## 5.3 Second-order systems and their responses

We recall from Section 2.1.2 that a second-order system is a dynamical system in which *two variables* are required and sufficient to describe the storage of position (linear or angular), velocity (or momentum), energy, mass, etc. As illustrated in Figure 5.5, example of second order systems include:

(i) the position of car on a road and the forced mass-spring-damper system (2.12),
(ii) the angular position of a rotating system (2.24),
(iii) the RLC circuit (2.45) (and any electric circuit where energy is stored in two elements, capacitors or inductors as they might be),
(iv) the linearized pendulum about either the down or up position (3.25) and (3.26), and
(v) the water height dynamics for two connected tanks.

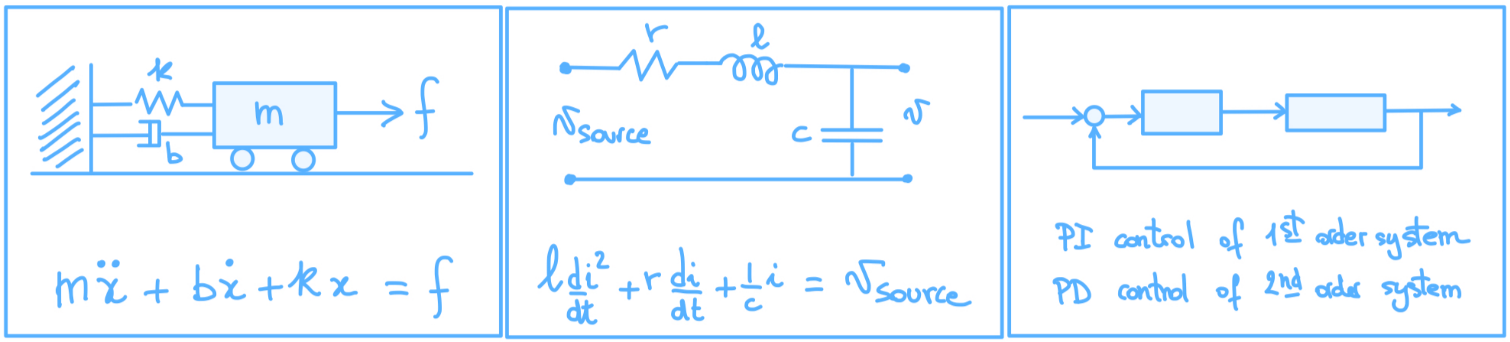



Figure 5.5: Illustrations of second-order systems from earlier chapters and from the later chapters on control systems.

### 5.3.1 Canonical form of second-order systems with canonical parameters $(\omega_{\mathsf{n}}, \zeta)$

We start by defining a *canonical form of a second-order system* with canonical parameters (just like we did for first-order systems with the time constant $\tau > 0$).

The *canonical form of a second order system* is

$$\ddot{y}(t) + 2\zeta\omega_{\mathsf{n}}\dot{y}(t) + \omega_{\mathsf{n}}^2 y(t) \;=\; \omega_{\mathsf{n}}^2 u(t) \tag{5.16}$$

with corresponding transfer function

$$G_{\text{second-order}}(s) = \frac{Y(s)}{U(s)} \;=\; \frac{\omega_{\mathsf{n}}^2}{s^2 + 2\zeta\omega_{\mathsf{n}} s + \omega_{\mathsf{n}}^2} \tag{5.17}$$

where, as usual, $u(t)$ and $y(t)$ are the input and output of the system, and where the canonical parameters are:

- $\omega_{\mathsf{n}} > 0$ is the *natural frequency* of the system, indicating how fast the system oscillates in the absence of damping; and
- $\zeta \geq 0$ is the *damping ratio*, a dimensionless measure of damping in the system.

### 5.3.2 The mass-spring-damper systems example

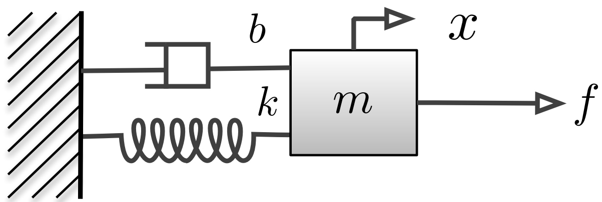



Figure 5.6: Generalizing equation (2.12), a mass-spring-damper system with parameters $m > 0$, $b \geq 0$, and $k > 0$, subject to a force $f(t)$.
In our discussion, the mass and the spring coefficient are always positive, but we do allow the damper to be present ($b > 0$) or not ($b = 0$).

Consider a forced mass-spring-damper system as in Figure 5.6:

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = f(t). \tag{5.18}$$

Taking the Laplace transform (at zero initial conditions) we obtain

$$(ms^2 + bs + k)X(s) = F(s). \tag{5.19}$$

In turn, the transfer function is written, can be manipulated, as follows:

$$G_{\text{mass-spring-damper}}(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} = \frac{1}{k}\frac{k/m}{s^2 + (b/m)s + k/m}. \tag{5.20}$$

Therefore, the natural frequency $\omega_{\text{n}}$ and damping ratio $\zeta$ can computed as functions of mass $m$, spring stiffness $k$ and damping coefficient $b$ by matching the denominators of $G_{\text{second-order}}(s)$ in (5.17) and $G_{\text{mass-spring-damper}}(s)$ in (5.20):

$$\omega_{\text{n}} = \sqrt{\frac{k}{m}} \qquad \text{and} \qquad \zeta = \frac{b}{2\sqrt{mk}}. \tag{5.21}$$

### 5.3.3 Placement of poles and classification of second-order systems

Next, we compute the two poles of the second-order system in canonical form (5.17), which we report for convenience:

$$G_{\text{second-order}}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.22}$$

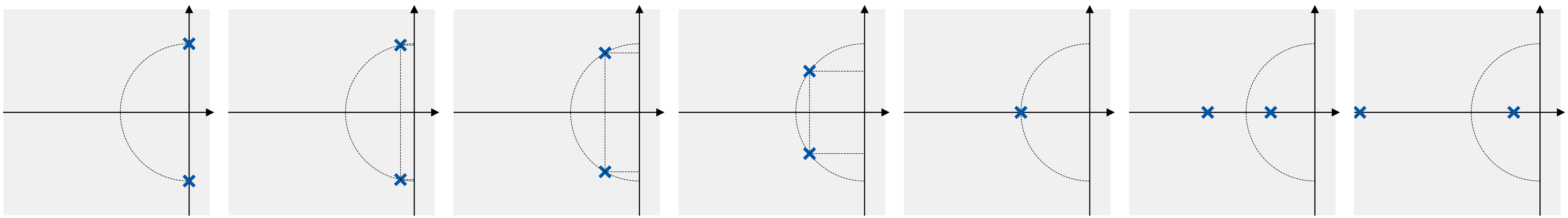
The poles are

$$\text{poles of } G_{\text{second-order}}(s) = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = -\omega_n\big(\zeta \pm \sqrt{\zeta^2 - 1}\big)$$

$$= \begin{cases} \underbrace{+\mathrm{i}\omega_n,\ -\mathrm{i}\omega_n}_{\text{both purely imaginary}} & \text{if } \zeta = 0 \\ \underbrace{\omega_n\big(-\zeta + \mathrm{i}\sqrt{1-\zeta^2}\big),\ \omega_n\big(-\zeta - \mathrm{i}\sqrt{1-\zeta^2}\big)}_{\text{complex conjugate}} & \text{if } 0 < \zeta < 1 \\ \underbrace{-\omega_n, -\omega_n}_{\text{repeated real}} & \text{if } \zeta = 1 \\ \underbrace{\omega_n\big(-\zeta + \sqrt{\zeta^2-1}\big),\ \omega_n\big(-\zeta - \sqrt{\zeta^2-1}\big)}_{\text{both real distinct}} & \text{if } \zeta > 1 \end{cases}$$

In other words, depending upon the damping ratio $\zeta$, the poles are purely imaginary, complex conjugate, real equal, or real distinct, see Figure 5.7. When $0 < \zeta < 1$, the complex conjugate poles of $G_{\text{second-order}}(s)$ are:

$$-\zeta\omega_n \pm \mathrm{i}\omega_d, \quad \text{where the } \textit{damped natural frequency} \text{ is} \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$



(a) $\zeta = 0.0 \implies$ two poles $= \pm i\omega_n$
*Case I: undamped system*

(b) $\zeta = 0.2$
*Case II: underdamped system*

(c) $\zeta = 0.5$
*Case II: underdamped system*

(d) $\zeta = 0.8$
*Case II: underdamped system*

(e) $\zeta = 1.0 \implies$ two poles $= -\omega_n$
*Case III: critically-damped system*

(f) $\zeta = 1.1$
*Case IV: overdamped system*

(g) $\zeta = 1.5$
*Case IV: overdamped system*

Figure 5.7: Poles of a second order system as a function of the damping ratio $\zeta$, at fixed natural frequency $\omega_n$. The dashed semicircle has radius $\omega_n$.
At $\zeta = 0$, the two poles are purely imaginary and equal to $\pm i\omega_n$.
As $\zeta$ increases from $0$ to $1$, the two complex conjugate poles move strictly inside the left half plane, sliding along the semicircle.
When $0 < \zeta < 1$, the two complex conjugate poles have real part $-\zeta\omega_n$ and imaginary part $\pm i\omega_d$.
At $\zeta = 1$, the two poles are coincident at the real value $-\omega_n$.
For $\zeta > 1$, the two poles split: one moves left towards $-\infty$ (the fast pole) and one moves right towards the imaginary axis (the slow dominant pole).
Image generated by `2ndorder-poles.py`

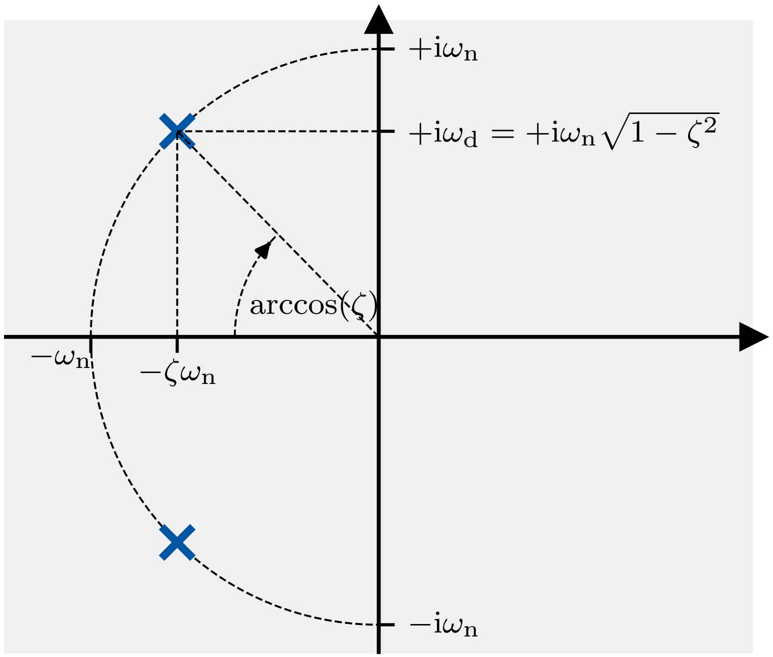



Figure 5.8: Poles of an underdamped second-order system, defined by a natural frequency $\omega_n$ and a damping ratio $0 < \zeta < 1$.
Note the damped natural frequency $\omega_d$ and the damping angle $\arccos(\zeta)$.
To verify that the complex conjugate poles of $G_{\text{second-order}}(s)$ move on the circle of radius $\omega_n$, it suffices to show that $|-\zeta\omega_n \pm i\omega_d| = \omega_n$.
Image generated by `2ndorder-pole-beta.py` .

**Remark 5.3 (The canonical form and the mass-spring-damper system: continued).** *For a mass-spring-damper system, the characteristic equation is* $ms^2 + bs + k = 0$ *and its solutions are*

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

*Therefore, the two roots are equal and real when* $b^2 = 4mk$*. We define the critical damping parameter as* $b_{\text{critical}} = 2\sqrt{mk}$*. Then*

- *the system is underdamped for* $b < b_{\text{critical}} = 2\sqrt{mk}$*,*
- *the system is critically damped for* $b = b_{\text{critical}} = 2\sqrt{mk}$*, and*
- *the system is overdamped for* $b > b_{\text{critical}} = 2\sqrt{mk}$*.*
- *It is now clear why* $\zeta$ *is called the damping ratio: for mass-spring-dampers systems,* $\zeta$ *is indeed a ratio:*

$$\zeta = \frac{b}{b_{\text{critical}}} = \frac{b}{2\sqrt{mk}}.$$

•

### 5.3.4 From poles to free response

There are four possible distinct placements of the two poles of a second order system. We review the four cases here and we show the free response (from unit initial position and zero initial velocity) in the following Table 5.2.

**Case I:** ***Undamped systems.*** When $\zeta = 0$, the system is *undamped* and exhibits persistent oscillatory behavior.

**Case I:** ***Underdamped systems.*** When $0 < \zeta < 1$, the system is *underdamped* and exhibits damped oscillatory behavior.

**Case III:** ***Critically-damped systems.*** When $\zeta = 1$, the system is *critically damped* and returns to equilibrium as quickly as possible without oscillating.

**Case IV:** ***Overdamped systems.*** When $\zeta > 1$, the system is *overdamped* and returns to equilibrium without oscillating, but more slowly than in the critically damped case.

Regarding the natural frequency $\omega_n$: this parameter determines only the speed of the response in each of the four cases.

