UC Santa Barbara, Department of Mechanical Engineering
ME103 Dynamical Systems. Chapter Slides.

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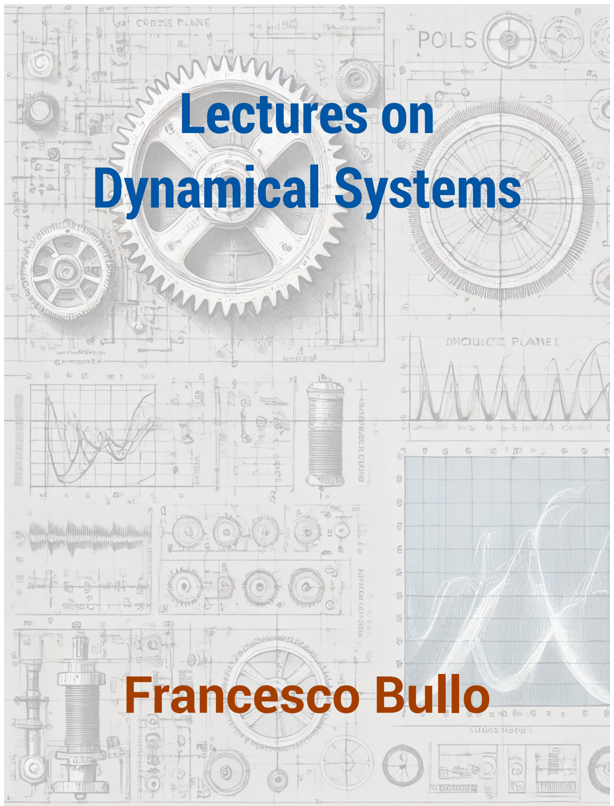
http://motion.me.ucsb.edu/ME103-Fall2025/syllabus.html



Contents

Chapter 5

The Transfer Function and Time Responses of Dynamical Systems

In this chapter, we introduce the fundamental concepts and analytical tools used to study dynamical systems, focusing on the role of the *transfer function* in relating inputs to outputs. We begin by showing how systems described by differential equations can be analyzed with the Laplace transform, which expresses dynamics in the frequency domain. System behavior and stability are determined by the poles of the transfer function in the complex plane, while the *impulse response* provides both the transfer function and insight into responses to different inputs.

The chapter progresses to specific system orders, starting with *first-order systems*, which are defined by a single state variable and characterized by their *time constant* τ. These systems exhibit stability with a real pole in the left half-plane, and their response to impulses, steps, and ramps is derived using inverse Laplace transforms. We then consider *second-order systems*, which involve two state variables and are exemplified by mass-spring-damper systems. These systems are described by parameters such as the *natural frequency* ω_{n} and *damping ratio* ζ, which dictate their response types. The step response of underdamped systems is particularly important for control system design, providing metrics like rise time and overshoot.

Higher-order systems are also discussed, with a focus on their *step response* and the influence of *dominant poles* on transient behavior. The steady-state gain, determined by the transfer function at zero frequency, is a crucial aspect of these systems.

The chapter concludes with appendices on specific topics such as the behavior of underdamped systems with zeros in different half-planes and the application of the *Routh-Hurwitz stability criterion* for assessing stability. This criterion provides a systematic approach to ensure that all roots of a characteristic polynomial are in the left half-plane, confirming system stability.

5.1 The transfer function and the impulse response

We consider a dynamical system with state $y(t)$ and input $u(t)$ in the form:

$$a_0 y(t) + a_1 \frac{dy}{dt}(t) + \cdots + a_n \frac{d^n y}{dt^n}(t) \;=\; b_0 u(t) + b_1 \frac{du}{dt}(t) + \cdots + b_m \frac{d^m u}{dt^m}(t) \qquad (5.1)$$

where

- $y(t)$ is the *output*, or *response*,
- $u(t)$ is the *input* applied to the system,
- $a_0, \ldots, a_n$ and $b_0, \ldots, b_m$ are constant coefficients.

