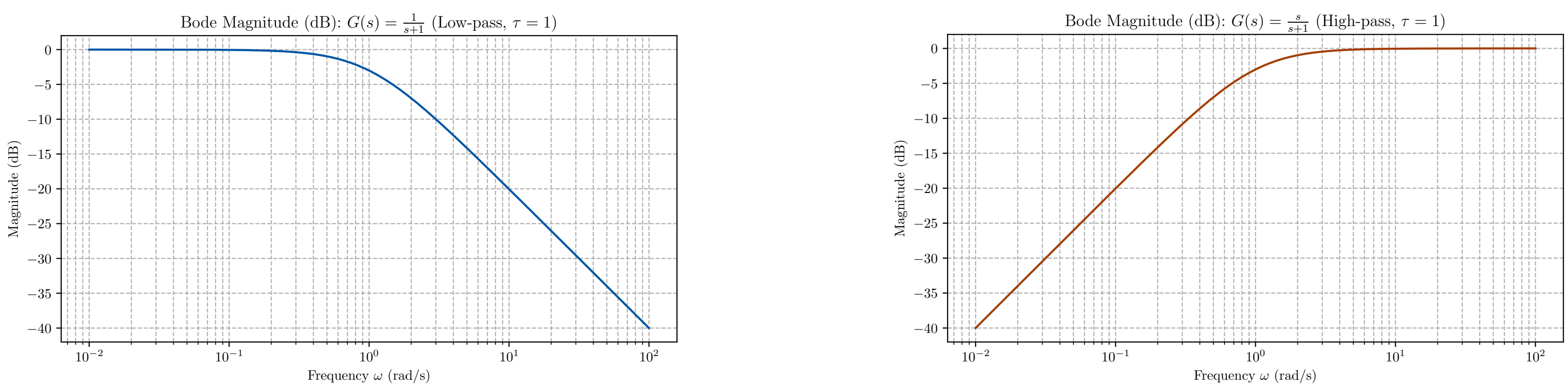
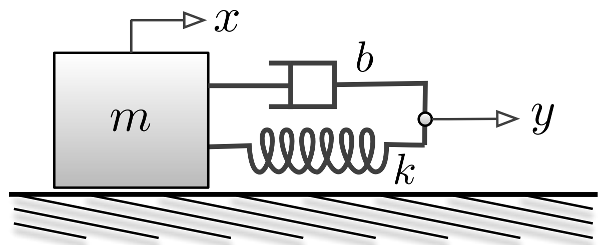


Figure 6.13: Bode magnitude plots for first-order filters with $\tau = 1$: (a) low-pass $G(s) = \frac{1}{s+1}$ and (b) high-pass $G(s) = \frac{s}{s+1}$. The low-pass filter preserves low frequencies and attenuates high frequencies at -20 dB/decade, while the high-pass filter behaves oppositely.

▼

E6.2 **Mass-spring-damper system connected to a moving point.** We study a second-order transfer function with a first-order polynomial in the numerator. The zero of the transfer function is the root of this polynomial. Given positive parameters m, b, k, consider a mass-spring-damper system with position $x(t)$ connected to a moving point $y(t)$, sliding without friction over a horizontal plane:



(i) Derive the equation of motion and its Laplace transform under zero initial conditions. Write the transfer function $G(s)$ from $Y(s)$ to $X(s)$. Identify the system's undamped natural frequency ω_{n} and the location of the zero of $G(s)$.

(ii) Compute the magnitude frequency response $|G(\mathrm{i}\omega)|$ and plot it using Python or another software tool. Use $m = 1$, $b = 0.5$, $k = 1$.

(iii) Determine:

(a) the approximate frequency ω where the resonant peak occurs,
(b) the approximate value of $|G(\mathrm{i}\omega)|$ at this peak,
(c) the frequency values where $|G(\mathrm{i}\omega)| = 1$.

(iv) Evaluate $\lim_{\omega\to\infty} |G(\mathrm{i}\omega)|$ and interpret this physically for the system.