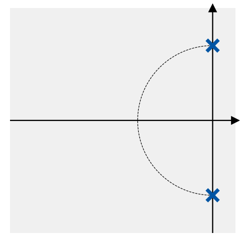
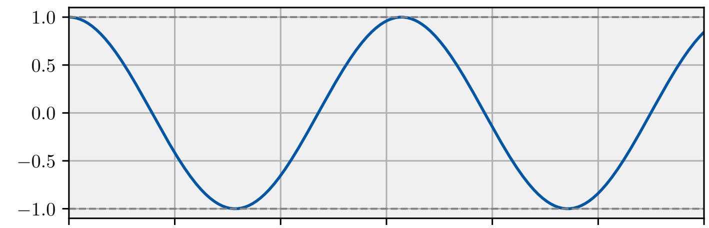
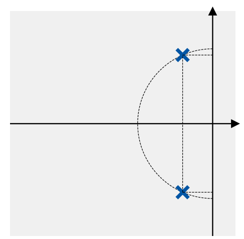
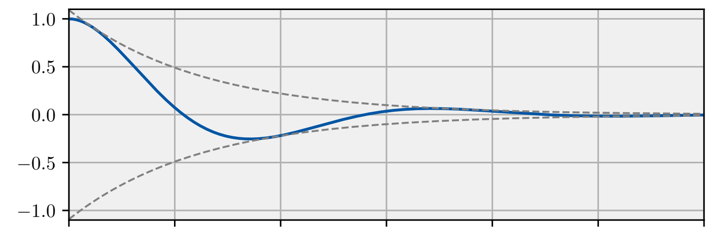
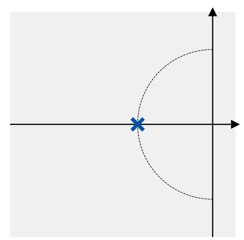
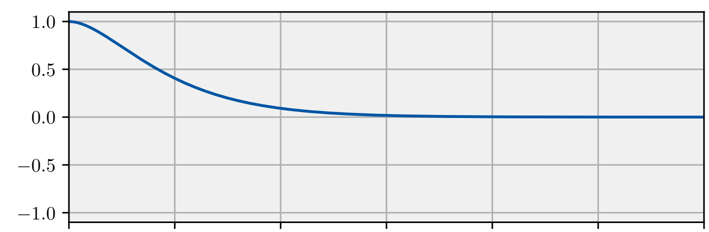
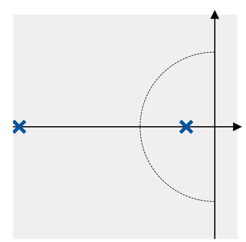
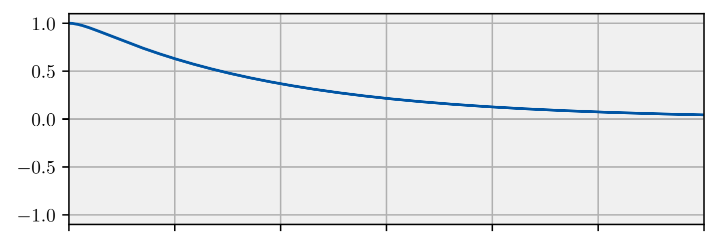
| | damping ratio $\zeta$ | poles of transfer function (5.22) and corresponding functions of time | poles location (dashed line = circle of radius $\omega_n$) | free response of $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$ with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$ |
|---|---|---|---|---|
| *Case I: undamped system* | $\zeta = 0$ | two poles $= \pm i\omega_n$<br>$\sin(\omega_n t)$, $\cos(\omega_n t)$<br>sinusoidal waves |  |  |
| *Case II: underdamped system* | $0 < \zeta < 1$ | two poles $= -\omega_n\zeta \pm i\omega_n\sqrt{1-\zeta^2}$<br>$e^{-\omega_n\zeta t}\sin(\omega_d t)$,<br>$e^{-\omega_n\zeta t}\cos(\omega_d t)$<br>where $\omega_d = \omega_n\sqrt{1-\zeta^2}$<br>damped sinusoidal waves |  |  |
| *Case III: critically-damped system* | $\zeta = 1$ | two poles $= -\omega_n$<br>$e^{-\omega_n t}$, $t\,e^{-\omega_n t}$<br>exponential decay (with transient) |  |  |
| *Case IV: overdamped system* | $\zeta > 1$ | two poles $= -\omega_n(\zeta \pm \sqrt{\zeta^2 - 1})$<br>slow pole: $e^{-\omega_n(\zeta-\sqrt{\zeta^2-1})t}$<br>fast pole: $e^{-\omega_n(\zeta+\sqrt{\zeta^2-1})t}$<br>exponential decay |  |  |

Table 5.2: Classification of a second order system into 4 classes: undamped, underdamped, critically-damped, and overdamped.
For $0 < \zeta < 1$ (Case II), the *damped frequency* is $\omega_d = \omega_n\sqrt{1-\zeta^2}$. Note $\omega_d < \omega_n$, so the presence of damping diminishes the frequency of oscillations.
In the overdamped case (Case IV), the pole close to the imaginary axis is the *slow pole*, whereas the pole moving towards $-\infty$ is the fast pole.

### 5.3.5 Impulse, step, and ramp responses of second-order systems

```python
import numpy as np; import matplotlib.pyplot as plt; import control as ctrl
plt.rcParams.update({"text.usetex": True, "font.family": "serif", "font.serif": ...
    ["Computer Modern Roman"], "font.size": 16 })

# Define the parameters of the system
natural_frequency = 1.0  # Natural frequency, omega_n
damping_ratios = [0.0, 0.2, 0.4, 0.8, 1.0, 1.5, 3.0]  # Damping ratios, zeta

# Define time range for the simulation
t = np.linspace(0, 12, 1000); ramp_input = t  # Unit ramp input
colors = ['#752d00', '#a43e00', '#d35000', '#ff6100', '#ff8800', '#ffaf00', '#ffcc00']

# Initialize the figure for impulse, step, and ramp response
fig, axs = plt.subplots(3, 1, figsize=(10, 10))

# Loop through each damping ratio and plot impulse, step, and ramp responses
for idx, zeta in enumerate(damping_ratios):
    # Define the transfer function of the second-order system
    num = [natural_frequency**2]; den = [1, 2 * zeta * natural_frequency, ...
        natural_frequency**2]
    system = ctrl.TransferFunction(num, den)

    # Compute and plot the impulse response
    t_impulse, y_impulse = ctrl.impulse_response(system, T=t)
    line_style = '--' if idx == 4 else '-'
    axs[0].plot(t_impulse, y_impulse, line_style, label=f'$\\zeta$ = {zeta}', ...
        color=colors[idx])

    # Compute and plot the step response
    t_step, y_step = ctrl.step_response(system, T=t)
    axs[1].plot(t_step, y_step, line_style, label=f'$\\zeta$ = {zeta}', color=colors[idx])

    # Compute and plot the ramp response
    t_ramp, y_ramp = ctrl.forced_response(system, T=t, U=ramp_input)
    axs[2].plot(t_ramp, y_ramp, line_style, label=f'$\\zeta = {zeta}$', color=colors[idx])

# Add labels, legends, grid, and set xlim for all subplots
for ax in axs:
    ax.legend(loc='lower right',fontsize=14); ax.grid(True); ax.set_xlim(0, 25);

# Set plot properties
axs[0].set_xlim(0, 12); axs[0].set_ylim(-1.1, 1.1); axs[0].set_ylabel('Impulse response')
axs[0].set_title('Impulse, step, and ramp response of 2nd order system')
axs[1].set_xlim(0, 12); axs[1].set_ylim(-0.1, 2.1);  axs[1].set_ylabel('Step response')
axs[2].set_xlim(0, 12); axs[2].set_ylim(0, 12); axs[2].set_xlabel('time $t$');  ...
    axs[2].set_ylabel('Ramp response')

# Add arrow with text
arrow_color = '#0055A4'
text_color = arrow_color  # Set text color same as arrow color

axs[0].text(1, 0.5, "increasing $\\zeta$", ha="center", va="center", rotation=-45, ...
    size=15, color=text_color, bbox=dict(boxstyle="rarrow,pad=0.3", fc="none", ...
    ec=arrow_color, lw=2, alpha=0.5))
axs[1].text(2, 0.75, "increasing $\\zeta$", ha="center", va="center", rotation=-45, ...
    size=15, color=text_color, bbox=dict(boxstyle="rarrow,pad=0.3", fc="none", ...
    ec=arrow_color, lw=2, alpha=0.5))
axs[2].text(7, 5, "increasing $\\zeta$", ha="center", va="center", rotation=-45, ...
    size=15, color=text_color, bbox=dict(boxstyle="rarrow,pad=0.3", fc="none", ...
    ec=arrow_color, lw=2, alpha=0.5))

# Save the plot to a PDF file
plt.tight_layout(); plt.savefig('2ndorder-responses.pdf', bbox_inches='tight')
```

Listing 5.2: Python script generating Figure 5.9. This script relies upon the Python Control Systems Library (Fuller et al., 2021). Available at `2ndorder-responses.py`

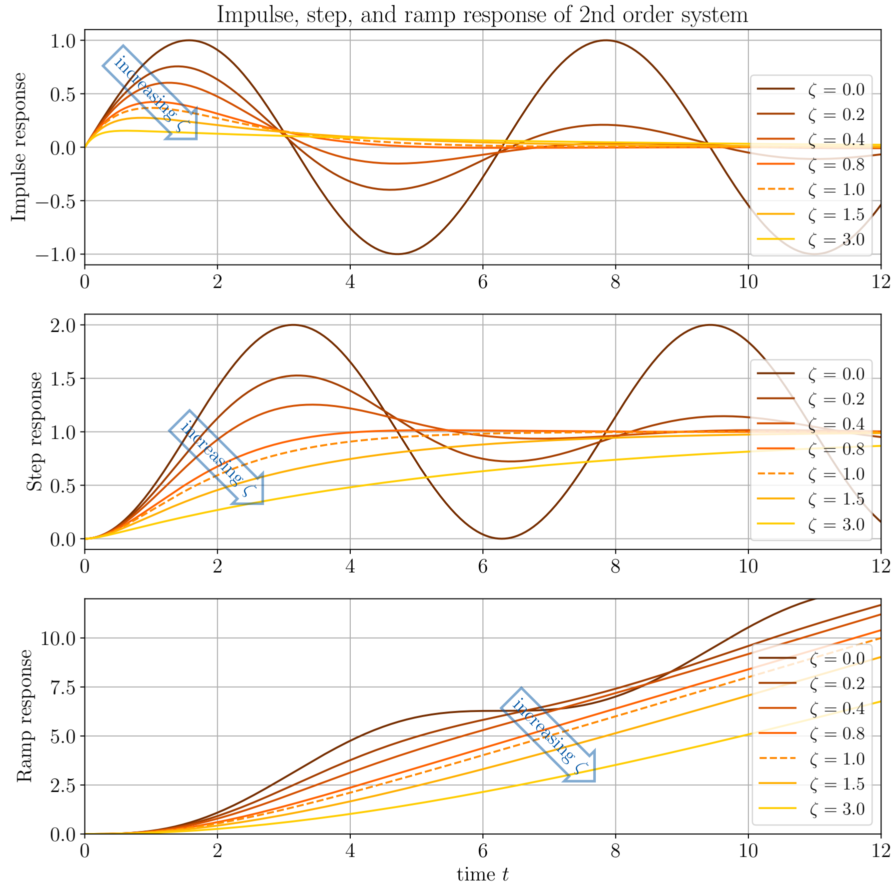


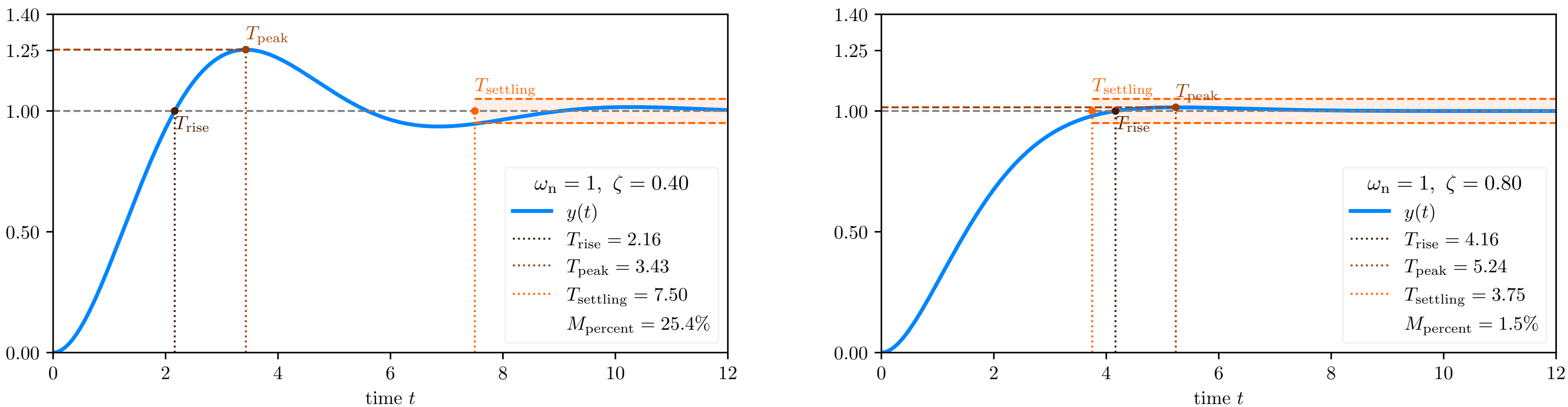

Figure 5.9: Impulse, step, and ramp responses of the 2nd order dynamics (5.16). From the step response plots, we note: (i) At $\zeta = 0.4$, the response is fast and the overshoot is 25.4%. Smaller damping ratios lead to large overshoot. (ii) At $\zeta = 0.8$, the overshoot is only 1.4% (but the response is slower than at $\zeta = 0.4$). Larger damping ratios (i.e., $\zeta > 1$) lead to slow responses.