in general $n \neq m$ // $n \geq m$

In this chapter we are mostly interested in the *forced response* where *all initial conditions are zero*. In this case, the response depends only upon the input:

$$\frac{d^i y}{dt^i}(0) = 0 \qquad \text{for } i = 0, 1, \ldots, n-1,$$

$$\frac{d^j u}{dt^j}(0) = 0 \qquad \text{for } j = 0, 1, \ldots, m-1.$$

Since the initial conditions are zero, the derivative property (P2) states $\mathcal{L}\left[\frac{d}{dt}y(t)\right] = sY(s)$ and $\mathcal{L}\left[\frac{d}{dt}u(t)\right] = sU(s)$. Taking the Laplace transform of left and right hand side of (5.1), we obtain:

$$\Big(a_0 + a_1 s + \cdots + a_n s^n\Big) Y(s) \;=\; \Big(b_0 + b_1 s + \cdots + b_m s^m\Big) U(s). \qquad (5.2)$$

same information:

1. $a_0 y + \cdots + a_n \frac{dy^n}{dt^n} = b_0 u + \cdots + b_m \frac{du^m}{dt^m}$
2. $(a_0, \ldots, a_n)$ and $(b_0, \ldots, b_m)$
3. $G(s) = (b_0 + b_1 s + \cdots + b_m s^m) / (a_0 + a_1 s + \cdots + a_n s^n)$

rational function

The *transfer function* of the control system is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n} = \left.\frac{\mathcal{L}[\text{output}]}{\mathcal{L}[\text{input}]}\right|_{\text{zero initial conditions}} \tag{5.3}$$

In other words, we have the *multiplication formula*

$$Y(s) = G(s)U(s) \tag{5.4}$$

Note:

- This result is simple to remember: in the Laplace domain,

$$\text{output} = \text{transfer function} \times \text{input} \tag{5.5}$$

- If $G(s)$ and $U(s)$ are *rational* functions, then also $Y(s)$ is a *rational* function.
- If $u(t)$ is an exponential signal (as in the Laplace transform Tables 4.2 and 4.3) and the ODE is linear, then also $y(t)$ is an exponential signal.
- Here are some simple examples (where k is a constant):

  (i) $y(t) = ku(t)$ implies $G(s) = k$, constant t.f.
  (ii) $y(t) = k\dot{u}(t)$ implies $G(s) = ks$, and derivative t.f.
  (iii) $y(t) = k \int_0^t u(\sigma)d\sigma$ implies $G(s) = \frac{k}{s}$. integral t.f.

Remarks 5.1. *Here are some comments and extensions.*

(i) Systems of the form (5.1) are said to be linear, because the input and state appear linearly, and time-invariant, because the coefficients are assumed constant, that is, time invariant.

(ii) The transfer function $G(s)$ is equivalent to the ODE model (5.1), in the sense that $G(s)$ contains the same information as the ODE model, i.e., the coefficients $a_0, \dots, a_n$ and $b_0, \dots, b_m$.

(iii) Many different physical systems may have the same transfer function. Therefore, it makes sense to define and study canonical systems, e.g., first-order, second-order, etc.

Stable and marginally stable dynamical systems

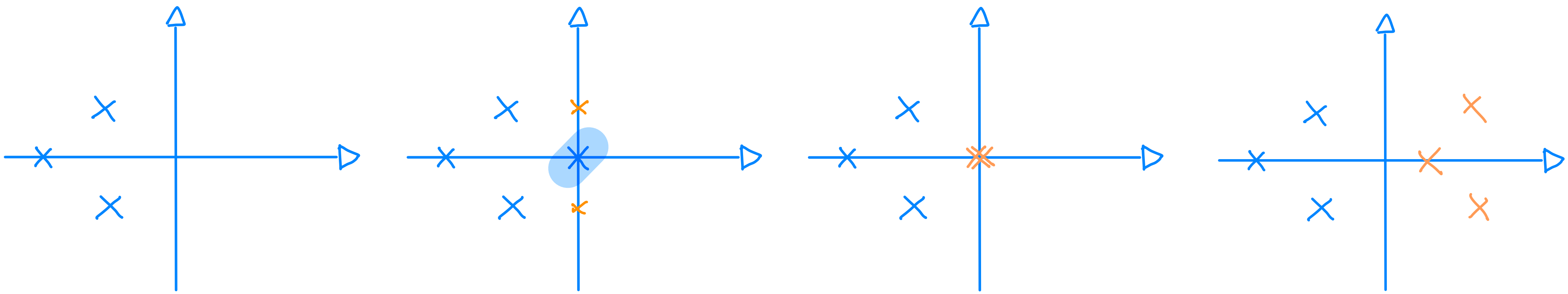
$G(s)$ has a pole s^* when $\text{den}(s^*)=0$.