Answer:

(i) The equation of motion is

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = b\dot{y}(t) + ky(t).$$

Taking the Laplace transform with zero initial conditions:

$$(ms^2 + bs + k)X(s) = (bs + k)Y(s).$$

The transfer function is

$$\boxed{G(s) = \frac{X(s)}{Y(s)} = \frac{bs + k}{ms^2 + bs + k}}.$$

The undamped natural frequency is

$$\omega_{\mathsf{n}} = \sqrt{\frac{k}{m}},$$

and the zero of $G(s)$ is at

$$s = -\frac{k}{b}.$$

(ii) Substituting $s = \mathrm{i}\omega$ gives

$$G(\mathrm{i}\omega) = \frac{\mathrm{i}b\omega + k}{-m\omega^2 + \mathrm{i}b\omega + k}.$$

The magnitude frequency response is

$$\boxed{|G(\mathrm{i}\omega)| = \sqrt{\frac{b^2\omega^2 + k^2}{m^2\omega^4 + (b^2 - 2km)\omega^2 + k^2}}}. \tag{6.38}$$

(iii) For $m = 1, b = 0.5, k = 1$:

(a) Resonant peak at $\boxed{\omega \approx 0.948}$,
(b) Peak magnitude $\boxed{\approx 2.283}$,
(c) $|G(\mathrm{i}\omega)| = 1$ at $\boxed{\omega = 0}$ and $\boxed{\omega \approx 1.414}$.

(iv) As $\omega \to \infty$, $\boxed{|G(\mathrm{i}\omega)| \to 0}$. Physically, a rapidly oscillating input at the moving point produces negligible motion in the mass because the system's inertia prevents it from following the high-frequency excitation.

▼

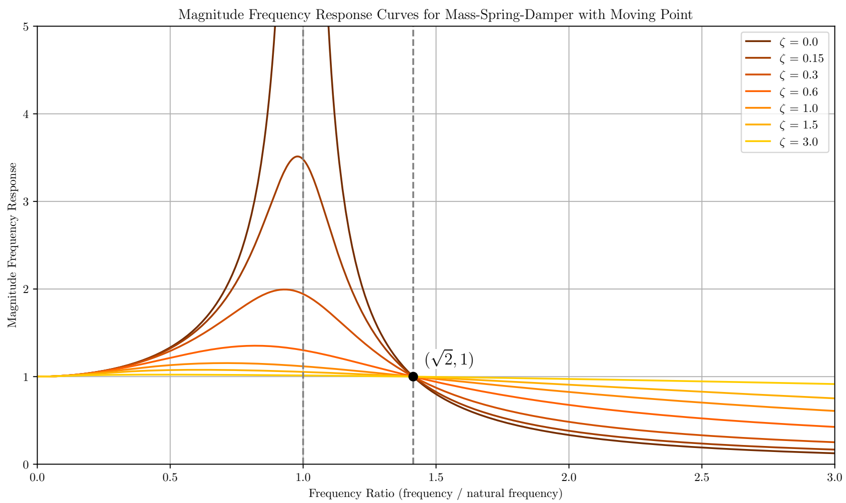
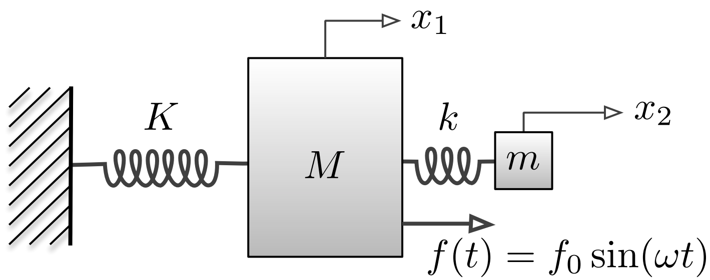


Figure 6.14: Magnitude frequency response of the mass-spring-damper system in (6.38). The horizontal axis shows ω/ω_n, where ω_n is the undamped natural frequency.

E6.3 **The undamped vibration absorber.** During operation, mechanical systems are often subjected to vibrations (e.g., an internal combustion engine at idle). These vibrations can be detrimental to performance and comfort, and it is often desirable to attenuate or eliminate them. A classical passive solution is the *dynamic vibration absorber* designed by Frahm (1911). Consider a main mass M connected to a wall via a spring of stiffness K and subject to an oscillatory force $f_0 \sin(\omega t)$. To attenuate the vibrations, a secondary smaller mass m is attached to the main mass via a spring of stiffness k (with $K \neq k$ in general):



(i) Derive the equations of motion for the system and compute the transfer function $G(s) = \dfrac{X_1(s)}{F(s)}$ in two cases:

(a) without absorber ($m = 0$), and
(b) with absorber ($m > 0$).

Identify the poles and zeros of both transfer functions and comment on the location of the zero when $\omega = \sqrt{k/m}$.