### 5.3.6 Step response of an underdamped system

For an underdamped second-order system with damping ratio $0 < \zeta < 1$, and arbitrary natural frequency $\omega_\mathrm{n}$, the step response is

$$y(t) \;=\; 1 - \mathrm{e}^{-\zeta\omega_\mathrm{n} t}\left(\cos(\omega_\mathrm{d} t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_\mathrm{d} t)\right) \tag{5.23}$$

where $\omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1-\zeta^2}$ is the *damped natural frequency*. (We refer to Appendix 5.5 for inverse Laplace transform calculations.)




(a) $\zeta = .4$ and $\omega_\mathrm{n} = 1$

(b) $\zeta = .8$ and $\omega_\mathrm{n} = 1$

Figure 5.10: Step response of an underdamped second order system from zero initial position and zero initial velocity, for unit natural frequency and varying damping ratios $\zeta$. Image generated by

The step response shows how different values of $\zeta$ affect key characteristics such as rise time, peak time, percent overshoot, and settling time. (i) *A low damping ratio* $\zeta = .4$ leads to fast response times, but also high overshoot and prolonged oscillations before settling. (ii) *A high damping ratio* $\zeta = .8$ provides a smooth slower response with minimal overshoot, but also slow reaction times.

## Time domain specifications as functions of natural frequency and damping ratio

- The *rise time* $T_{\text{rise,0\%-100\%}}$ (respectively, $T_{\text{rise,10\%-90\%}}$) is the time required for the response to rise from 0% to 100% (respectively, from 10% to 90%) of the final value. Some calculations and approximation show:

$$T_{\text{rise,0\%-100\%}} = \frac{\pi - \arccos(\zeta)}{\omega_n\sqrt{1-\zeta^2}}, \qquad \text{and} \qquad T_{\text{rise,10\%-90\%}} \approx \frac{1.8}{\omega_n}.$$

- The *peak time* $T_{\text{peak}}$ is the time it takes for the response to reach the maximum overshoot value (this is the first peak in the oscillatory response, at which the overshoot is maximum). Some calculations show:

$$T_{\text{peak}} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{5.24}$$

- The *settling time* $T_{\text{settling}}$ is the time it takes for the response to remain within a certain range (typically 1% or 5%) of the steady-state value. For the 1% ad 5% criteria, approximate formulas are:

$$T_{\text{settling }1\%} \approx \frac{5}{\zeta\omega_n} \qquad \text{and} \qquad T_{\text{settling }5\%} \approx \frac{3}{\zeta\omega_n} \tag{5.25}$$

  On a related note, the *time constant* of the underdamped system is

$$\tau = \frac{1}{\zeta\omega_n} \tag{5.26}$$

- The *percent overshoot* $M_{\text{percent}}$ is the maximum amount the system response overshoots its final value, divided by its final value. Some calculations show:

$$M_{\text{percent}} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \tag{5.27}$$

It is useful to verify that the values of $T_{\text{rise}}$, $T_{\text{peak}}$, $T_{\text{settling}}$ and $M_{\text{percent}}$ in Figure 5.10 are correct, for values of $\omega = 1$ and $\zeta \in \{0.4, 0.8\}$.

## 5.4 Higher-order systems and their step response

In this section we consider higher-order systems of the form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n} \tag{5.28}$$

Assume $G(s)$ has distinct real poles $-p_1, \ldots, -p_n$, meaning that the denominator of $G(s)$ can be factored as $(s+p_1)(s+p_2)\ldots(s+p_n)$. We assume the poles are in the strict left half plane, that is, all $p_i$ are strictly positive.

**In class assignment**

Why do we assume that the poles are in the left half plane?

For a stable transfer function $G(s)$,

(i) if the input is a unit step, then the steady-state output (the output after all decaying signals have decayed) is a step of magnitude $G(0)$:

$$u(t) = \mathbf{1}(t) \quad \Longrightarrow \quad y_{\text{steady-state}}(t) = G(0)\mathbf{1}(t), \tag{5.29}$$

(ii) $G(0)$ is the *steady-state gain* (or *DC gain*) since it is the amplification (or attenuation) of the input signal at the output.

Note: $G(0) = 1$ for the canonical forms of first and second-order systems.

We now verify these statements. When $u(t) = \mathbf{1}(t)$ and $U(s) = \frac{1}{s}$, the partial fraction expansion of $Y(s) = G(s) \cdot \frac{1}{s}$ is

$$Y(s) = \frac{r}{s} + \sum_{i=1}^{n} \frac{r_i}{s + p_i} \tag{5.30}$$

for appropriate residues $r, r_1, \ldots, r_n$. Therefore, the output is the sum of a step function and $n$ exponentially decaying terms:

$$y(t) = r + \sum_{i=1}^{n} r_i \,\mathrm{e}^{-p_i t} \tag{5.31}$$

We are particularly interested in the behavior for large times $t$, when the exponentially decaying terms are below 1% of their initial value. To study this asymptotic behavior, we compute $r$ using the single-pole residue formula:

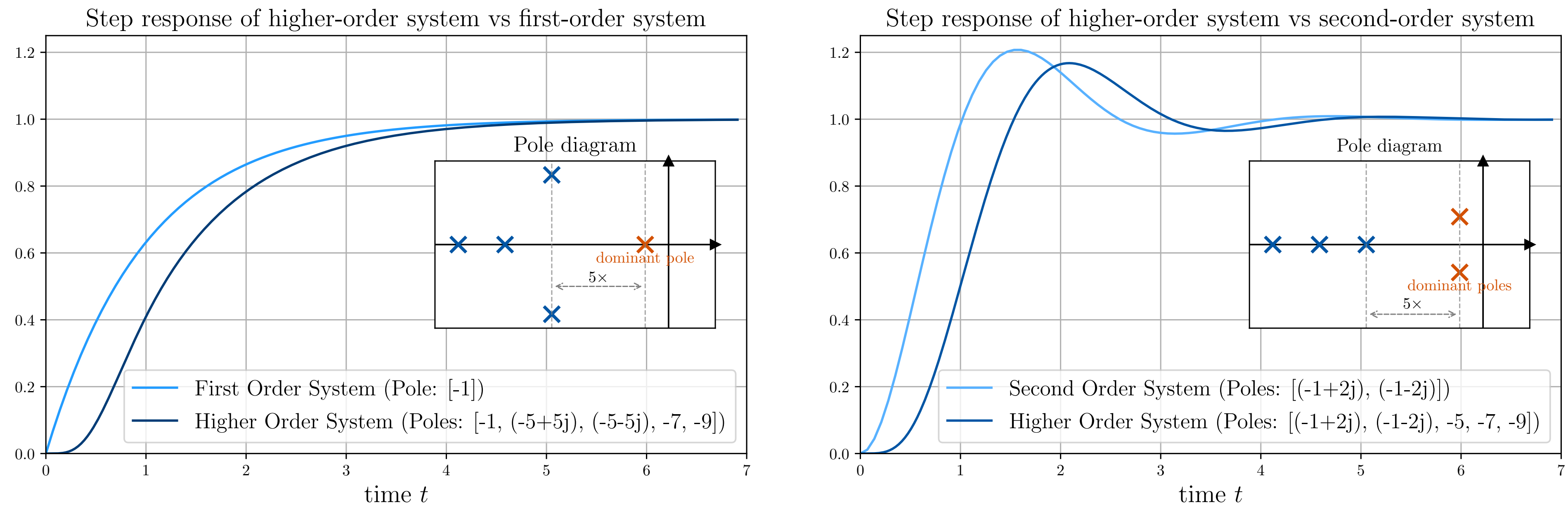
$$r \;=\; sY(s)\Big|_{s=0} \;=\; s \cdot \frac{1}{s} G(s)\Big|_{s=0} \;=\; G(0) \tag{5.32}$$

(An alternative equivalent approach to computing the behavior for large times is given by the final value theorem in Section 4.4.)

In systems with multiple poles, one or a few *dominant poles* might primarily determine the system's transient response. The *dominant poles are the ones closest to the imaginary axis* (i.e., with the smallest real parts) which decay more slowly:

- if the dominant pole is a single real pole, the system's response resembles that of a first-order system, characterized by a single exponential decay, and
- if the dominant poles are a pair of complex conjugate poles, the response resembles that of a second-order system, featuring oscillatory behavior with a decay rate governed by the real part of the dominant poles.

This approximation is accurate when the dominant pole(s) are significantly slower (e.g., 5x slower) than the remaining poles.




(a) one dominant pole

(b) complex conjugate poles

Figure 5.11: Step responses of higher-order systems with either a single dominant pole or a pair of dominant complex conjugate poles.
In both cases, the dominant pole approximation has numerator set to have the same DC gain as the original system (unit DC gain in these examples).
Note that the gap is $5\times$, that is, the multiplicative difference between the real part of the dominant pole and the real part of the other poles.
Note that the approximation is perhaps acceptable, but not great. Image generated by `higherorder-comparison.py` .

## 5.5 Appendix: Derivation of the free and step response for second order systems

In this appendix we report some useful calculations that explain some of the formulas and plots presented earlier.

### 5.5.1 Step response for underdamped system

We consider an underdamped second-order system with zero initial conditions ($x(0) = \dot{x}(0) = 0$) subject to a step input:

$$\ddot{y} + 2\zeta\omega_{\mathsf{n}}\dot{y} + \omega_{\mathsf{n}}^2 y = \omega_{\mathsf{n}}^2 \mathbf{1}(t) \tag{5.33}$$

with natural frequency $\omega_{\mathsf{n}}$ and damping ratio $\zeta$. Since $U(s) = 1/s$, we compute

$$Y(s) = \frac{\omega_{\mathsf{n}}^2}{s(s^2 + 2\omega_{\mathsf{n}}\zeta s + \omega_{\mathsf{n}}^2)} \tag{5.34}$$

Since $s^2 + 2\omega_{\mathsf{n}}\zeta s + \omega_{\mathsf{n}}^2 = (s + \omega_{\mathsf{n}}\zeta)^2 + \omega_{\mathsf{d}}^2$ for $\omega_{\mathsf{d}} = \omega_{\mathsf{n}}\sqrt{1-\zeta^2}$, we now expand this rational function in a partial fraction expansion using the terms corresponding to unit step and damped sine and cosine waves:

$$Y(s) = \frac{\alpha}{s} + \beta\frac{\omega_{\mathsf{d}}}{(s + \omega_{\mathsf{n}}\zeta)^2 + \omega_{\mathsf{d}}^2} + \gamma\frac{s + \zeta\omega_{\mathsf{n}}}{(s + \omega_{\mathsf{n}}\zeta)^2 + \omega_{\mathsf{d}}^2} \tag{5.35}$$

To compute $\alpha$, we can use the residue's formula:

$$\alpha = sY(s)\Big|_{s=0} = 1. \tag{5.36}$$

Using the numerators matching method, we can compute the coefficients $\beta$ and $\gamma$ and obtain

$$y(t) \;=\; 1 - \mathrm{e}^{-\zeta\omega_{\mathsf{n}}t}\Big(\cos(\omega_{\mathsf{d}}t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_{\mathsf{d}}t)\Big) \tag{5.37}$$

As this response is equal to the one given in equation (5.23).

### 5.5.2 Free response for underdamped system

For $0 < \zeta < 1$, we consider

$$\ddot{x} + 2\zeta\omega_{\mathsf{n}}\dot{x} + \omega_{\mathsf{n}}^2 x = 0$$

with initial position $x(0)$ and initial velocity $\dot{x}(0)$. We take the Laplace transform to obtain:

$$\Big(s^2X(s) - sx(0) - \dot{x}(0)\Big) + 2\zeta\omega_{\mathsf{n}}\Big(sX(s) - x(0)\Big) + \omega_{\mathsf{n}}^2X(s) = 0. \tag{5.38}$$

From here we compute $X(s)$ as follows

$$X(s) = \frac{(s + 2\zeta\omega_{\mathsf{n}})x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_{\mathsf{n}}s + \omega_{\mathsf{n}}^2} \tag{5.39}$$

Since the system is underdamped, we define the damped frequency by $\omega_{\mathsf{d}} = \omega_{\mathsf{n}}\sqrt{1-\zeta^2}$ and we note

$$s^2 + 2\zeta\omega_{\mathsf{n}}s + \omega_{\mathsf{n}}^2 = (s + \zeta\omega_{\mathsf{n}})^2 + (\omega_{\mathsf{n}}\sqrt{1-\zeta})^2 \overset{\text{by definition}}{=} (s + \zeta\omega_{\mathsf{n}})^2 + \omega_{\mathsf{d}}^2 \tag{5.40}$$

With this denominator, recalling rows (7) and (8) of Table 4.2, we compute the partial fraction expansion:

$$X(s) = \frac{\zeta\omega_{\mathsf{n}}x(0) + \dot{x}(0)}{\omega_{\mathsf{d}}} \cdot \frac{\omega_{\mathsf{d}}}{(s + \zeta\omega_{\mathsf{n}})^2 + \omega_{\mathsf{d}}^2} + x(0) \cdot \frac{s + \zeta\omega_{\mathsf{n}}}{(s + \zeta\omega_{\mathsf{n}})^2 + \omega_{\mathsf{d}}^2} \tag{5.41}$$

so that the inverse Laplace transform is immediate:

$$x(t) \;=\; \frac{\zeta\omega_{\mathsf{n}}x(0) + \dot{x}(0)}{\omega_{\mathsf{d}}} \cdot \mathrm{e}^{-\zeta\omega_{\mathsf{n}}t}\sin(\omega_{\mathsf{d}}t) \;+\; x(0) \cdot \mathrm{e}^{-\zeta\omega_{\mathsf{n}}t}\cos(\omega_{\mathsf{d}}t) \tag{5.42}$$

When $\dot{x}(0) = 0$, we simplify this expression to

$$x(t) = x(0)\,\mathrm{e}^{-\zeta\omega_{\mathsf{n}}t}\Big(\cos(\omega_{\mathsf{d}}t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_{\mathsf{d}}t)\Big). \tag{5.43}$$

This solution is shown in Figure 5.13, for varying values of the damping ratio $\zeta$.

