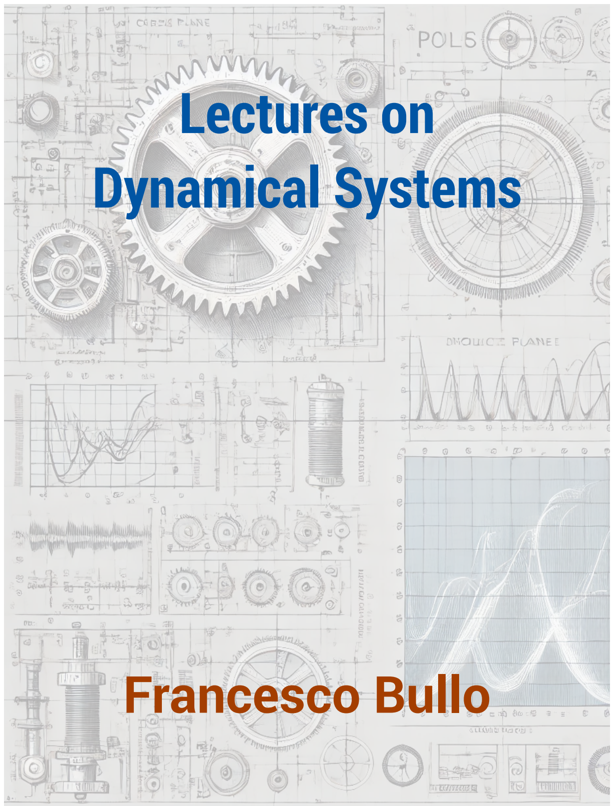
UC Santa Barbara, Department of Mechanical Engineering

ME103 Dynamical Systems. Chapter Slides.

Francesco Bullo
http://motion.me.ucsb.edu/ME103-Fall2025/syllabus.html



Contents

Chapter 4

The Laplace Transform

In this chapter, we explore the application of the *Laplace transform* in analyzing linear time-invariant systems, providing a bridge between time-domain functions and their frequency-domain counterparts.

The *Laplace transform* converts time-domain signals into functions of a complex variable s, where algebraic methods can replace differential operations. Its key properties, such as linearity, time differentiation, and integration, make it especially powerful for handling ordinary differential equations. The inverse transform is computed using lookup tables, *partial fraction expansion*, or symbolic software like SymPy. The location of poles and their residues determining the structure of the solution. In solving differential equations, the Laplace transform reduces them to algebraic equations in $Y(s)$, whose solutions are found by partial fraction expansion and inversion. The resulting time-domain response separates into a *free response*, governed by the system's characteristic polynomial and initial conditions, and a *forced response*, determined by external inputs.

Additional properties broaden the method's applicability. The *Initial* and *Final Value Theorems* connect limiting time-domain values to Laplace-domain expressions, while the time delay property shows how shifts in time map to exponentials in s. Standard *Laplace transform pairs*, particularly for exponential signals, illustrate how single, repeated, or distinct poles in the s-domain correspond to exponential or polynomially scaled exponentials in the time domain. Together, these results show that the Laplace transform is a key method for analyzing transient and steady-state behavior in dynamical systems.

To support these discussions, an appendix offers a brief review of complex numbers, emphasizing the *imaginary unit* and the representation of complex numbers in polar form using the *Euler formula*.

Illustrating the content of this chapter

The Laplace transform simplifies the modeling and analysis of linear time-invariant systems by converting a function of time t into a function of a complex variable s, making differential equations easier to solve.

Given two positive constants a and b, consider the ordinary differential equation (ODE):

$$\ddot{x} + (a+b)\dot{x} + abx = 0 \tag{4.1}$$

The Laplace transform aids in understanding this ODE by converting it into the simpler algebraic equation:

$$s^2 + (a+b)s + ab = 0. \tag{4.2}$$

This algebraic equation can be rewritten as

$$s^2 + (a+b)s + ab = (s+a)(s+b) = 0, \tag{4.3}$$

yielding two solutions: $s_1 = -a$ and $s_2 = -b$.

As we will explore in this chapter, the properties of the Laplace transform ensure that each solution to the ODE (4.1) is of the form

$$x(t) = c_1 \mathrm{e}^{-at} + c_2 \mathrm{e}^{-bt} \tag{4.4}$$

where the constants c_1 and c_2 are determined by the initial conditions. In other words,

a zero $-a$ of the algebraic equation (4.2) $\quad\rightarrow\quad$ a term e^{-at} in the solution to the differential equation (4.1).