A linear time-invariant system with a transfer function $G(s)$ is

- *stable* when all poles of $G(s)$ are in the strict left half plane,
- *unstable* when at least one pole of $G(s)$ lies in the strict right half plane,
- *marginally stable* when
  - all poles of $G(s)$ are in the strict half plane or on the imaginary axis,
  - the poles of $G(s)$ on the imaginary axis (~~if any~~) are not repeated.

$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1(t)$ bounded

$G(s) = \frac{1}{s^2}$; $\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$ unstable



(a) *Stable system*: All poles lie in the left half of the complex plane. This ensures each system mode decays exponentially over time, resulting in bounded output for any bounded input.

(b) *Marginally stable system*: Each pole lies in the left half plane or on the imaginary axis. No poles on the imaginary axis are repeated. As illustrated in Exercise E5.2, there exist marginally stable systems and bounded inputs such that the output response is unbounded.

(c) ~~*Marginally*~~ *unstable system*: A repeated pole exists on the imaginary axis and every other pole lies in the left half plane. Although all poles are in the left half plane or on the axis, the repeated pole causes unbounded responses.

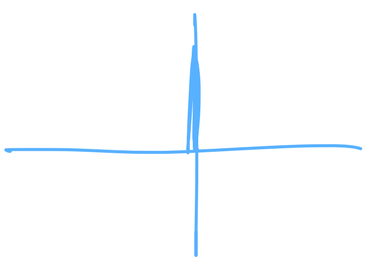
(d) *Unstable system*: At least one pole lies in the right half plane. This leads to exponentially growing system modes, making the output unbounded even for bounded inputs.

Canonical transfer functions and canonical inputs

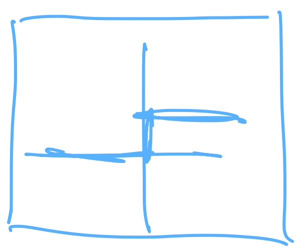
In this and the next chapter we study the responses of canonical systems (i.e., canonical transfer functions) to canonical inputs.

transfer function:	canonical form	impulse response, step response, and ramp response	frequency response (i.e., response to a sinusoidal input)
first order:	$\dfrac{1}{\tau s + 1}$	Section 5.2	Chapter 6
second order:	$\dfrac{\omega_\mathrm{n}^2}{s^2 + 2\zeta\omega_\mathrm{n} s + \omega_\mathrm{n}^2}$	Section 5.3	Chapter 6
higher order:	no typical form	Section 5.4	Chapter 6

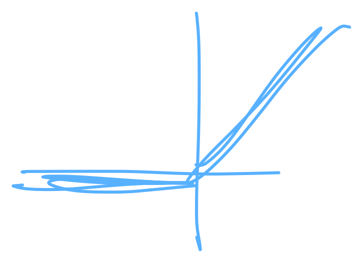
Table 5.1: Transfer functions for canonical systems. Their responses to canonical inputs are discussed in this chapter and the next.



impulse



step



ramp

Responses to canonical inputs: impulse, step, and ramp

Given a transfer function $G(s)$, we wish to compute how the system responds to *canonical inputs*. Specifically, we consider:

impulse response: the response $y_{\text{impulse}}(t)$ from zero initial condition when the input $u(t) = \delta(t)$ is a unit impulse,

step response: the response $y_{\text{step}}(t)$ from zero initial condition when the input $u(t) = \mathbf{1}(t)$ is a unit step, and

ramp response: the response $y_{\text{ramp}}(t)$ from zero initial condition when the input $u(t) = t \cdot \mathbf{1}(t)$ is a unit ramp.