## 5.6 Appendix: Visualization of the free response of an underdamped system

In the undamped and underdamped regime, when $0 \leq \zeta < 1$, consider

$$\ddot{x} + 2\zeta\omega_{\mathsf{n}}\dot{x} + \omega_{\mathsf{n}}^2 x = 0$$

with positive initial position $x(0) = x_0 > 0$ and zero initial velocity $\dot{x}(0) = 0$. Via the inverse Laplace transform calculations in Appendix 5.5, the free response of an underdamped system

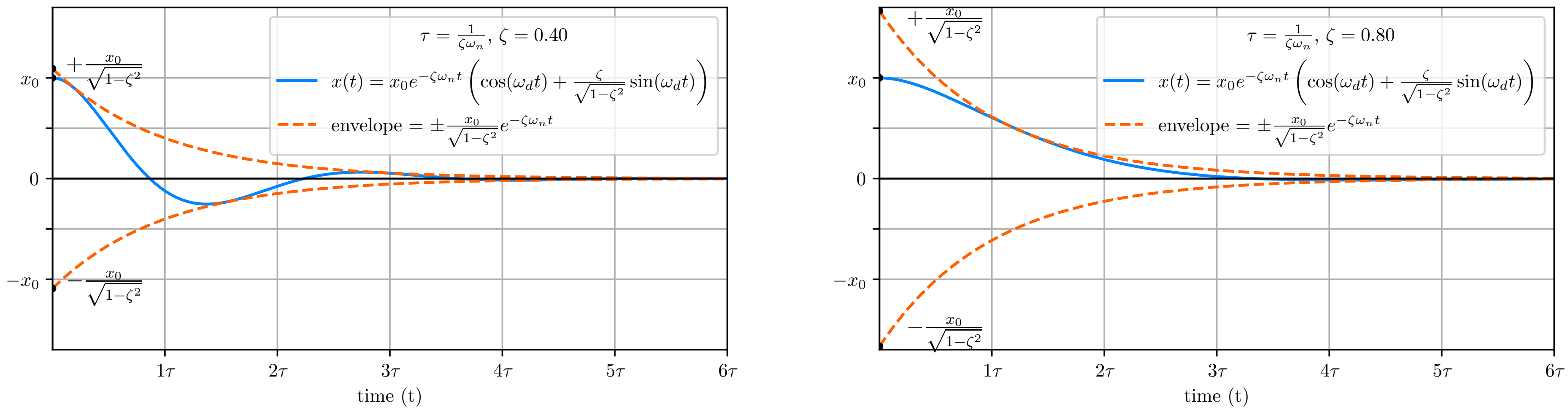
$$x(t) = x_0\,\mathrm{e}^{-\zeta\omega_{\mathsf{n}}t}\Big(\cos(\omega_{\mathsf{d}}t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_{\mathsf{d}}t)\Big) \tag{5.44}$$

where the *damped frequency* is $\omega_{\mathsf{d}} = \omega_{\mathsf{n}}\sqrt{1-\zeta^2}$. Using trigonometric equalities, we can rewrite the solution as

$$x(t) \;=\; \underbrace{\frac{x_0}{\sqrt{1-\zeta^2}}\,\mathrm{e}^{-\zeta\omega_{\mathsf{n}}t}}_{\text{exponentially-decaying envelope}} \cdot \cos\Big(\omega_{\mathsf{d}}t + \arctan\frac{\zeta}{\sqrt{1-\zeta^2}}\Big) \tag{5.45}$$

The expression (5.45) is useful because the precise expression of the exponentially-decaying envelope is now clear.

As for first order systems, after time equal to $5 \cdot \tau$, the free response is guaranteed to be below $1\%$ of the initial value $\dfrac{x_0}{\sqrt{1-\zeta^2}}$.




(a) $\zeta = .4$

(b) $\zeta = .8$

Figure 5.12: Free response of an underdamped second order system from initial position $x_0 > 0$ and zero initial velocity.
Note: the the exponentially-decaying envelope starts at $\pm\dfrac{x_0}{\sqrt{1-\zeta^2}}$.

Note: after time equal to $5 \cdot \tau = 5/(\zeta\omega_\mathrm{n})$, the solution is guaranteed to be below $1\%$ of the initial value $\dfrac{x_0}{\sqrt{1-\zeta^2}}$.

Note however: for $0 < \zeta < 1$, the factor $\frac{1}{\sqrt{1-\zeta^2}}$ is always greater than 1 and approximately $2.3$, $7.1$ and $22.4$ at $\zeta = .9, .99, .999$, respectively.

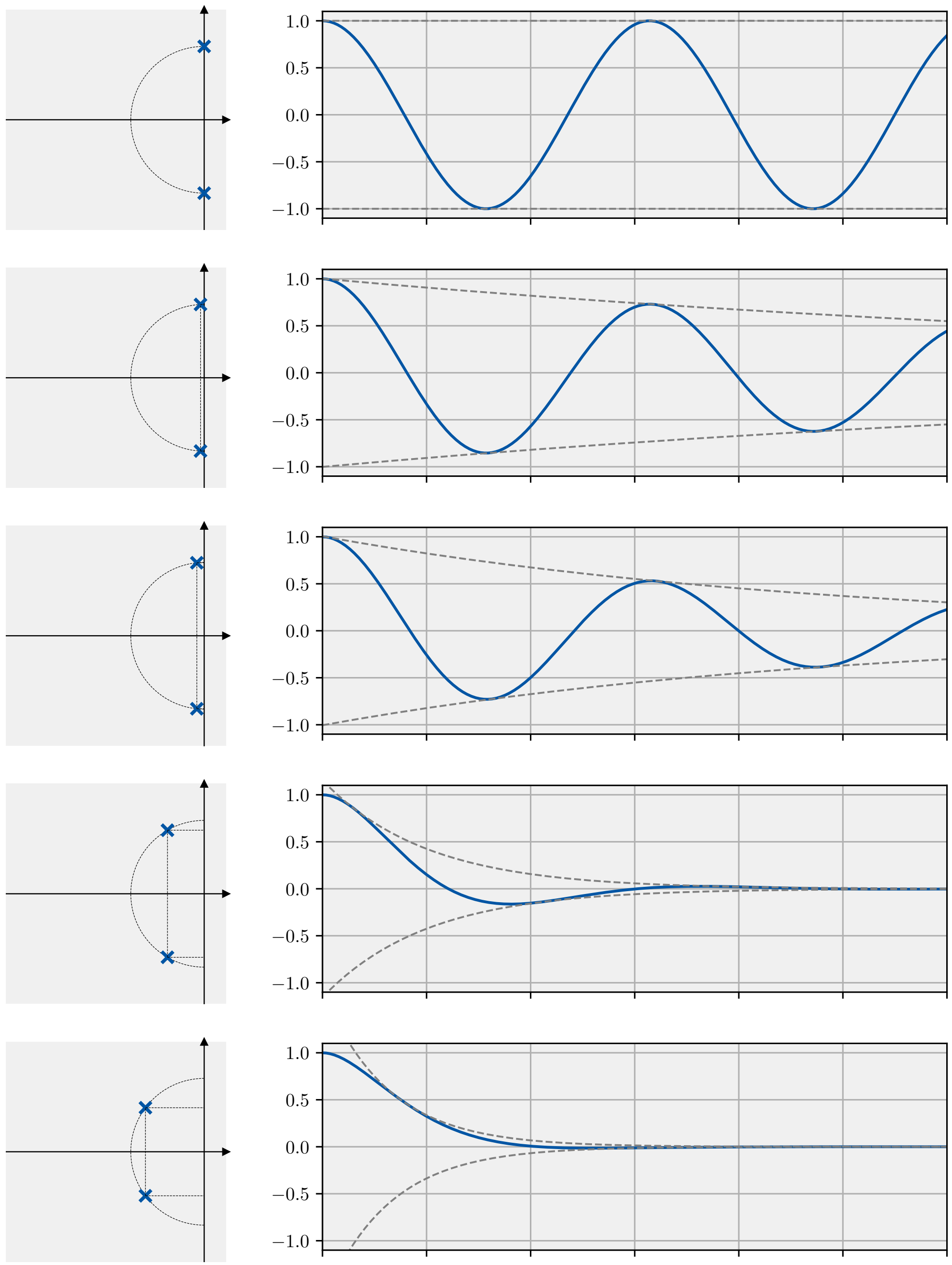



Figure 5.13: Illustrations of the free response of undamped and underdamped second-order systems.
Left panels: location of the two poles and semicircle of radius $\omega_\mathrm{n}$, we let $\omega_\mathrm{n} = 1$.
Right panels: the free response from zero initial velocity (solid blue line) and the exponentially-decaying envelope (dashed gray lines): $\pm \dfrac{x_0}{\sqrt{1-\zeta^2}}\,\mathrm{e}^{-\zeta\omega_\mathrm{n} t}$.

## 5.7 Appendix: Underdamped systems with zeros in the left and right half plane

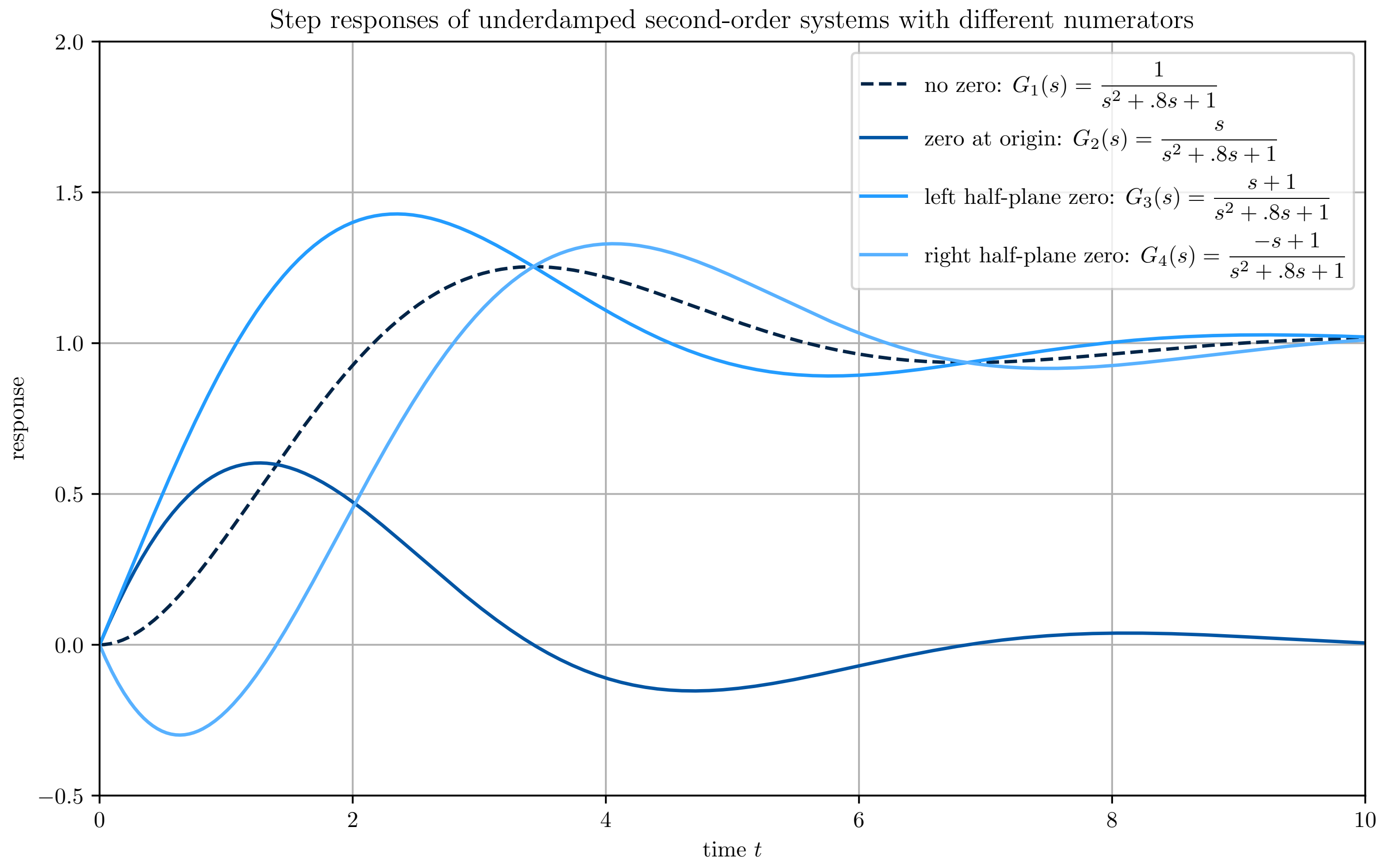



Figure 5.14: Step responses $y_1(t), \ldots, y_4(t)$, of underdamped second-order systems with different numerators: $y_i(t) = \mathcal{L}^{-1}[G_i(s)/s]$.
We leave it to the reader to explain why at each instant of time $t^*$ such that $y_2(t^*) = 0$, we have $y_1(t^*) = y_3(t^*) = y_4(t^*)$. (Hint: think about the correctness and implications of the equality $y_2(t) = \frac{d}{dt} y_1(t)$.)
Image generated by `2ndorder-underdamped-threestep.py` .