(ii) Suppose that the steady-state response is

$$x_1(t) = a_1 \sin(\omega t), \qquad x_2(t) = a_2 \sin(\omega t), \tag{6.39}$$

with constant amplitudes a_1, a_2 and forcing frequency ω. Substitute into the equations from part (i) to obtain an algebraic system for (a_1, a_2) in terms of M, K, m, k, ω and f_0.

(iii) Show that when $\omega = \sqrt{k/m}$, the amplitude $a_1 = 0$, i.e., the main mass does not vibrate. Interpret physically: the secondary oscillator (m, k) exerts a spring force that cancels the external force $f_0 \sin(\omega t)$ exactly.

(iv) Determine initial conditions $x_1(0), \dot{x}_1(0), x_2(0), \dot{x}_2(0)$ such that $x_1(t) \equiv 0$ and $x_2(t) = a_2 \sin(\omega t)$ solves the system.

(v) Using your software of choice, plot the Bode magnitude (in dB) of the transfer functions from the external force $f(t)$ to the displacement $x_1(t)$ for both cases in part (i)b. For the absorber case, take $\omega = \sqrt{k/m}$. Comment on the vibration attenuation near the tuning frequency and the asymptotic slope at high frequency.

Answer:

(i) The equations of motion are

$$M\ddot{x}_1 + Kx_1 + k(x_1 - x_2) = f_0 \sin(\omega t),$$
$$m\ddot{x}_2 + k(x_2 - x_1) = 0.$$

Taking the Laplace transform with zero initial conditions, the equations become

$$Ms^2X_1 + KX_1 + k(X_1 - X_2) = F(s),$$
$$ms^2X_2 + k(X_2 - X_1) = 0.$$

Case (a): without absorber ($m = 0$). The second equation vanishes and the first gives

$$(Ms^2 + K)X_1 = F(s),$$

hence

$$G_{\text{no absorber}}(s) = \frac{X_1(s)}{F(s)} = \frac{1}{Ms^2 + K}.$$

This is a single mass–spring system with two complex-conjugate poles at $\pm j\sqrt{K/M}$ and no zeros.
Case (b): with absorber ($m > 0$). Eliminating X_2 from the two equations yields

$$\Big(Ms^2 + K + k - \frac{k^2}{ms^2 + k}\Big)X_1 = F(s),$$

so that, after simple manipulations:

$$G_{\text{absorber}}(s) = \frac{s^2 + \frac{k}{m}}{\frac{Mm}{1}s^4 + (Mk + m(K + k))s^2 + Kk}.$$

There is a zero at $s = \pm j\sqrt{k/m}$, which corresponds to $\sqrt{k/m}$ in the frequency domain. Intuitively, this zero will cancel the response of x_1 when the input frequency $\omega = \sqrt{k/m}$.

FBTodo: define the notion and effect of a “transmission zero” : a certain mode gets killed – then cite it here. (Astrom had it + ask chatGPT + find it in Ogata. I forget: left-half place zero, imaginary axis zero, or right-half plane = nonminimum phase)

(ii) Calculating the second derivative of the given solutions (6.39) we find:

$$\ddot{x}_1 = -a_1\omega^2 \sin\omega t, \qquad \ddot{x}_2 = -a_2\omega^2 \sin\omega t.$$

Substituting these equations into the differential equations from part (i) we obtain

$$-a_1 M\omega^2 \sin\omega t + a_1(K+k)\sin\omega t - a_2 k \sin\omega t = P_0 \sin\omega t$$
$$-a_2 m\omega^2 \sin\omega t + k(a_2 - a_1)\sin\omega t = 0$$

We can simplify by eliminating the $\sin\omega t$ terms, finally yielding:

$$a_1(K + k - M\omega^2) - ka_2 = P_0 \tag{6.40}$$
$$-ka_1 + a_2(k - m\omega^2) = 0 \tag{6.41}$$

(iii) Substituting $\omega = \sqrt{k/m}$ into equation (6.41) we obtain

$$-ka_1 + a_2\big(k - m\tfrac{k}{m}\big) = -ka_1 + 0 = 0,$$

which implies $a_1 = 0$. Thus, the main mass M does not oscillate; the absorber cancels the external force at all times.

(iv) Since $a_1 = 0$, equation (6.40) gives $-ka_2 = P_0$, so $a_2 = -P_0/k$. The solution is $x_2(t) = -\frac{P_0}{k}\sin(\omega t)$ with $\omega = \sqrt{k/m}$. The initial conditions are

$$x_1(0) = 0, \quad \dot{x}_1(0) = 0, \quad x_2(0) = 0, \quad \dot{x}_2(0) = -\frac{\omega P_0}{k}. \tag{6.42}$$

We verify this result in a numerical simulation reported in Figure 6.15.