These canonical input have a very simple physical intuition: in a mechanical example, the impulse corresponds to a hammer hitting a nail, the step corresponds to a constant force applied to a vehicle (like in the car velocity system), and the ramp corresponds to a growing signal with constant (like a thermometer in a tank that is warming up).

From the Laplace transform Table 4.2 recall that $\mathcal{L}[\delta(t)] = 1$, $\mathcal{L}[\mathbf{1}(t)] = \frac{1}{s}$, and $\mathcal{L}[t] = \frac{1}{s^2}$ so that, from $Y(s) = G(s)U(s)$,

$$Y_{\text{impulse}}(s) = \mathcal{L}[y_{\text{impulse}}(t)] = G(s) \tag{5.6}$$

$y_{\text{impulse}}(t) = \mathcal{L}^{-1}[G(s)]$.

$$Y_{\text{step}}(s) = \mathcal{L}[y_{\text{step}}(t)] = \frac{1}{s}G(s) \tag{5.7}$$

$$Y_{\text{ramp}}(s) = \mathcal{L}[y_{\text{ramp}}(t)] = \frac{1}{s^2}G(s) \tag{5.8}$$

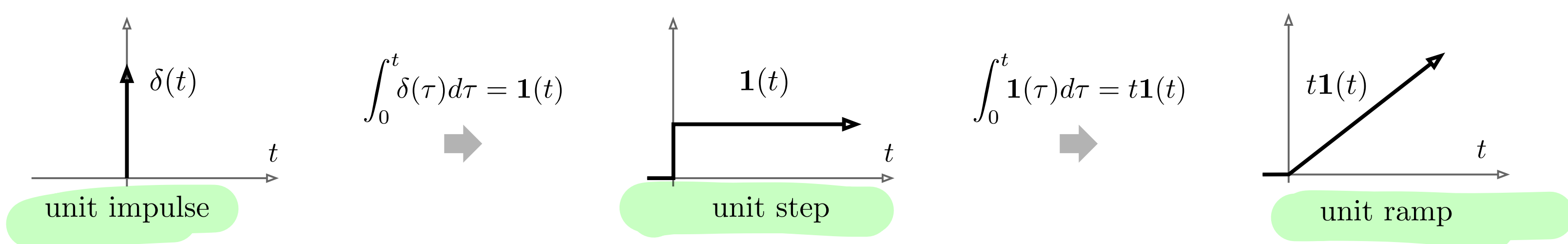


Figure 5.1: Unit impulse, unit step, and unit ramp functions

The impulse response

Therefore, the impulse response is

$$Y_{\text{impulse}}(s) = \mathcal{L}[y_{\text{impulse}}(t)] = G(s) \tag{5.9}$$

This simple equation has a surprising implication. Taking the inverse Laplace transform of both left and right hand side we obtain:

$$y_{\text{impulse}}(t) = \mathcal{L}^{-1}[G(s)] = g(t) \tag{5.10}$$

where, following our convention, we use $g(t)$ denote the function of time whose Laplace transform is $G(s)$.

We have learned:

(i) *the Laplace transform of the impulse response is the transfer function*,

(ii) to learn the transfer function of an unknown system, *(1) apply an impulse and (2) take the Laplace transform of the response*

(iii) *the impulse response contains all information about the input/output control system*

Note: the following representations are all equivalent:

(i) two vectors of coefficients $a_0, \dots, a_n$ and $b_0, \dots, b_m$,

(ii) the differential equation (5.1), $a_0 y(t) + a_1 \dot{y}(t) + \dots + a_n \frac{dy^n}{dt^n} = b_0 u + b_1 + \dots b_m \frac{du^m}{dt^m}$

(iii) the transfer function $G(s) = \dfrac{b_0 + b_1 s + \cdots + b_n s^m}{a_0 + a_1 s + \cdots + a_n s^n}$, and

(iv) the impulse response $y_{\text{impulse}}(t) = \mathcal{L}^{-1}[G(s)]$

5.1.1 Detour: The impulse response in vehicle dynamics and audio system analysis

Remark 5.2 (The impulse response in acoustics). *In the field of acoustics and audio engineering, measuring the impulse response of a room (like a concert hall or a living room) or an audio system (like a speaker or a microphone) is very useful. Measuring impulse response is the first step towards optimizing them for audio quality and thereby designing audio-related products and technologies.*

In the context of acoustics, the impulse response is the sound received at a specific location B in response to a brief large-magnitude input signal at location A.