## 5.8 Appendix: Routh-Hurwitz stability tests for low-order transfer functions

The *Routh-Hurwitz stability criterion* provides a method to determine the stability of a transfer function $G(s)$ by examining the signs and values of the coefficients of the denominator of $G(s)$, that is, its characteristic polynomial. We refer for example to (DiStefano et al., 1997) for a complete treatment[1] and here we focus on low-order transfer functions.

The criterion (which ensures that all poles of $G(s)$ are in the left-half plane) is summarized for first, second, and third-order polynomials as follows:

(i) A first-order polynomial

$$P(s) = a_1 s + a_0,$$

has a zero with strictly negative real part if $a_0 > 0$ and $a_1 > 0$.

(ii) A second-order polynomial

$$P(s) = a_2 s^2 + a_1 s + a_0$$

has zeros with strictly negative real part if $a_0 > 0$, $a_1 > 0$, and $a_2 > 0$.

(iii) A third-order polynomial

$$P(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

has zeros with strictly negative real part if $a_0 > 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, and

$$a_2 a_1 - a_3 a_0 > 0. \tag{5.46}$$

For example, consider the polynomial $s^3 + 5s^2 + 6s + 1$. Clearly, every coefficient is positive and additionally: $a_2 a_1 - a_3 a_0 = 5 \cdot 6 - 1 \cdot 1 > 0$ so that the Routh-Hurwitz criterion states that the solutions of $s^3 + 5s^2 + 6s + 1 = 0$ have strictly negative real part. Indeed, the following Python code numerically computes the roots to be `-3.2469796 -1.55495813 -0.19806226`.

```
# Python code to compute numerically the roots of a polynomial
import numpy as np
```

[1]See also https://en.wikipedia.org/wiki/RouthâĂŞHurwitz_stability_criterion

```
# Define the polynomial coefficients: x^3 + 5x^2 + 6x + 1
coeffs = [1, 5, 6, 1]
# Calculate the roots
roots = np.roots(coeffs)
print("Roots:", roots)
```

# 5.9 Exercises

## Section 5.1: The transfer function and the impulse response

E5.1 **From poles to transfer function and differential equation.** The poles of a transfer function $G(s)$ are drawn in Figure 5.15.

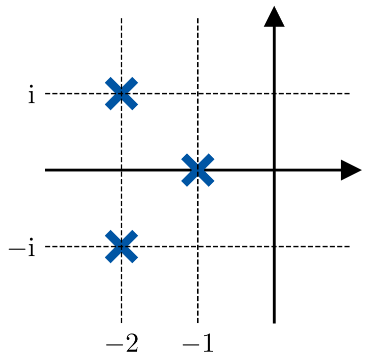



Figure 5.15: Complex plane with poles of a transfer function
The poles are at $(-2 \pm \mathrm{i})$ and $-1$.

(i) Under the additional assumption that $G(0) = 1/5$, compute the transfer function $G(s)$.

(ii) Compute the damping ratio $\zeta$ and natural frequency $\omega_{\mathrm{n}}$ for the two complex poles of $G(s)$.

(iii) Let $X(s) = G(s)U(s)$ and write the differential equation associated to $G(s)$, governing $x(t)$ as a function of $u(t)$.

E5.2 **Marginally stable systems.** Find an example of a marginally stable systems and a bounded input signal with the property that the system's output response is unbounded. Verify that the output response is unbounded.

***Answer:*** We consider the integrator, which is a marginally stable system: $G(s) = 1/s$, and the step input so that $U(s) = 1/s$.
Then the output is:

$$Y(s) = G(s) \cdot U(s) = \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}$$

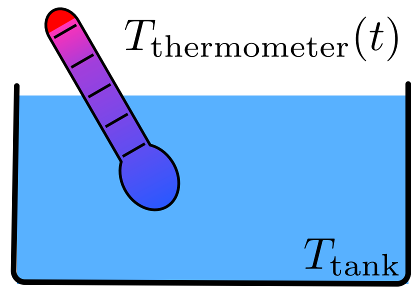
which corresponds in the time domain to:

$$y(t) = t.$$

This is unbounded, even though the input is bounded (a step input). Thus, the system has an unbounded output response to a bounded input. ▼

## Section 5.2: First-order systems and their responses

E5.3 **Thermometer transfer function and ramp response in a warming tank.** Consider a thermometer with temperature $\theta(t)$ immersed in a water tank with temperature $\theta_{\text{tank}}(t)$. Let $c$ and $r$ denote the thermal capacity of the thermometer and the tank-thermometer thermal resistance, respectively.




(i) Derive the governing equation for the system.

(ii) Take the Laplace transform of the equation you found in part (i), assuming zero initial conditions.

(iii) Compute the transfer function from $\Theta_{\text{tank}}(s) = \mathcal{L}[\theta_{\text{tank}}(t)]$ to $\Theta(s) = \mathcal{L}[\theta(t)]$.

(iv) Is this a first-order or a second-order system? If it is first order, compute the time constant. Otherwise, if it is a second-order system, compute the natural frequency and the damping ratio.

(v) Compute the step response of this system in the time domain.

(vi) Finally, assume $\theta_{\text{tank}}(t) = t$ is the unit ramp function. Compute the asymptotic value $e_{\text{steady-state}} = \lim_{t\to\infty} e(t)$, where the error $e(t) := \theta(t) - \theta_{\text{tank}}(t)$.

**Hint:** In exercise E6.1, we computed $\dfrac{1}{s^2(s\tau+1)} = -\dfrac{\tau}{s} + \dfrac{1}{s^2} + \dfrac{\tau^2}{s\tau+1}$.

***Answer:***

(i) Fourier's Law of Heat Conduction gives us $q(t) = \frac{1}{r}(\theta(t) - \theta_{\text{tank}}(t))$, and the temperature of the thermometer is governed by the equation $c\dot{\theta}(t) = -q(t)$. Combining these gives the dynamics

$$\dot{\theta}(t) \;=\; \frac{1}{cr}(\theta_{\text{tank}}(t) - \theta(t))$$

(ii) Taking the Laplace transform gives

$$s\Theta(s) \;=\; \frac{1}{cr}(\Theta_{\text{tank}}(s) - \Theta(s))$$

(iii) The transfer function is found to be

$$G(s) \;=\; \frac{\Theta(s)}{\Theta_{\text{tank}}(s)} \;=\; \frac{1}{crs+1}$$

(iv) This is a first-order system. The time constant is $\tau = cr$.

(v) To compute the step response, we need to compute the inverse Laplace transform

$$\theta(t) = \mathcal{L}^{-1}\Big[G(s)\cdot\mathcal{L}[\mathbf{1}(t)]\Big] = \mathcal{L}^{-1}\left[\frac{1}{crs+1}\cdot\frac{1}{s}\right].$$

The partial fraction expansion yields

$$\frac{1}{crs+1}\cdot\frac{1}{s} \;=\; \frac{-1}{s+\frac{1}{cr}} + \frac{1}{s}$$

so that, taking the inverse Laplace transform, we obtain

$$\theta(t) \;=\; 1 - \mathrm{e}^{-t/(cr)}$$

(vi) For a first-order system with a unit ramp input, the magnitude of the steady-state error is equal to the time constant. Based on how we have defined the error, we have $e_{\text{steady-state}} = -cr$.

Alternatively, from the hint, it is clear that $\theta(t) = -\tau + t + \tau^2\,\mathrm{e}^{-t/\tau}$. Subtracting the ramp $t$ and waiting until the decaying exponential decays, we obtain $e_{\text{steady-state}} = -\tau = -cr$.

▼

E5.4 **First-order system subject to sinusoidal input.** Consider a stable first-order system with time constant $\tau$.

(i) Write the dynamics for this system subject to a sinusoidal input $u(t) = \sin(\omega t)$.

(ii) Apply the Laplace transform and compute $X(s)$.

(iii) Write the partial fraction decomposition of $X(s)$ assuming zero initial condition $x(0) = 0$ (do not yet compute the coefficients).

(iv) Write an expression for $x(t)$ corresponding to the partial fraction expansion of $X(s)$ (do not yet compute the coefficients).

(v) Compute the coefficients for the partial fraction expansion and write an explicit expression for $x(t)$.

***Answer:***

(i) With sinusoidal input, the system dynamics are

$$\tau \dot{x} = -x + \sin(\omega t). \tag{5.47}$$

(ii) Applying the Laplace transform (with $x(0) = 0$) gives

$$\tau s X(s) = -X(s) + \frac{\omega}{s^2 + \omega^2}. \tag{5.48}$$

Rearranging:

$$X(s) = \frac{\omega}{(s^2 + \omega^2)(\tau s + 1)}. \tag{5.49}$$

(iii) The partial fraction decomposition has the form

$$X(s) = \alpha \frac{\omega}{s^2 + \omega^2} + \beta \frac{s}{s^2 + \omega^2} + \frac{\gamma}{\tau s + 1}. \tag{5.50}$$

(iv) Using known inverse Laplace transforms,

$$\mathcal{L}^{-1}\left[\frac{\omega}{s^2 + \omega^2}\right] = \sin(\omega t), \qquad \mathcal{L}^{-1}\left[\frac{s}{s^2 + \omega^2}\right] = \cos(\omega t), \qquad \mathcal{L}^{-1}\left[\frac{1}{\tau s + 1}\right] = \mathrm{e}^{-t/\tau},$$

we have

$$x(t) = \alpha \sin(\omega t) + \beta \cos(\omega t) + \gamma\, \mathrm{e}^{-t/\tau}\,. \tag{5.51}$$

(v) To determine $\alpha, \beta, \gamma$, start with

$$\frac{\omega}{(s^2 + \omega^2)(\tau s + 1)} = \alpha \frac{\omega}{s^2 + \omega^2} + \beta \frac{s}{s^2 + \omega^2} + \frac{\gamma}{\tau s + 1}. \tag{5.52}$$

Multiply through by $(s^2 + \omega^2)(\tau s + 1)$:

$$\omega = \alpha\omega(\tau s + 1) + \beta s(\tau s + 1) + \gamma(s^2 + \omega^2). \tag{5.53}$$

Comparing coefficients of like powers of $s$:

$$s^2 : \quad 0 = \beta\tau + \gamma, \tag{5.54}$$

$$s^1 : \quad 0 = \alpha\omega\tau + \beta, \tag{5.55}$$

$$s^0 : \quad \omega = \alpha\omega + \gamma\omega^2. \tag{5.56}$$

From the first two:

$$\gamma = -\beta\tau, \qquad \beta = -\alpha\omega\tau, \tag{5.57}$$

so

$$\gamma = \alpha\omega\tau^2. \tag{5.58}$$

Substituting into the constant term equation:

$$\omega = \alpha\omega + \alpha\omega^3\tau^2, \tag{5.59}$$

$$1 = \alpha(1 + \omega^2\tau^2), \tag{5.60}$$

$$\alpha = \frac{1}{1 + \omega^2\tau^2}. \tag{5.61}$$

Then

$$\beta = -\frac{\omega\tau}{1 + \omega^2\tau^2}, \qquad \gamma = \frac{\omega\tau^2}{1 + \omega^2\tau^2}. \tag{5.62}$$

Therefore

$$X(s) = \frac{\omega}{1 + \omega^2\tau^2} \cdot \frac{1}{s^2 + \omega^2} - \frac{\omega\tau}{1 + \omega^2\tau^2} \cdot \frac{s}{s^2 + \omega^2} + \frac{\omega\tau^2}{1 + \omega^2\tau^2} \cdot \frac{1}{\tau s + 1}. \tag{5.63}$$

Inverting term by term:

$$x(t) = \frac{1}{1 + \omega^2\tau^2}\left(\sin(\omega t) - \omega\tau\cos(\omega t) + \omega\tau^2\,\mathrm{e}^{-t/\tau}\right). \tag{5.64}$$

▼

## Section 5.3: Second-order systems and their responses

E5.5 **Step and free response of a critically-damped second-order system.** Consider a critically damped second-order system in canonical form.

(i) Compute the unit step response $y(t)$ analytically.

(ii) Compute the free response for the same system with initial conditions $y(0) = y_0$ and $\dot{y}(0) = v_0$.

(iii) Write Python code based on `sympy` that computes the same responses from $G(s)$ and the input, and verifies both the step response and the free response.