(v) The Bode magnitude of the transfer function from the external force $P(t)$ to the displacement $x_1(t)$ is shown in Figure 6.16, both without the absorber ($m = 0$) and with the absorber tuned to $\omega = \sqrt{k/m}$. The tuned vibration absorber introduces a zero in the transfer function at the excitation frequency, eliminating vibration of the main mass. At high frequencies, both configurations exhibit the expected $-40\,\mathrm{dB/decade}$ slope due to the two poles of the system.

▼

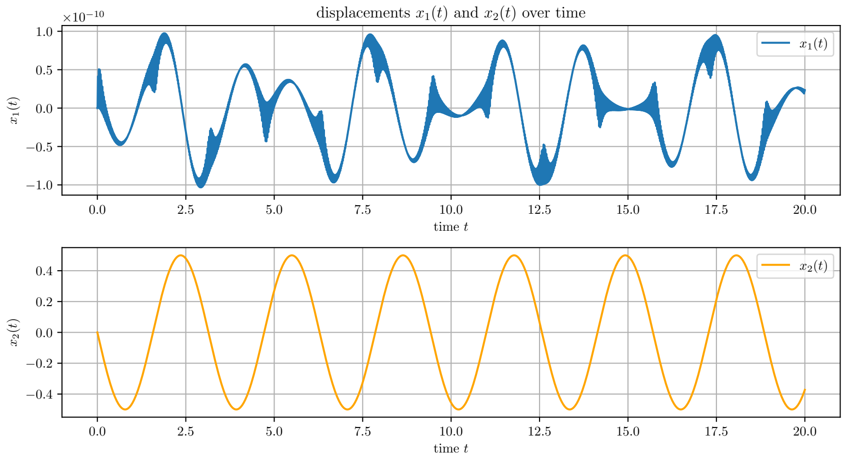


Figure 6.15: Simulation of Frahm's undamped vibration absorber for $\omega = \sqrt{k/m}$ and the initial conditions from equation (6.42) in part (iv). The numerical integrator is implemented with high accuracy parameters. Top panel: the displacement $x_1(t)$ remains of order e^{-10}, consistent with the analytical prediction $x_1(t) \equiv 0$.

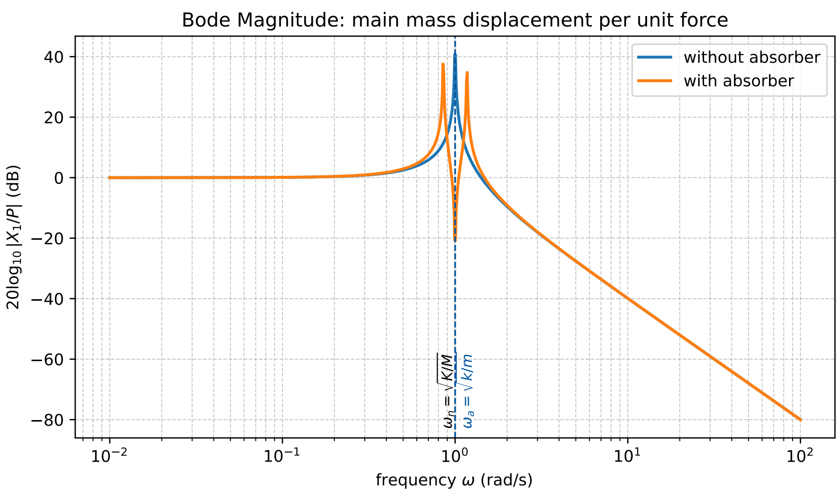


Figure 6.16: Bode magnitude plots for the displacement $x_1(t)$ without an absorber (solid red) and with a tuned absorber (solid blue) for $\omega = \sqrt{k/m}$. The tuned absorber introduces a zero in the transfer function at the excitation frequency, eliminating vibration of the main mass. At high frequencies, both configurations exhibit the expected $-40\,\mathrm{dB/decade}$ slope due to the relative degree two both systems.

E6.4 **Transfer function and frequency response of an RC circuit.** In the diagram below of an RC circuit, let $v_{\text{input}}(t)$ be the voltage at the input, $r > 0$ be a resistance in Ohms, $c > 0$ be a capacitance in Farads, and $v_{\text{output}}(t)$ be the voltage at the output.