Sound Quality Assessment: By analyzing the impulse response, engineers can determine the reverberation characteristics of a room. This helps in assessing how sound is reflected and absorbed, affecting the quality of audio heard in the space.

Speaker and Microphone Design: Understanding the impulse response of speakers and microphones allows designers to optimize their products for clarity, frequency response, and distortion characteristics.

Audio Mixing and Mastering: In music production, the impulse response of different spaces (like concert halls, studios, etc.) can be used to digitally simulate how music would sound in those environments.

Noise Reduction and Echo Cancellation: In telecommunications, the impulse response of devices and environments helps in developing algorithms for noise reduction and echo cancellation.

5.2 First-order systems and their responses

In this and the next section we study examples of canonical transfer functions and their responses. We start with the canonical form of the transfer function of first order systems.

We recall from Section 2.1.1 that a first-order system is a dynamical system in which *one variable* is required and sufficient to describe the storage of position (linear or angular), velocity (or momentum), energy, mass, etc. As illustrated in Figure 5.2, examples of first order systems include:

(i) the linear growth/decay model (1.1),
(ii) the car velocity system (2.4),
(iii) the RC circuit (2.44) (and any electric circuit where energy storage is one capacitor or one inductor),
(iv) the thermal dynamics (3.4) of a thermometer (or of any single body with uniform temperature), and
(v) the height dynamics (3.15) of a water tank.

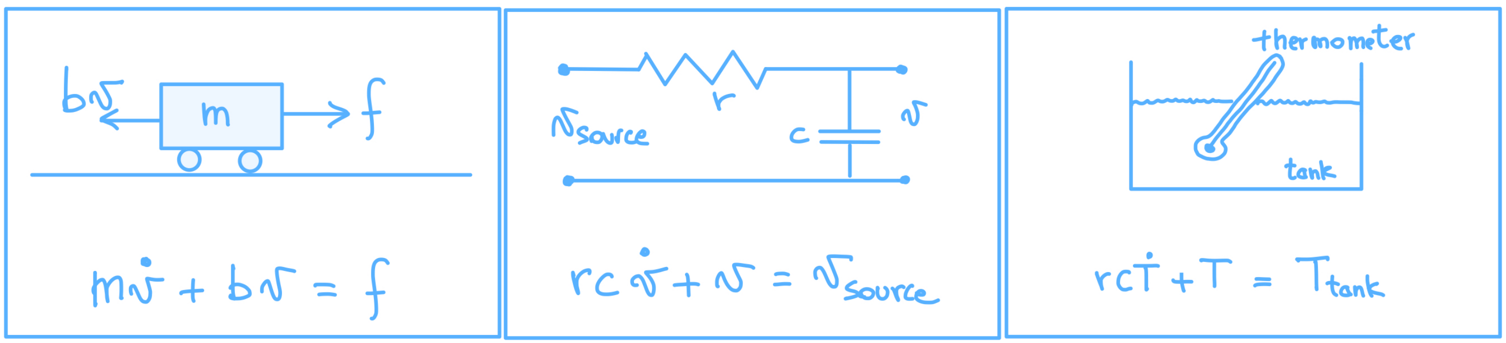


Figure 5.2: Illustrations of first order systems from earlier and later chapters.