***Answer:***

(i) For damping ratio $\zeta = 1$, the system has transfer function

$$G(s) = \frac{\omega_\mathrm{n}^2}{s^2 + 2\omega_\mathrm{n} s + \omega_\mathrm{n}^2}. \tag{5.65}$$

With a unit step input $U(s) = \frac{1}{s}$, the output in the Laplace domain is

$$Y(s) = \frac{\omega_\mathrm{n}^2}{s(s + \omega_\mathrm{n})^2}. \tag{5.66}$$

The partial fraction decomposition gives

$$\frac{\omega_\mathrm{n}^2}{s(s + \omega_\mathrm{n})^2} = \frac{1}{s} - \frac{1}{s + \omega_\mathrm{n}} - \frac{\omega_\mathrm{n}}{(s + \omega_\mathrm{n})^2}. \tag{5.67}$$

The inverse Laplace transform yields

$$y(t) = 1 - \mathrm{e}^{-\omega_\mathrm{n} t} - \omega_\mathrm{n} t\, \mathrm{e}^{-\omega_\mathrm{n} t}. \tag{5.68}$$

Therefore, for $t \geq 0$, the step response is

$$\boxed{y(t) = 1 - (1 + \omega_\mathrm{n} t)\, \mathrm{e}^{-\omega_\mathrm{n} t}.} \tag{5.69}$$

(ii) The free response is obtained from the homogeneous equation

$$\ddot{y}(t) + 2\omega_\mathrm{n} \dot{y}(t) + \omega_\mathrm{n}^2 y(t) = 0. \tag{5.70}$$

The characteristic equation has a repeated root $s = -\omega_\mathrm{n}$, so the general solution is

$$y(t) = (A + Bt)\, \mathrm{e}^{-\omega_\mathrm{n} t}. \tag{5.71}$$

Applying $y(0) = y_0$ and $\dot{y}(0) = v_0$:

$$A = y_0, \quad B - \omega_\mathrm{n} A = v_0 \quad \Rightarrow \quad B = v_0 + \omega_\mathrm{n} y_0. \tag{5.72}$$

Therefore, for $t \geq 0$, the free response is

$$\boxed{y(t) = \big(y_0 + (v_0 + \omega_\mathrm{n} y_0) t\big)\, \mathrm{e}^{-\omega_\mathrm{n} t}.} \tag{5.73}$$

(iii) Python code to compute and verify both responses:

```python
import sympy as sp

# symbols
```

```
s, t, omega_n, y0, v0 = sp.symbols('s t omega_n y0 v0', real=True, positive=True)

# transfer function and step input
G_s = omega_n**2 / (s**2 + 2*omega_n*s + omega_n**2)
U_s = 1/s

# step response
Y_s_step = sp.simplify(G_s * U_s)
y_t_step = sp.inverse_laplace_transform(Y_s_step, s, t)
y_t_step_simplified = sp.simplify(y_t_step)
y_t_step_target = 1 - (1 + omega_n*t)*sp.exp(-omega_n*t)
check_step = sp.simplify(y_t_step_simplified - y_t_step_target)

# free response
Y_s_free = (s*y0 + v0 + 2*omega_n*y0) / (s + omega_n)**2
y_t_free = sp.inverse_laplace_transform(Y_s_free, s, t)
y_t_free_simplified = sp.simplify(y_t_free)
y_t_free_target = (y0 + (v0 + omega_n*y0)*t) * sp.exp(-omega_n*t)
check_free = sp.simplify(y_t_free_simplified - y_t_free_target)

# Output both verifications
y_t_step_simplified, check_step, y_t_free_simplified, check_free
```

▼

E5.6 **From complex conjugate poles to canonical parameters and functions of time.** Consider a function of time $x(t)$ with Laplace transform $X(s)$. The rational function $X(s)$ has two poles drawn in Figure 5.16.

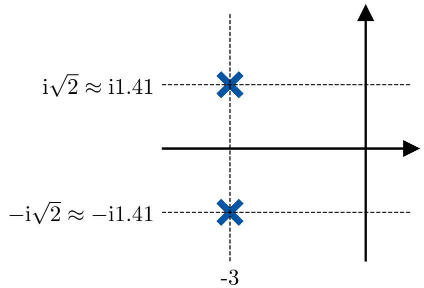



Figure 5.16: Complex plane with two complex conjugate poles at $-3 \pm \mathrm{i}\sqrt{2}$.
Recall that $s_{1,2} = -\omega_{\mathrm{n}}\big(\zeta \pm \sqrt{\zeta^2 - 1}\big)$ and that the poles belong to a semicircle of radius $\omega_{\mathrm{n}}$.

(i) Compute the damping ratio $\zeta$, natural frequency $\omega_{\mathrm{n}}$, damped natural frequency $\omega_{\mathrm{d}}$, and time constant $\tau$ for these poles.

(ii) What are the two functions of time $f_1(t)$ and $f_2(t)$ associated to the two poles? Substitute in the values of $\zeta$ and $\omega_{\mathrm{n}}$.

(iii) Assume that $x(t) = \alpha f_1(t) + \beta f_2(t)$ and that $x(0) = 0$ and $\dot{x}(0) = 10$. Write a formula for $x(t)$.

E5.7 **Pendulum-tuned mass damper.** A pendulum tuned mass damper is a device used in tall structures to reduce vibrations caused by seismic activity or wind. The simplified system in Figure 5.17 consists of a pendulum of mass $m$ and length $\ell$ attached to a structural beam that moves horizontally. The horizontal displacement of the beam is denoted by $u$, the horizontal displacement of the pendulum mass relative to the beam is denoted by $x$, and the acceleration due to gravity is $g$.

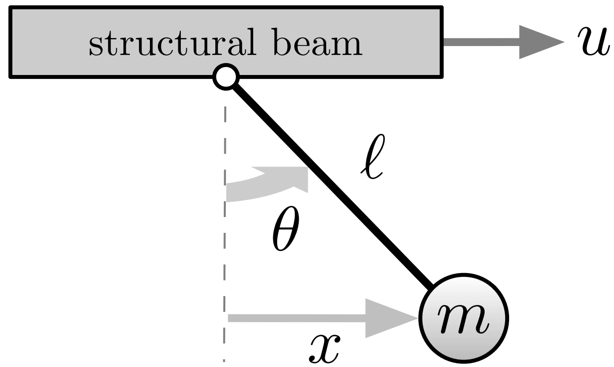



Figure 5.17: Pendulum-tuned mass damper suspended from a structural beam.

The goal is to derive a simplified model of the pendulum's horizontal motion and to characterize its input–output behavior.

(i) Show that the equation of horizontal motion for the pendulum is
$$\ddot{x} + g\tan\theta = -\ddot{u}. \tag{5.74}$$

(ii) Use the small-angle approximation to eliminate $\theta$ from the equation.

(iii) Compute the transfer function from $u$ to $x$.

(iv) Determine the impulse response $x(t)$ of the pendulum.

E5.8 **Transfer function, step response, and final value of a mass-spring-damper plus extra damper.** Consider a mass-spring-damper system (with parameters $m$, $b_1$ and $k$) connected to an additional damper (with parameter $b_2$), as illustrated in Figure 5.18. Let $z(t)$ be the position of the right-most point connected to the additional damper. At $t = 0$, a unit-step input is applied to position $z(t)$. Assume the initial conditions are $x(0) = \dot{x}(0) = z(0) = 0$.

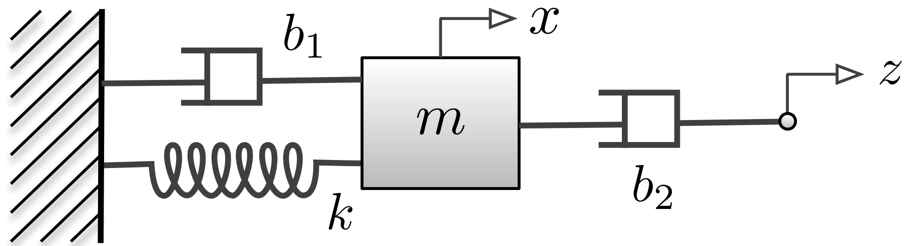



Figure 5.18: A mass-spring-damper system with extra damper

***Note:*** *Does this final value behavior of this mechanical system makes physical sense to you?*

***Answer:***

(i) The equation of motion is

$$m\ddot{x} + b_1\dot{x} + b_2(\dot{x} - \dot{z}) + kx = 0$$

so that

$$m\ddot{x} + (b_1 + b_2)\dot{x} + kx = b_2\dot{z}$$

(ii) We compute the Laplace transform with zero initial conditions:

$$\big(ms^2 + (b_1 + b_2)s + k\big)X(s) = b_2 s Z(s)$$

so that

$$\frac{X(s)}{Z(s)} = \frac{b_2 s}{ms^2 + (b_1 + b_2)s + k}$$

(iii) We now apply a unit step $Z(s) = \frac{1}{s}$ to compute

$$X(s) = \frac{b_2}{ms^2 + (b_1 + b_2)s + k}$$

and substitute in the parameter values:

$$X(s) = \frac{5}{s^2 + 10s + 50} = \frac{5}{(s+5)^2 + 5^2}$$

(iv) The inverse Laplace transform of $X(s)$ is precisely (no need to perform the partial fraction expansion in this case):

$$x(t) = \mathrm{e}^{-5t}\sin(5t)$$

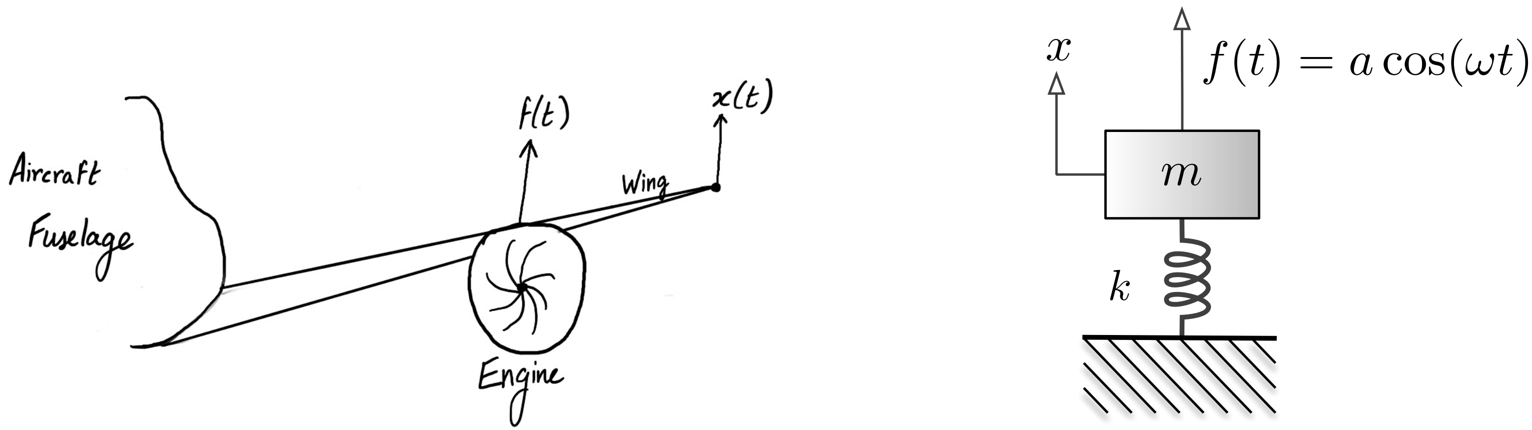
(v) Since $x(t)$ is a damped sinusoidal wave, $\lim_{t\to+\infty} x(t) = 0$.

(vi) It is immediate to see that, from the Final Value Theorem,

$$\lim_{t\to+\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{b_2 s}{ms^2 + (b_1 + b_2)s + k} = 0$$

▼

E5.9 **Oscillatory response of an aircraft wing to turbine-induced vibrations (DiStefano et al., 1997).** The structural integrity of a wing is critical in the design of turbine-driven jet aircraft. A source of failure on certain aircraft is the oscillatory nature of the vertical position of some jet turbines, as illustrated in Figure 5.19. As a




(a) Aircraft schematic showing turbine and wing structure.   (b) Spring-mass model representing the wing.

Figure 5.19: Modeling the wing oscillations caused by turbine-induced vibrations.

simple model, we ignore aerodynamic forces and consider only a sinusoidal excitation generated by the turbine:

$$f(t) = a\cos(\omega t)$$

We describe the aircraft wing as a spring-mass system (no damper), subject to the force $f(t)$, obtaining:

$$m\ddot{x} + kx = f(t).$$

Here $x$ is the position of the wing tip, $m$ is the equivalent mass of the wing, and $k$ is the spring constant related to the stiffness of the wing. In what follows, we design $k$ such that the vibration amplitude is below a threshold.

(i) Write the Laplace transform of $f(t)$, given by $F(s)$.
(ii) Assuming zero initial velocity and position, find the Laplace transform of $x(t)$ in terms of $a$, $\omega$, $m$, and $k$.
(iii) Perform a partial fraction expansion for $X(s)$: first set up the expansion and then solve for the coefficients.
(iv) Compute $x(t)$ via the inverse Laplace transform of $X(s)$.
(v) Now, let us consider a model airplane, whose mass $m = 1$ kg. Let $a = 25$ N, and $\omega = 10$ rad/s. Assuming we don't want the wing to oscillate by more than 2 m, what is the largest permissible value of $k$?
**Hint:** For the purpose of identifying the maximum value of $x(t)$, assume the difference of two cosines at different angles is bounded above by the number 2.

E5.10 **Transfer function of DC motor.** In this exercise, we compute the transfer function of the DC motor in Section 2.5. We recall the governing equations (2.47):

$$I_{\mathrm{m}}\ddot{\theta}_{\mathrm{m}}(t) + b\dot{\theta}_{\mathrm{m}}(t) = k_{\mathrm{torque}} i_{\mathrm{cond}}(t) \tag{5.75a}$$

$$\ell \frac{d}{dt} i_{\mathrm{cond}}(t) + r i_{\mathrm{cond}}(t) = v_{\mathrm{source}}(t) - k_{\mathrm{velocity}}\dot{\theta}_{\mathrm{m}}(t) \tag{5.75b}$$

and refer to Section 2.5 for the definition of all terms.

Let $\omega_{\mathrm{m}} = \dot{\theta}_{\mathrm{m}}$ be the shaft angular velocity. Use the following notation: $V_{\mathrm{source}}(s) = \mathcal{L}[v_{\mathrm{source}}(t)]$, $\Omega_{\mathrm{m}}(s) = \mathcal{L}[\omega_{\mathrm{m}}(t)]$, and $I_{\mathrm{cond}}(s) = \mathcal{L}[i_{\mathrm{cond}}(t)]$.