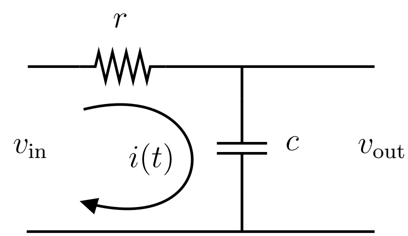


Figure 6.17: First-order RC circuit

(i) Find the governing equation of the circuit in terms of the variables v_{input} and v_{output}.

(ii) Compute the Laplace transform of the governing equation assuming $v_{\text{output}}(0) = 0$.

(iii) Obtain the input-to-output transfer function $G(s) = \frac{V_{\text{output}}(s)}{V_{\text{input}}(s)}$ for the system.

(iv) Show that the pole of the system is stable.

(v) Compute the magnitude frequency response of the transfer function.

(vi) In applications where signal quality is crucial (e.g., telecommunications and music), circuits such as the above RC circuit are often employed. These circuits serve as either high-pass filters (i.e., removing low-frequency noise) or low-pass filters (i.e., removing high-frequency noise). Is the above circuit a high-pass filter or a low-pass filter? Explain your reasoning using the magnitude-frequency response of the transfer function.

Answer:

(i) Applying Kirchoff's laws, the differential equation governing the system is given by

$$v_{\text{output}}(t) = v_{\text{input}}(t) - rc\frac{\mathrm{d}v_{\text{output}}}{\mathrm{d}t}$$

(ii) The Laplace transform is given by

$$\begin{aligned}\mathcal{L}[v_{\text{output}}] &= V_{\text{input}}(s) - rc(sV_{\text{output}}(s) - v_{\text{output}}(0)) \\ \implies \quad & V_{\text{output}}(s) = v_{\text{input}}(s) - srcV_{\text{output}}(s)\end{aligned}$$

(iii) The input-to-output transfer function is found by rearranging the equation found in the solution of part (ii). It is as follows:

$$\begin{aligned}V_{\text{output}}(s) &= v_{\text{input}}(s) - srcV_{\text{output}}(s) \\ \implies \quad & V_{\text{output}}(s)(src + 1) = v_{\text{input}}(s) \\ \implies \quad & G(s) = \frac{V_{\text{output}}(s)}{v_{\text{input}}(s)} = \frac{1}{src + 1}\end{aligned}$$

(iv) By inspecting the transfer function, we find a pole at $s = -1/rc$. Because $r > 0$, $c > 0$, then $rc > 0$ and this pole must be in the left half of the complex plane and is therefore stable.

(v) The magnitude-frequency response is found by

$$\begin{aligned}|G(s)| = |G(\mathrm{i}\omega)| &= \left|\frac{1}{\mathrm{i}\omega rc + 1}\right| \\ &= \frac{1}{\sqrt{1 + (\omega rc)^2}}\end{aligned}$$

(vi) By inspection of the magnitude-frequency response, as the frequency ω becomes larger (i.e., high-frequency) then $|G(\mathrm{i}\omega)| \to 0$. Therefore, this would eliminate high-frequency noise indicating that the circuit behaves as a low-pass filter.

▼

Section 6.2: Lightly damped systems and the beating phenomenon

E6.5 **Sinusoidal forcing of an undamped second-order system.** A second-order system with zero damping ratio $\zeta = 0$ and natural frequency $\omega_n > 0$ is given by

$$\ddot{y}(t) + \omega_n^2 y(t) = \omega_n^2 u(t)$$

(i) Compute the transfer function $G(s) = Y(s)/U(s)$.

(ii) Compute the poles of $G(s)$. Is the system stable, marginally stable, or unstable?

(iii) Assume the input is a unit-magnitude sinusoidal signal $u(t) = \sin(\omega_u t)$ and use the partial fraction expansion to compute the forced response $Y(s)$ from zero initial conditions.

Hint: A correct answer needs to have the correct expansion, with all potential terms (even those that, in the end, have zero coefficient). There should be 4 terms and 4 free coefficients.

(iv) Compute the inverse Laplace transform of $Y(s)$ to obtain $y(t)$.

Answer:

(i) The undamped second order system is $\ddot{y} + \omega_n^2 y = \omega_n^2 u(t)$, so that

$$s^2 Y(s) + \omega_n^2 Y(s) = \omega_n^2 U(s) \qquad \Longrightarrow \qquad G(s) = \frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

(ii) We compute

$$s^2 + \omega_n^2 = 0 \qquad \Longrightarrow s^2 = -\omega_n^2$$

so that

$$s = \pm\sqrt{-\omega_n^2} \qquad \Longrightarrow \qquad s = \pm i\omega_n$$

The system is marginally stable, because the poles are on the imaginary axis and not repeated.