We revisit and slightly expand the discussion on unforced first-order systems in Section 2.1.1. Given a time constant $\tau > 0$, the *canonical form of a first order system* is

$$\tau\dot{y}(t) + y(t) = u(t) \quad \longrightarrow \quad \tau s Y(s) + Y(s) = U(s) \tag{5.11}$$

where, as usual, $u(t)$ and $y(t)$ are the input and output of the system. The transfer function is

$(\tau s + 1)Y(s) = U(s)$

$\tau s + 1 = 0 \Leftrightarrow s = -\frac{1}{\tau}$

$$G_{\text{first-order}}(s) = \frac{Y(s)}{U(s)} = \frac{1}{\tau s + 1} \tag{5.12}$$

$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{\tau s + 1}$

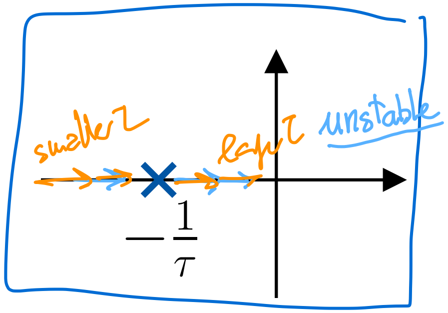


Figure 5.3: The transfer function (5.12) of a first order system has a single real pole at $s = -1/\tau$.
Since $\tau > 0$ is always positive, the pole is always on the strict left half plane.
When the time constant τ increases, the pole $s = -1/\tau$ moves towards the imaginary axis and the system response (both free and forced) becomes slower.

larger τ $\Rightarrow$ slower response

$y_{\text{impulse}}(0) = \frac{1}{\tau}$

Via the inverse Laplace transform methods, we compute the impulse, step, and ramp response of a first-order system to be:

$$y_{\text{impulse}}(t) = \mathcal{L}^{-1}\left[\frac{1}{s\tau + 1}\right] = \frac{1}{\tau}\mathrm{e}^{-t/\tau} \tag{5.13}$$

$$y_{\text{step}}(t) = \mathcal{L}^{-1}\left[\frac{1}{s\,(s\tau + 1)}\right] = 1 - \mathrm{e}^{-t/\tau} \tag{5.14}$$

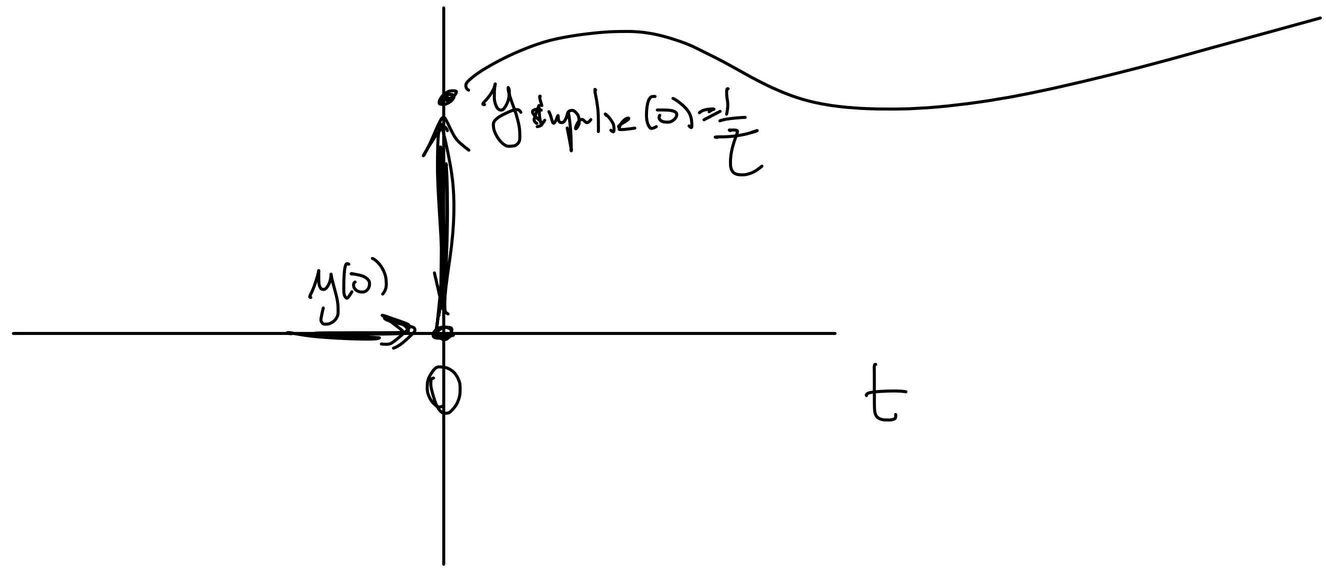
$$y_{\text{ramp}}(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2\,(s\tau + 1)}\right] = t - \tau(1 - \mathrm{e}^{-t/\tau}) \tag{5.15}$$

when $t \geq 5\tau$
$e^{-t/\tau} \approx 0$
$y_{\text{step}}(t) \simeq 1.$

These calculations are left to Exercise E4.4.