(i) Take the Laplace transforms of the two equations, assuming zero initial conditions and using only the shaft angular velocity (and not the shaft angle).

(ii) Compute the transfer function from the voltage source $V_{\mathrm{source}}(s)$ to the angular velocity $\Omega_{\mathrm{m}}(s)$.

(iii) What is the order of this transfer function?

(iv) Explain why the system is underdamped for large values of $k_{\mathrm{velocity}}$ and $k_{\mathrm{torque}}$.

***Answer:***

(i) We compute

$$(sI_{\mathrm{m}} + b)\Omega_{\mathrm{m}}(s) = k_{\mathrm{torque}} I_{\mathrm{cond}}(s) \tag{5.76a}$$

$$(s\ell + r) I_{\mathrm{cond}}(s) = V_{\mathrm{source}}(s) - k_{\mathrm{velocity}}\Omega_{\mathrm{m}}(s) \tag{5.76b}$$

(ii) We eliminate $I_{\mathrm{cond}}(s)$ to obtain

$$(sI_{\mathrm{m}} + b)\Omega_{\mathrm{m}}(s) = \frac{k_{\mathrm{torque}}}{s\ell + r}\Big(V_{\mathrm{source}}(s) - k_{\mathrm{velocity}}\Omega_{\mathrm{m}}(s)\Big) \tag{5.77}$$

$$\implies \quad (s\ell + r)(sI_{\mathrm{m}} + b)\Omega_{\mathrm{m}}(s) = k_{\mathrm{torque}} V_{\mathrm{source}}(s) - k_{\mathrm{torque}} k_{\mathrm{velocity}}\Omega_{\mathrm{m}}(s) \tag{5.78}$$

$$\implies \quad \frac{\Omega_{\mathrm{m}}(s)}{V_{\mathrm{source}}(s)} = \frac{k_{\mathrm{torque}}}{(s\ell + r)(sI_{\mathrm{m}} + b) + k_{\mathrm{torque}} k_{\mathrm{velocity}}} \tag{5.79}$$

(iii) This is a second-order system.

(iv) Looking at the formula for the damping ratio, the larger are $k_{\mathrm{torque}} k_{\mathrm{velocity}}$, the smaller is the damping ratio $\zeta$.

▼

E5.11 **Two-compartment chemical reactor with reaction and mixing.** Consider a two-tank chemical reactor with volumes $v_1$ and $v_2$. Let $c_1(t)$ and $c_2(t)$ be the concentrations of a certain chemical substance in tanks 1 and 2. Liquid flows at a constant rate $q$ from tank 1 to tank 2 and at the same rate from tank 2 to tank 1. Each tank is perfectly mixed, and the substance undergoes a first-order decay with rate constants $k_1, k_2 > 0$. The input $u(t)$ is the injection concentration entering tank 1 through an additional inflow at rate $q_{\text{in}}$, and the output is $y(t) = c_2(t)$. In summary, the dynamics are

$$v_1\dot{c}_1 = -qc_1 + qc_2 - k_1 v_1 c_1 + q_{\text{in}} u(t), \tag{5.80a}$$

$$v_2\dot{c}_2 = -qc_2 + qc_1 - k_2 v_2 c_2 \tag{5.80b}$$

(i) Show that for zero input $u(t) \equiv 0$ and no decay $k_1 = k_2 = 0$, the total amount of chemical substance in both tanks is constant.

(ii) Find all equilibrium points $(c_1^*, c_2^*)$ for constant $u(t) \equiv u_0$.

(iii) Derive the transfer function $G(s) = Y(s)/U(s)$.

(iv) Check the stability by locating the poles of $G(s)$ and write them explicitly.

***Answer:***

(i) For $u(t) \equiv 0$ and $k_1 = k_2 = 0$, summing the two equations gives

$$v_1\dot{c}_1 + v_2\dot{c}_2 = -qc_1 + qc_2 - qc_2 + qc_1 = 0. \tag{5.81}$$

Therefore

$$\frac{d}{dt}\big(v_1c_1 + v_2c_2\big) = 0, \tag{5.82}$$

which means the total amount of chemical substance $v_1c_1 + v_2c_2$ is constant in time.

(ii) For $u(t) \equiv u_0$ and $\dot{c}_1 = \dot{c}_2 = 0$,

$$\begin{cases} 0 = -qc_1^* + qc_2^* - k_1v_1c_1^* + q_{\text{in}}u_0, \\ 0 = -qc_2^* + qc_1^* - k_2v_2c_2^*. \end{cases} \tag{5.83}$$

Solving,

$$c_1^* = \frac{(q + k_2v_2)q_{\text{in}}u_0}{(q + k_1v_1)(q + k_2v_2) - q^2}, \quad c_2^* = \frac{q\,q_{\text{in}}u_0}{(q + k_1v_1)(q + k_2v_2) - q^2}. \tag{5.84}$$

(iii) With zero initial conditions, the Laplace-domain equations are

$$\begin{cases} (v_1s + q + k_1v_1)C_1(s) - qC_2(s) = q_{\text{in}}U(s), \\ -qC_1(s) + (v_2s + q + k_2v_2)C_2(s) = 0. \end{cases} \tag{5.85}$$

From the second equation,

$$C_1(s) = \frac{v_2s + q + k_2v_2}{q}C_2(s). \tag{5.86}$$

Substitution into the first equation gives

$$\big(v_1s + q + k_1v_1\big)\big(v_2s + q + k_2v_2\big)C_2(s) - q^2C_2(s) = q_{\text{in}}qU(s). \tag{5.87}$$

Therefore

$$G(s) = \frac{Y(s)}{U(s)} = \frac{q_{\text{in}}q}{\big(v_1s + q + k_1v_1\big)\big(v_2s + q + k_2v_2\big) - q^2}. \tag{5.88}$$

(iv) The poles solve

$$\big(v_1s + q + k_1v_1\big)\big(v_2s + q + k_2v_2\big) - q^2 = 0. \tag{5.89}$$

Expanding yields the quadratic $as^2 + bs + c = 0$ with

$$a = v_1v_2, \quad b = v_1(q + k_2v_2) + v_2(q + k_1v_1), \quad c = (q + k_1v_1)(q + k_2v_2) - q^2. \tag{5.90}$$

The three coefficients are positive, so by the Routh–Hurwitz criterion for a second-order polynomial in Appendix 5.8, both roots have negative real parts. Therefore, $G(s)$ has all poles in the open left-half plane, and the equilibrium is asymptotically stable.

▼

## Section 5.4: Higher-order systems and their step response

E5.12 **Transfer function of suspension system.** Consider the suspension system described in Section 2.2 and Figure 2.9. Recall that the equations of motion for the system were found to be:

$$\begin{aligned} m_{\mathrm{s}}\ddot{x}_{\mathrm{s}} + b\left(\dot{x}_{\mathrm{s}} - \dot{x}_{\mathrm{us}}\right) + k_{\mathrm{s}}\left(x_{\mathrm{s}} - x_{\mathrm{us}}\right) &= 0 \\ m_{\mathrm{us}}\ddot{x}_{\mathrm{us}} + b\left(\dot{x}_{\mathrm{us}} - \dot{x}_{\mathrm{s}}\right) + k_{\mathrm{s}}\left(x_{\mathrm{us}} - x_{\mathrm{s}}\right) + k_{\mathrm{w}}(x_{\mathrm{us}} - r(t)) &= 0. \end{aligned}$$

where $x_{\mathrm{s}}(t)$ is the vertical position of the sprung mass, $x_{\mathrm{us}}(t)$ the vertical position of the sprung mass, and $r(t)$ is the height of the road surface. Assume that the initial positions and velocities of both masses are equal to zero: $x_{\mathrm{s}}(0) = x_{\mathrm{us}}(0) = \dot{x}_{\mathrm{s}}(0) = \dot{x}_{\mathrm{us}}(0) = 0$

Define the Laplace transforms: $X_{\mathrm{us}}(s) = \mathcal{L}[x_{\mathrm{us}}(t)]$, $X_{\mathrm{s}}(s) = \mathcal{L}[x_{\mathrm{s}}(t)]$ and $R(s) = \mathcal{L}[r(t)]$.

(i) Using the properties of Laplace transforms, find the Laplace transforms of the two equations.

(ii) Use the two equations to eliminate the intermediate variable $X_{\mathrm{us}}(s)$ to obtain an expression for $X_{\mathrm{s}}(s)$ in terms of $R(s)$.

***Answer:***

(i) Applying the Laplace transform yields

$$m_s s^2 X_{\mathsf{s}}(s) + bsX_{\mathsf{s}}(s) - bsX_{\mathsf{us}}(s) + k_s X_{\mathsf{s}}(s) - k_s X_{\mathsf{us}}(s) = 0$$
$$m_{us} s^2 X_{\mathsf{us}}(s) + bsX_{\mathsf{us}}(s) - bsX_{\mathsf{s}}(s) + k_s X_{\mathsf{us}}(s) - k_s X_{\mathsf{s}}(s) + k_w X_{\mathsf{us}}(s) - k_w R(s) = 0.$$

Grouping like terms, we get

$$(m_s s^2 + bs + k_s)X_{\mathsf{s}}(s) - (bs + k_s)X_{\mathsf{us}}(s) = 0$$
$$(m_{us} s^2 + bs + k_s + k_w)X_{\mathsf{us}}(s) - (bs + k_s)X_{\mathsf{s}}(s) - k_w R(s) = 0$$

(ii) We solve for $X_{\mathsf{us}}(s)$ in the first equation, ans substitute this into the second equation to obtain

$$(m_{us} s^2 + bs + k_s + k_w)\frac{(m_s s^2 + bs + k_s)}{(bs + k_s)}X_{\mathsf{s}}(s) - (bs + k_s)X_{\mathsf{s}}(s) - k_w R(s) = 0.$$

Solving for $X_{\mathsf{s}}(s)$ in terms of $R(s)$ yields

$$X_{\mathsf{s}}(s) = \frac{k_w(bs + k_s)}{(m_{us} s^2 + bs + k_s + k_w)(m_s s^2 + bs + k_s) - (bs + k_s)^2}R(s)$$

▼

E5.13 **Transfer function of building system.** Recall the dynamics of the building system (without air conditioner) studied in Section 3.1:

$$\begin{aligned} c_1\dot{\theta}_1 &= \frac{1}{r_{12}}(\theta_2-\theta_1)+\frac{1}{r_{1,\text{ext}}}(\theta_{\text{ext}}-\theta_1)\\ c_2\dot{\theta}_2 &= \frac{1}{r_{12}}(\theta_1-\theta_2)+\frac{1}{r_{23}}(\theta_3-\theta_2)\\ c_3\dot{\theta}_3 &= \frac{1}{r_{23}}(\theta_2-\theta_3). \end{aligned}$$

Note that we changed notation: we let $\theta_i(t)$ denote the temperature in room $i$ and $\Theta_i(s)=\mathcal{L}[\theta_i(t)]$ be its Laplace transform. Similarly, we let $\Theta_{\text{ext}}(s)=\mathcal{L}[\theta_{\text{ext}}(t)]$.

We aim to compute the transfer function of the building system (without air conditioner) from the external temperature to the temperature in room 3. Assume all resistances and all thermal capacities are equal. Let $r_{12}=r_{23}=r_{1,\text{ext}}=r$ and $c_1=c_2=c_3=c$.

(i) Take the Laplace transforms of the three equations, assuming zero initial conditions.

(ii) Explain, in words, how to find the overall transfer function from $\Theta_{\text{ext}}(s)$ to $\Theta_3(s)$.

(iii) Compute the transfer function from $\Theta_{\text{ext}}(s)$ to $\Theta_3(s)$. You may use Matlab or Python for this if you wish, but be sure to include your code if you choose to do so.

(iv) What is the order of this transfer function?

(v) Using the Routh-Hurwitz criterion in Appendix 5.8, verify that the system is stable.

E5.14 **Transfer function of a (spring-mass)$^2$-damper system.** Consider the system composed of two masses, two springs and a damper in Figure 5.20. As usual, let $X_1(s) = \mathcal{L}[x_1(t)]$, $X_2(s) = \mathcal{L}[x_2(t)]$, and $Y(s) = \mathcal{L}[y(t)]$.

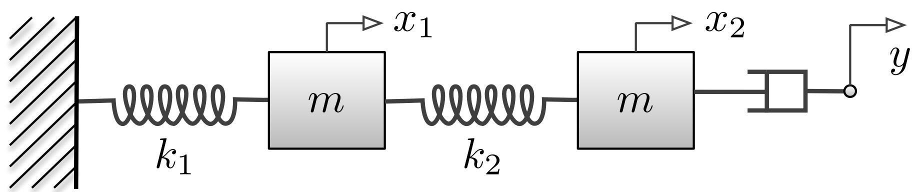



Figure 5.20: A system of two masses interconnected by springs and a damper

(i) Derive the equations of motion for this system.

(ii) Take the Laplace transforms of the equations you derived in part (i), assuming zero initial conditions.

(iii) Compute the transfer function from $Y(s)$ to $X_1(s)$.

(iv) What is the order of this transfer function?