(iii) We compute

$$Y(s) = G(s) \cdot \mathcal{L}[\sin(\omega_u t)] = \frac{\omega_u \omega_n^2}{(s^2 + \omega_n^2)(s^2 + \omega_u^2)} \tag{6.43}$$

Therefore, we setup the partial fraction expansion:

$$\frac{\omega_u \omega_n^2}{(s^2 + \omega_n^2)(s^2 + \omega_u^2)} = A_1 \frac{\omega_n}{(s^2 + \omega_n^2)} + A_2 \frac{s}{(s^2 + \omega_n^2)} \quad + \quad B_1 \frac{\omega_u}{(s^2 + \omega_u^2)} + B_2 \frac{s}{(s^2 + \omega_u^2)} \tag{6.44}$$

Next, we compute

$$\omega_u \omega_n^2 = A_1 \omega_n (s^2 + \omega_u^2) + A_2 s (s^2 + \omega_u^2) + B_1 \omega_u (s^2 + \omega_n^2) + B_2 s (s^2 + \omega_n^2) \tag{6.45}$$

$$= A_1 \omega_n s^2 + A_1 \omega_n \omega_u^2 + A_2 s^3 + A_2 \omega_u^2 s + B_1 \omega_u s^2 + B_1 \omega_u \omega_n^2 + B_2 s^3 + B_2 \omega_n^2 s \tag{6.46}$$

$$= (A_2 + B_2) s^3 + (A_1 \omega_n + B_1 \omega_u) s^2 + (A_2 \omega_u^2 + B_2 \omega_n^2) s + A_1 \omega_n \omega_u^2 + B_1 \omega_u \omega_n^2 \tag{6.47}$$

We now setup 4 linear equations in 4 unknowns (A_1, A_2, B_1, and B_2):

$$A_2 + B_2 = 0 \tag{6.48}$$

$$A_1 \omega_n + B_1 \omega_u = 0 \tag{6.49}$$

$$A_2 \omega_u^2 + B_2 \omega_n^2 = 0 \tag{6.50}$$

$$A_1 \omega_n^2 \omega_u^2 + B_1 \omega_u \omega_n^2 = \omega_u \omega_n^2 \tag{6.51}$$

After some calculations, we obtain

$$A_1 = \frac{\omega_u \omega_n}{\omega_u^2 - \omega_n^2}, \quad A_2 = 0, \quad B_1 = \frac{-\omega_n^2}{\omega_u^2 - \omega_n^2}, \quad B_2 = 0,$$

and, finally:

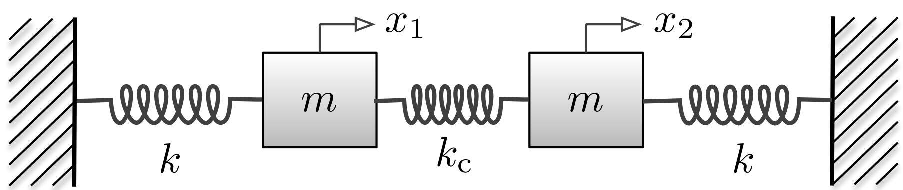
$$Y(s) = \frac{\omega_n}{\omega_u^2 - \omega_n^2}\left(\frac{\omega_u \omega_n}{s^2 + \omega_n^2} - \frac{\omega_u \omega_n}{s^2 + \omega_u^2}\right)$$

(iv) We are now ready to compute the inverse Laplace transform

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{\omega_n}{\omega_u^2 - \omega_n^2}\Big(\omega_u \sin(\omega_n t) - \omega_n \sin(\omega_u t)\Big)$$

▼

E6.6 **Beating in weakly-coupled identical harmonic oscillators.** Consider two identical harmonic oscillators with mass m and spring stiffness k interconnected by a spring with stiffness k_c. (We assume the system has no dampers and no friction).



(i) Write the equations of motion for $x_1(t)$ and $x_2(t)$, possibly using a free body diagram.

(ii) Define the *sum position* $x_{\text{sum}}(t) = x_1(t) + x_2(t)$. Summing the equations of motion for $x_1(t)$ and $x_2(t)$, obtain a differential equation for x_{sum}.
Define the *difference position* $x_{\text{diff}}(t) = x_1(t) - x_2(t)$. Subtracting the equations for $x_2(t)$ from the equation for $x_1(t)$, obtain a differential equation for x_{diff}.