$t \geq 5\tau$

$y_{\text{ramp}}(t) \simeq t - \tau$


$y_{\text{impulse}}(0) = \frac{1}{\tau}$
$y(0)$
0
t

Impulse, step and ramp responses of first-order systems

```python
import numpy as np; import matplotlib.pyplot as plt; import control as ctrl
plt.rcParams.update({"text.usetex": True, "font.family": "serif", "font.serif": ...
    ["Computer Modern Roman"], "font.size": 16 })

# Define a range of time constants and time range for the simulation
time_constants = [10, 8, 5, 3, 2, 1, 0.5]
t = np.linspace(0, 25, 1000)

# Define your preferred color vector
colors = ['#752d00', '#a43e00', '#d35000', '#ff6100', '#ff8800', '#ffaf00', '#ffcc00']

# Initialize the figure for impulse, step, and ramp responses
fig, axs = plt.subplots(3, 1, figsize=(10, 10))

# Loop through each time constant and plot the impulse, step, and ramp responses
for idx, tau in enumerate(time_constants):
    # Define the transfer function of the first-order system
    num = [1]
    den = [tau, 1]
    system = ctrl.TransferFunction(num, den)

    # Compute and plot the impulse response
    t_impulse, y_impulse = ctrl.impulse_response(system, T=t)
    axs[0].plot(t_impulse, y_impulse, label=f'$\\tau = {tau}$', color=colors[idx])

    # Compute and plot the step response
    t_step, y_step = ctrl.step_response(system, T=t)
    axs[1].plot(t_step, y_step, label=f'$\\tau = {tau}$', color=colors[idx])

    # Compute and plot the ramp response
    ramp_input = t
    t_ramp, y_ramp = ctrl.forced_response(system, T=t, U=ramp_input)
    axs[2].plot(t_ramp, y_ramp, label=f'$\\tau = {tau}$', color=colors[idx])

# Add labels, legends, grid, and set xlim for all subplots
for ax in axs:
    ax.legend(fontsize=14); ax.grid(True); ax.set_xlim(0, 25); ax.set_ylabel('Response')
axs[0].set_xlim(0, 5);  axs[0].set_xticklabels([]); axs[1].set_xticklabels([]);

# Add arrow with text
arrow_color = '#0055A4'
text_color = arrow_color

axs[0].text(0.5, 0.5, "increasing $\\tau$", ha="center", va="center", rotation=45, ...
     size=15, color=text_color, bbox=dict(boxstyle="larrow,pad=0.3", fc="none", ...
     ec=arrow_color, lw=2, alpha=0.5))
axs[1].text(2.5, 0.75, "increasing $\\tau$", ha="center", va="center", rotation=-45, ...
     size=15, color=text_color, bbox=dict(boxstyle="rarrow,pad=0.3", fc="none", ...
     ec=arrow_color, lw=2, alpha=0.5))
axs[2].text(15, 12, "increasing $\\tau$", ha="center", va="center", rotation=-45, ...
     size=15, color=text_color, bbox=dict(boxstyle="rarrow,pad=0.3", fc="none", ...
     ec=arrow_color, lw=2, alpha=0.5))

# Plot unit step and unit ramp in gray
axs[1].plot(t, np.ones_like(t), label='Unit Step', color='gray', linestyle='--')
axs[2].plot(t, t, label='Unit Ramp', color='gray', linestyle='--')

axs[0].set_title(r'Impulse, step, and ramp response of 1st order system');
# axs[1].set_title(r'Step response of 1st order system, for varying time constant $\tau$');
# axs[2].set_title(r'Ramp response of 1st order system, for varying time constant ...
     $\tau$'); axs[2].set_xlabel('time $t$')

# Adjust layout and save the plot to a PDF file
plt.tight_layout(); plt.savefig('1storder-responses.pdf', bbox_inches='tight')
```