This chapter is dedicated to understanding concepts and methods to generalize this result to arbitrary ODEs with inputs.

4.1 The Laplace transform

The *Laplace transform* of a function $f(t)$ is a function $F(s)$ formally defined by

$$F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} \mathrm{e}^{-st} f(t)\, dt, \tag{4.5}$$

where

- $f(t)$ is a function of time, such that $f(t) = 0$ for all $t < 0$,
- s is a complex variable,
- $\mathcal{L}[\cdot]$ is the symbol indicating the Laplace transform of its argument.

Because $f(t)$ can be discontinuous at $t = 0$, we interpret $f(0)$ in the formula (4.5) as the limit from the right: $f(0) = \lim_{t \to 0^+} f(t)$.

Note: Not all functions admit a well-defined Laplace transform, as the integral could be unbounded or nonexistent. However, all functions encountered in this context do.

The reverse process of finding the function of time $f(t)$ from its Laplace transform $F(s)$ is called the *inverse Laplace transform* and is denoted by

$$\mathcal{L}^{-1}[F(s)] = f(t). \tag{4.6}$$

There exists an integral formula[1] for the inverse Laplace transform, but it will not be needed here.

[1] For example, see https://en.wikipedia.org/wiki/Inverse_Laplace_transform

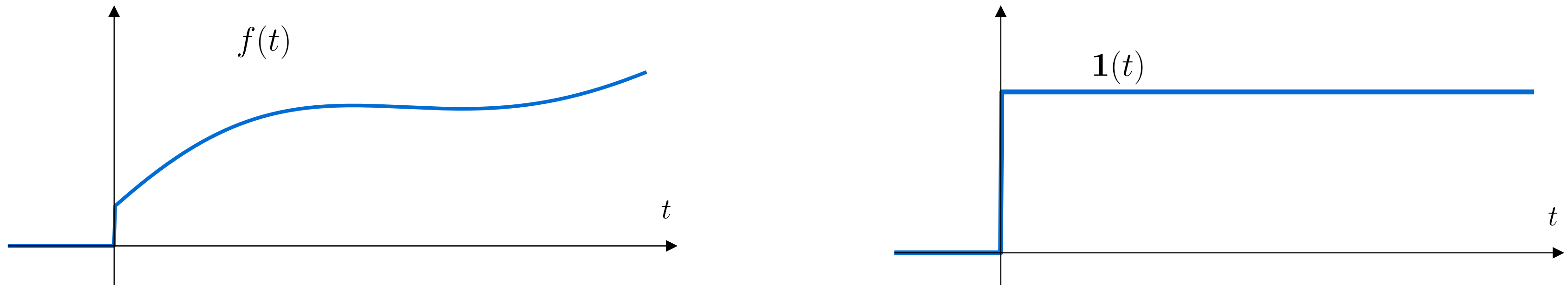


Figure 4.1: Left image: Laplace transforms are defined for functions that are zero for negative time. Right image: the unit step function.

In what follows, every function is to be understood as being zero for negative time. Hence, the function $f(t) = 1$ is understood to be the *unit step function* $\mathbf{1}(t)$ defined by

$$\mathbf{1}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0. \end{cases} \tag{4.7}$$

4.1.1 A useful example

It is relatively straightforward to gain some intuition for the Laplace transform formula (4.5). For any scalar real number a, we have

$$\mathcal{L}[\mathrm{e}^{-at}\,\mathbf{1}(t)] = \mathcal{L}[\mathrm{e}^{-at}] = \frac{1}{s+a} \tag{4.8}$$

To prove formula (4.8), we compute

$$\mathcal{L}[\mathrm{e}^{-at}] = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{e}^{-at}\;dt = \int_0^\infty \mathrm{e}^{-(a+s)t}\;dt \tag{4.9}$$

$$= \left[\frac{-1}{a+s}\,\mathrm{e}^{-(a+s)t}\right]_0^{+\infty} \tag{4.10}$$

$$= \lim_{t\to+\infty}\left(\frac{-1}{a+s}\,\mathrm{e}^{-(a+s)t}\right) - \left.\frac{-1}{a+s}\,\mathrm{e}^{-(a+s)t}\right|_{t=0} \tag{4.11}$$

$$\overset{(*)}{=} 0 + \frac{1}{s+a}. \tag{4.12}$$

Here, the step $(*)$ is valid when the real part of s is greater than $-a$, but, by other means, one can show that the formula holds even without that assumption.

4.1.2 Some nomenclature

The function $\mathcal{L}[\mathrm{e}^{-at}] = \dfrac{1}{s+a}$ is a fraction of polynomials.