***Answer:***

(i) Using free-body diagrams, the equations of motion for this system are

$$\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 &= k_2 x_2 \\ m_2\ddot{x}_2 + b\dot{x}_2 + k_2 x_2 &= k_2 x_1 + b\dot{y}\,. \end{aligned}$$

(ii) Taking the Laplace transforms of the two equations yields

$$\begin{aligned} (m_1 s^2 + k_1 + k_2)X_1(s) &= k_2 X_2(s) \\ (m_2 s^2 + bs + k_2)X_2(s) &= k_2 X_1(s) + bsY(s)\,. \end{aligned}$$

(iii) Solving the first equation for $X_2(s)$, substituting into the second equation, and solving for the transfer function $G(s) = X_1(s)/Y(s)$ yields

$$G(s) = \frac{k_2 b s}{(m_2 s^2 + bs + k_2)(m_1 s^2 + k_1 + k_2) - k_2^2}$$

(iv) The order of this transfer function is $4$.

▼

E5.15 **Transfer function of a suspended mass-spring$^2$-damper system.** Consider the suspended mass-spring-damper system shown in Figure 5.21. The positions $x$ and $y$ are measured from the initial equilibrium position. Let the force $f(t)$ be the input to the system and let the displacement $x(t)$ be the output. Assume zero initial conditions and *ignore the force of gravity on the mass*.

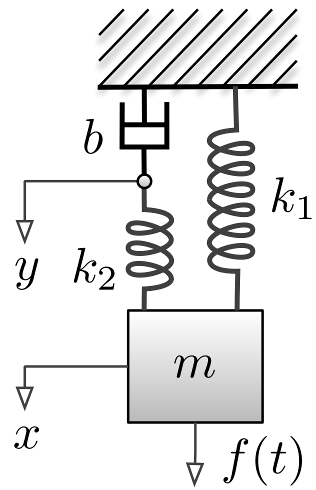



Figure 5.21: Suspended mass-spring-damper system with forcing input $f(t)$. Ignore gravity.

(i) Write down the equations of motion for the system.
(ii) Compute the Laplace transform of the equations of motion
(iii) Obtain the input-to-output transfer function of the system. Your expression should be the ratio of two polynomials. *Be sure to fully expand both the numerator and denominator polynomials*.

***Answer:***

(i) The equations of motion for the system are:

$$m\ddot{x} + k_1 x + k_2(x - y) = f(t), \tag{5.91}$$

$$k_2(x - y) = b\dot{y}. \tag{5.92}$$

(ii) The Laplace transforms (at zero initial condition) of both equation are:

$$\left(ms^2 + k_1 + k_2\right) X(s) = k_2 Y(s) + F(s), \tag{5.93}$$

$$k_2 X(s) = (k_2 + bs)\, Y(s). \tag{5.94}$$

(iii) The input is $f(t)$ and the output is $x(t)$. Therefore, we want a transfer function from $F(s)$ to $X(s)$. We can solve equation (5.94) for $Y(s)$ and substitute into equation (5.93) to obtain

$$(k_2 + bs)\left(ms^2 + k_1 + k_2\right) X(s) = k_2^2 X(s) + (k_2 + bs)\, F(s) \tag{5.95}$$

Finally, we solve for the transfer function $X(s)/F(s)$:

$$\frac{X(s)}{F(s)} = \frac{k_2 + bs}{mbs^3 + mk_2 s^2 + (k_1 + k_2)bs + k_1 k_2}. \tag{5.96}$$

▼

E5.16 **A slender beam discretized into $n$ rotary segments: from base torque to tip angle.** A slender beam is modeled as $n$ rigid segments joined by torsional springs of stiffness $k$ and rotary dampers of coefficient $b$. Each segment has moment of inertia $j$ about its joint axis. Let $\theta_i(t)$ denote the small angular displacement of segment $i$ for $i \in \{1, \dots, n\}$. The base is fixed, $\theta_0(t) = 0$. A torque input $u(t)$ is applied at the base joint onto segment 1. The tip is free, with no spring or damper to ground at segment $n$. The goal is to obtain and study the transfer function from base torque to tip angle:

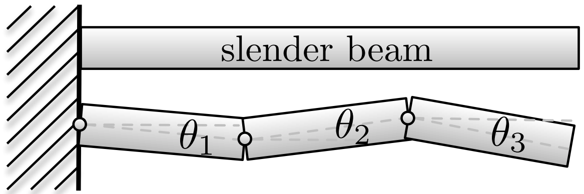



Figure 5.22: Discretized slender beam: each interconnection between two rotatory segment is described by a torsional spring of stiffness $k$ and a rotary damper of coefficient $b$.

$$G_n(s) = \frac{\Theta_n(s)}{U(s)},$$

where $\Theta_i(s) = \mathcal{L}[\theta_i(t)]$ and $U(s) = \mathcal{L}[u(t)]$.

**Hint:** You might find the following shorthand convenient. Define the operator $K(\partial_t) = k + b\partial_t$, which represents the combined effect of each segments' spring and damper (in other words, $K$ represents the mechanical impedance of the spring–damper element). For any time-dependent quantity $\phi(t)$,

$$K(\partial_t)\big(\phi(t)\big) = k\phi(t) + b\dot{\phi}(t),$$

and, in the Laplace domain (with zero initial conditions),

$$K(s) = k + bs \qquad \text{so that} \qquad \mathcal{L}[K(\partial_t)\big(\phi(t)\big)] = K(s)\Phi(s).$$

**Hint:** This model is called a nearest-neighbor chain (discrete Laplacian) with a fixed base and a free tip, i.e., Dirichlet boundary condition at $i = 0$ and Neumann boundary condition at $i = n$. This model corresponds to a torsional rod (or a lumped torsional spring–damper chain), not to a full Euler–Bernoulli bending-beam.

(i) *Time domain modeling* For $n \geq 3$, write ODEs regulating the angular displacement of each segment. Specifically, write the equation for the base segment $i = 1$, the recursive equations for the intermediate segments $i \in \{2, \dots, n-1\}$, and the equation for the tip segment $i = n$.

(ii) *Laplace domain modeling* Compute the Laplace transform at zero initial conditions and show:

$$js^2\Theta_1(s) = -K(s)\big(2\Theta_1(s) - \Theta_2(s)\big) + U(s), \tag{5.97a}$$

$$js^2\Theta_i(s) = -K(s)\big(2\Theta_i(s) - \Theta_{i-1}(s) - \Theta_{i+1}(s)\big), \qquad \text{for } i \in \{2, \dots, n-1\} \tag{5.97b}$$

$$js^2\Theta_n(s) = -K(s)\big(\Theta_n(s) - \Theta_{n-1}(s)\big). \tag{5.97c}$$

(Note that these are $n$ equations in $n+1$ variables $\Theta_1(s), \dots, \Theta_n(s)$ and $U(s)$, and that it is possible to obtain an expression for $G_n(s) = \frac{\Theta_n(s)}{U(s)}$. This requires recursive computations and Chebyshev polynomials. We do not pursue this approach here.)

(iii) *Compact matrix form (optional).* Stack $x(t) = (\theta_1(t), \dot{\theta}_1(t), \ldots, \theta_n(t), \dot{\theta}_n(t))^\mathsf{T}$. Write the state equations $\dot{x}(t) = Ax(t) + Bu(t)$ and the output $y(t) = \theta_n(t) = Cx(t)$. Write the transfer function $G_n(s) = C(sI - A)^{-1}B$ and identify the tridiagonal stiffness and damping structures inside $A$.

***Answer:***

(i) The torques from the springs and dampers on segment $i$ due to joint $(i-1, i)$ are $k(\theta_{i-1}(t) - \theta_i(t))$ and $b(\dot{\theta}_{i-1}(t) - \dot{\theta}_i(t))$, and analogously for joint $(i, i+1)$. Summing torques on the base segment 1 gives

$$j\ddot{\theta}_1(t) = -k\big(2\theta_1(t) - \theta_2(t)\big) - b\big(2\dot{\theta}_1(t) - \dot{\theta}_2(t)\big) + u(t). \tag{5.98}$$

For each intermediate segment $i \in \{2, \dots, n-1\}$,

$$j\ddot{\theta}_i(t) = -k\big(2\theta_i(t) - \theta_{i-1}(t) - \theta_{i+1}(t)\big) - b\big(2\dot{\theta}_i(t) - \dot{\theta}_{i-1}(t) - \dot{\theta}_{i+1}(t)\big). \tag{5.99}$$

For the free tip segment,

$$j\ddot{\theta}_n(t) = -k\big(\theta_n(t) - \theta_{n-1}(t)\big) - b\big(\dot{\theta}_n(t) - \dot{\theta}_{n-1}(t)\big). \tag{5.100}$$

Adopting the shorthand for the operator $K$, we get:

$$j\ddot{\theta}_1(t) = -K(\partial_t)\big(2\theta_1(t) - \theta_2(t)\big) + u(t), \tag{5.101a}$$

$$j\ddot{\theta}_i(t) = -K(\partial_t)\big(2\theta_i(t) - \theta_{i-1}(t) - \theta_{i+1}(t)\big), \qquad \text{for } i \in \{2, \dots, n-1\}, \tag{5.101b}$$

$$j\ddot{\theta}_n(t) = -K(\partial_t)\big(\theta_n(t) - \theta_{n-1}(t)\big). \tag{5.101c}$$

(ii) The equations (5.97) are an immediate consequence of the time-domain equations (5.101).

(iii) In matrix formulations, we define

$$M = jI_n \tag{5.102}$$

as the $n \times n$ diagonal mass matrix,

$$K_{\text{tri}} = \begin{bmatrix} 2k & -k & 0 & \dots & 0 \\ -k & 2k & -k & \dots & 0 \\ 0 & -k & 2k & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -k \\ 0 & \dots & 0 & -k & k \end{bmatrix}, \qquad D_{\text{tri}} = \begin{bmatrix} 2b & -b & 0 & \dots & 0 \\ -b & 2b & -b & \dots & 0 \\ 0 & -b & 2b & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -b \\ 0 & \dots & 0 & -b & b \end{bmatrix}, \tag{5.103}$$

as the $n \times n$ tridiagonal stiffness and damping matrices for a chain of $n$ rotational degrees of freedom with the base fixed and the tip free. With these definitions, the state equations are

$$\dot{x}(t) = \begin{bmatrix} 0 & I_n \\ -M^{-1}K_{\text{tri}} & -M^{-1}D_{\text{tri}} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ M^{-1}b_f \end{bmatrix} u(t), \qquad y(t) = \theta_n(t) = Cx(t), \tag{5.104}$$

where $b_f \in \mathbb{R}^n$ is the input force vector, and

$$C = [0, \dots, 0, 1, 0] \tag{5.105}$$

selects the angular displacement $\theta_n$ of the last joint. The transfer function from $u$ to $y$ is

$$G_n(s) = C(sI - A)^{-1}B \tag{5.106}$$

and its poles are illustrated in Figure 5.23.

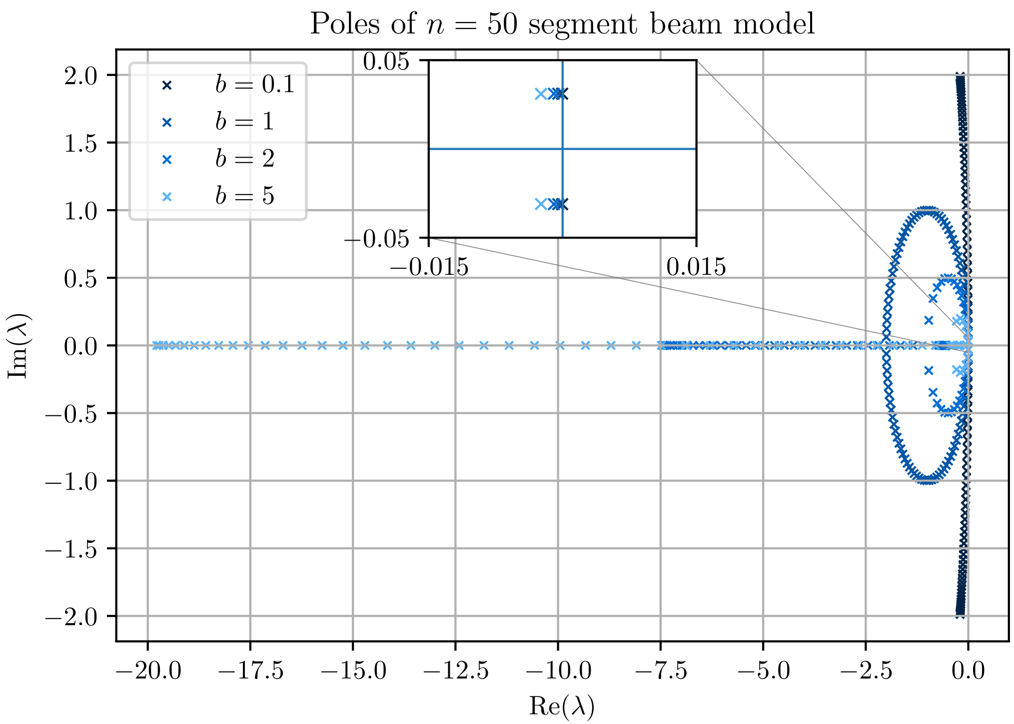



Figure 5.23: Complex plane with poles of the transfer function for a slender beam discretized into $n = 50$ segments. Each segment has moment of inertia $j = 1.0$, each spring has stiffness $k = 1.0$, and each rotary damper has damping coefficient $b$ selected in a range of values. The inset shows a zoom near the origin, confirming that all poles lie strictly in the open left half-plane (for all values of $b$).

▼

# Bibliography

J. J. DiStefano, , A. R. Stubberu, and I. J. Williams. *Schaum's Outline of Feedback and Control Systems*. McGraw-Hill, 2 edition, 1997.

S. Fuller, B. Greiner, J. Moore, R. M. Murray, R. van Paassen, and R. Yorke. The Python Control Systems library (python-control). In *IEEE Conf. on Decision and Control*, pages 4875–4881, 2021. doi.