Listing 5.1: Python script generating Figure 5.4. This script relies upon the Python Control Systems Library (Fuller et al., 2021). Available at 1storder-responses.py

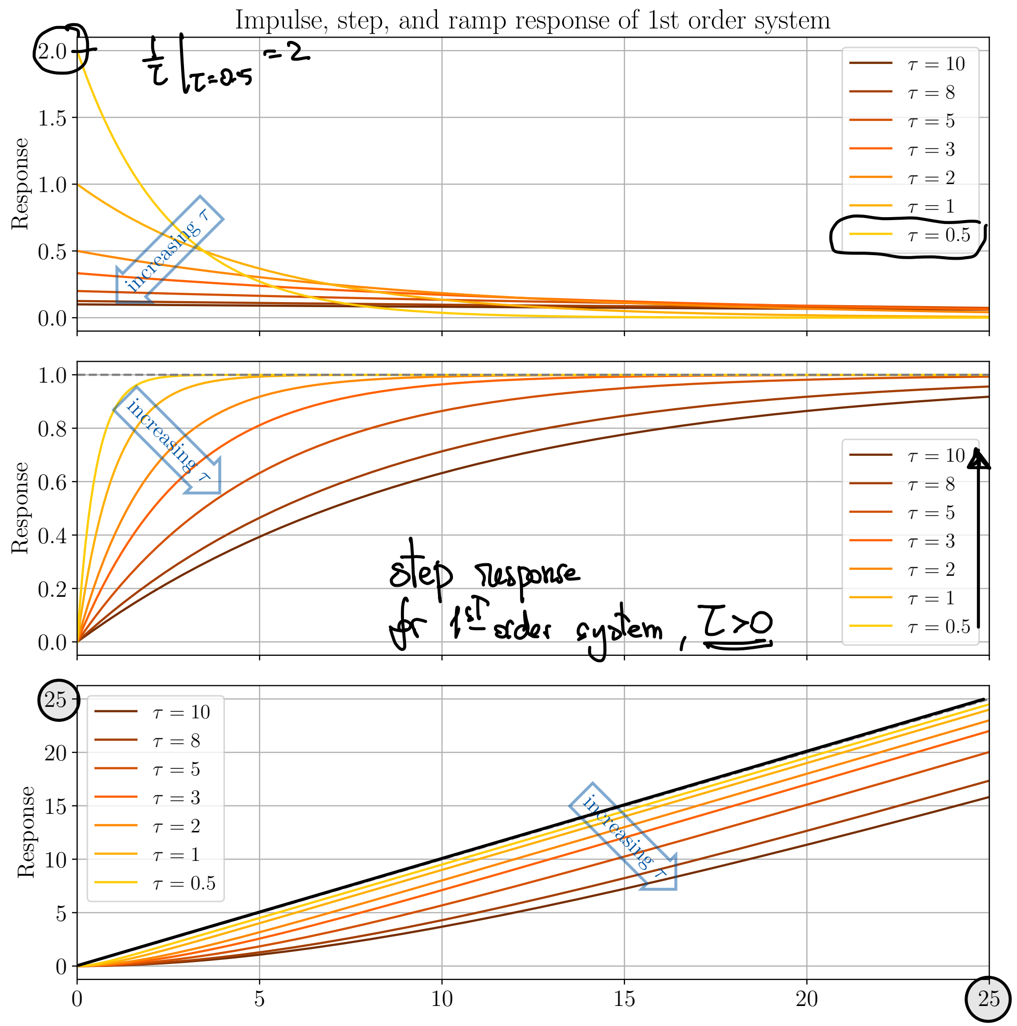


Figure 5.4: Canonical responses of the first order dynamics (5.11), when the input is a unit impulse, a unit step, and a unit ramp.
For increasing time constant τ, the system response become slower for all three inputs and, for the ramp response, the difference between input and output (tracking error) becomes larger.