(iii) What is the natural frequency of the second order dynamics of x_{sum}? What is the natural frequency of the second order dynamics of x_{diff}?

Next, assume $x_1(0) = 1, \dot{x}_1(0) = 0$ and $x_2(0) = \dot{x}_2(0) = 0$, that is, only the first mass is displaced and zero initial velocities.

(iv) What are corresponding initial conditions for $x_{\text{sum}}(0), \dot{x}_{\text{sum}}(0)$ and $x_{\text{diff}}(0), \dot{x}_{\text{diff}}(0)$? Write the solutions for $x_{\text{sum}}(t)$ and $x_{\text{diff}}(t)$.
Hint: Recall the solutions to the harmonic oscillator from Section 2.1.2.

(v) Write the solutions for $x_1(t)$ and $x_2(t)$.

Note: *We have learned that*
(1) in a mechanical system with two degrees of freedom there exist two natural frequencies,
(2) x_1 and x_2 are the sum and difference of sinusoidal functions, and
(3) when $k_c \ll k$, the two frequency satisfy $\omega_{\text{sum}} \approx \omega_{\text{diff}}$ and the system exhibits the beating phenomenon.

FBMaybe: add picture from goodnote with parallel and symmetric oscillation, where both masses move up and down or move towards away from each other

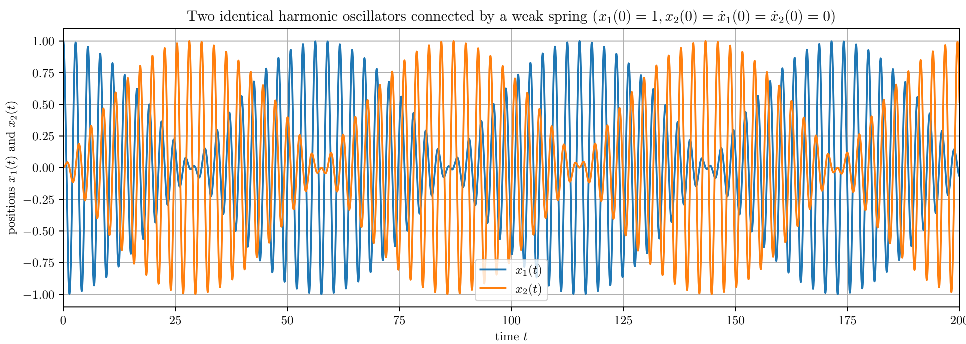


Figure 6.18: Two identical harmonic oscillators coupled by a weak spring ($m = 1$, $k = 5$, and $k_c = 0.25$) display the beating phenomenon.
(1) From physics viewpoint, the potential energy in the initial displacement of the first mass leaks into the second mass, in the sense that at each time t^* when $x_2(t^*) = 1$, energy conservation implies $\dot{x}_2(t^*) = 0$ and $x_1(t^*) = \dot{x}_1(t^*) = 0$.
(2) Even if the coupling between the two masses is weak, the cumulative effect of the dynamics is not weak!
Python code available at `ex-coupled-oscillators.py`

Answer:

(i) Using Newton's law, we obtain

$$m\ddot{x}_1 + kx_1 = k_{\mathsf{c}}(x_2 - x_1) \tag{6.52}$$

$$m\ddot{x}_2 + kx_2 = k_{\mathsf{c}}(x_1 - x_2) \tag{6.53}$$

(ii) Summing the equations, we obtain

$$m(\ddot{x}_1 + \ddot{x}_2) + k(x_1 + x_2) = 0 \qquad \Longrightarrow \qquad m\ddot{x}_{\mathsf{sum}} + kx_{\mathsf{sum}} = 0 \tag{6.54}$$

Subtracting the second equation from the first, we obtain

$$m(\ddot{x}_1 - \ddot{x}_2) + k(x_1 - x_2) = 2k_{\mathsf{c}}(x_2 - x_1) \qquad \Longrightarrow \qquad m\ddot{x}_{\mathsf{diff}} + (k + 2k_{\mathsf{c}})x_{\mathsf{diff}} = 0 \tag{6.55}$$

(iii) The natural frequency for x_{sum} is $\omega_{\mathsf{sum}} = \sqrt{k/m}$.

The natural frequency for x_{diff} is $\omega_{\mathsf{diff}} = \sqrt{(k + 2k_{\mathsf{c}})/m}$.

(iv) The initial conditions are $x_{\mathsf{sum}}(0) = 1$, $\dot{x}_{\mathsf{sum}}(0) = 0$ and $x_{\mathsf{diff}}(0) = 1$, $\dot{x}_{\mathsf{diff}}(0) = 0$.

Given these initial conditions, recall from Section 2.1.3 that the harmonic oscillator $\ddot{y} + \omega_{\mathsf{n}} y = 0$ has a solution of the form $y(t) = a\sin(\omega_{\mathsf{n}} t) + b\cos(\omega_{\mathsf{n}} t)$. Since $y(0) = 1$ and $\dot{y}(0) = 0$, one can calculate that the solution is $y(t) = \cos(\omega_{\mathsf{n}} t)$.

Since the two dynamics are simple harmonic oscillators, starting with a unit displacement and zero initial velocity, we have

$$x_{\mathsf{sum}}(t) = \cos(\omega_{\mathsf{sum}} t) \qquad \text{and} \qquad x_{\mathsf{diff}}(t) = \cos(\omega_{\mathsf{diff}} t) \tag{6.56}$$

(v) Summing and subtracting:

$$x_1(t) = \frac{1}{2}\big(x_{\mathsf{sum}}(t) + x_{\mathsf{diff}}(t)\big) = \frac{1}{2}\big(\cos(\omega_{\mathsf{sum}} t) + \cos(\omega_{\mathsf{diff}} t)\big) \tag{6.57}$$

$$x_2(t) = \frac{1}{2}\big(x_{\mathsf{sum}}(t) - x_{\mathsf{diff}}(t)\big) = \frac{1}{2}\big(\cos(\omega_{\mathsf{sum}} t) - \cos(\omega_{\mathsf{diff}} t)\big) \tag{6.58}$$

▼

E6.7 **Resonance and steady-state response.** Given positive parameters b and c, consider the second-order system with state variable x and input u:

$$\ddot{x} + b\dot{x} + cx = cu.$$

(i) Write the transfer function for this system.
(ii) Consider a sinusoidal input $u = \sin(\omega t)$. At approximately what frequency ω (as function of the system parameters) does resonance occur?
(iii) Write a condition on the system parameters such that sinusoidal inputs at each possible frequency $\omega > 0$ are attenuated.
(iv) Compute the steady-state response $x_{\text{ss}}(t)$ of the system to a sinusoidal input $u = \sin(\omega t)$.

Answer:

(i) The transfer function is

$$\frac{X(s)}{U(s)} = \frac{c}{s^2 + bs + c}.$$

(ii) Resonance occurs when the system is excited at a frequency near its natural frequency, i.e., $\omega = \omega_n$. Compare the transfer function to the canonical second-order system. The natural frequency is given by

$$\omega_n^2 = c \quad \Longrightarrow \quad \omega_n = \sqrt{c} = \omega.$$

(iii) To prevent resonance and attenuate all frequencies, we need the system to be overdamped. Compare the transfer function to the canonical second-order system. The damping ratio is given by

$$2\zeta\omega_n = b \implies \zeta = \frac{b}{2\sqrt{c}}.$$

The corresponding condition is

$$\frac{b}{2\sqrt{c}} > 1 \implies b > 2\sqrt{c}.$$

(iv) We can find the steady-state response using Eq. (6.1) from the lecture slides:

$$x_{\text{ss}}(t) = \big|G(\mathrm{i}\omega)\big| \sin\Big(\omega t + \arg\big(G(\mathrm{i}\omega)\big)\Big)$$

where $G(\mathrm{i}\omega) = X(\mathrm{i}\omega)/U(\mathrm{i}\omega)$. We will need to calculate the magnitude and angular frequency response of the transfer function. The sinusoidal transfer function is given by

$$G(\mathrm{i}\omega) = \frac{c}{-\omega^2 + c + \mathrm{i}b\omega}.$$

The magnitude frequency response is

$$|G(\mathrm{i}\omega)| = \frac{c}{\sqrt{\omega^4 + (b^2 - 2c)\omega^2 + c^2}}.$$

The angular frequency response is

$$\arg\big(G(\mathrm{i}\omega)\big) = -\arctan\left(\frac{b\omega}{c - \omega^2}\right).$$

Therefore, the steady-state response of the system is

$$x_{\text{ss}}(t) = \frac{c}{\sqrt{\omega^4 + (b^2 - 2c)\omega^2 + c^2}} \sin\left(\omega t - \arctan\left(\frac{b\omega}{c - \omega^2}\right)\right).$$

▼

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