- A function is *rational* if it is the quotient of two polynomial functions. In other words, $F(s)$ is rational if there exist two polynomials $\mathrm{Num}(s)$ and $\mathrm{Den}(s)$ such that
$$F(s) = \frac{\mathrm{Num}(s)}{\mathrm{Den}(s)}$$
- The domain of a rational function includes all complex numbers except for the values of s such that $\mathrm{Den}(s) = 0$.
- The points where the denominator equals zero are called the *poles* of the rational function.

4.1.3 Properties of Laplace transforms

The Laplace transform possesses several properties that greatly simplify its use in analysis. In particular, these properties often allow the computation of transforms without directly invoking the definition (4.5).

Properties of the Laplace Transform: In what follows, let $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

(P1) **Linearity:** $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$

(P2) **Derivative with respect to time:** $\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0)$

(P3) **Integral with respect to time:** $\mathcal{L}\left[\int_0^t f(u)\,du\right] = \frac{1}{s}F(s)$

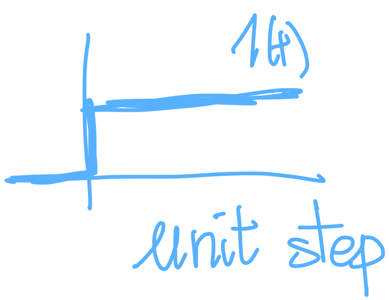
(P4) **Complex translation:** $\mathcal{L}[e^{-at} f(t)] = F(s + a)$.

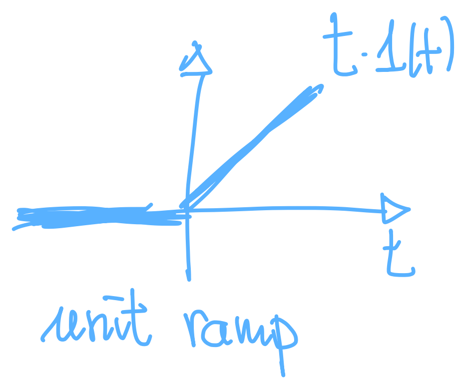
The inverse Laplace transform inherits analogous properties; for instance, it is also linear.

$$\mathcal{L}[t\,1(t)] = \mathcal{L}\left[\int_0^t 1(u)\,du\right] = \frac{1}{s}\cdot\mathcal{L}[1(t)] = \frac{1}{s^2}$$

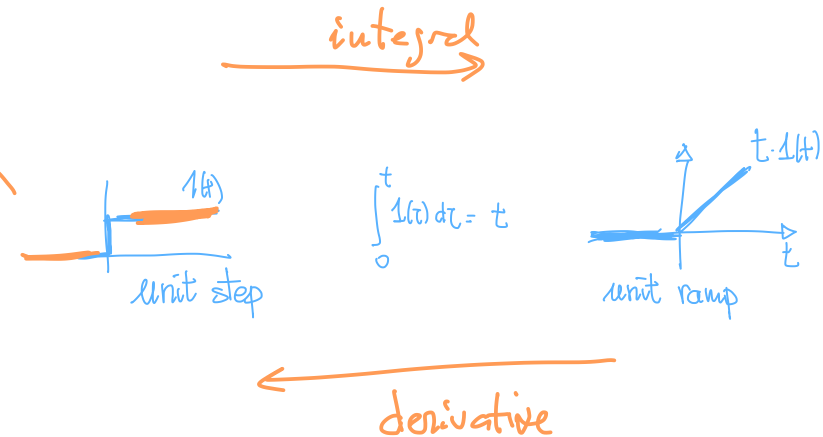
$$\mathcal{L}[1(t)] = \frac{1}{s}$$

$$\int_0^t 1(\tau)\,d\tau = t$$





function whose integral is unit step



$$\frac{d}{dt}1(t) = \begin{cases} 0 & t>0 \\ ? & t=0 \\ 0 & t<0 \end{cases}$$

4.1.4 The Laplace transform of exponential signals: Computation via linearity and other properties

We begin by considering the Laplace transform of the *exponential function*:

$$\mathcal{L}[\mathrm{e}^{-at}\,\mathbf{1}(t)] = \mathcal{L}[\mathrm{e}^{-at}] = \frac{1}{s+a} \tag{4.13}$$

In the following, we compute the Laplace transform of *exponential signals*, which are functions of time characterized by: (i) exponential functions with real or complex exponents, and (ii) their integrals, derivatives, and linear combinations. These exponential signals are depicted in Figure 4.1.

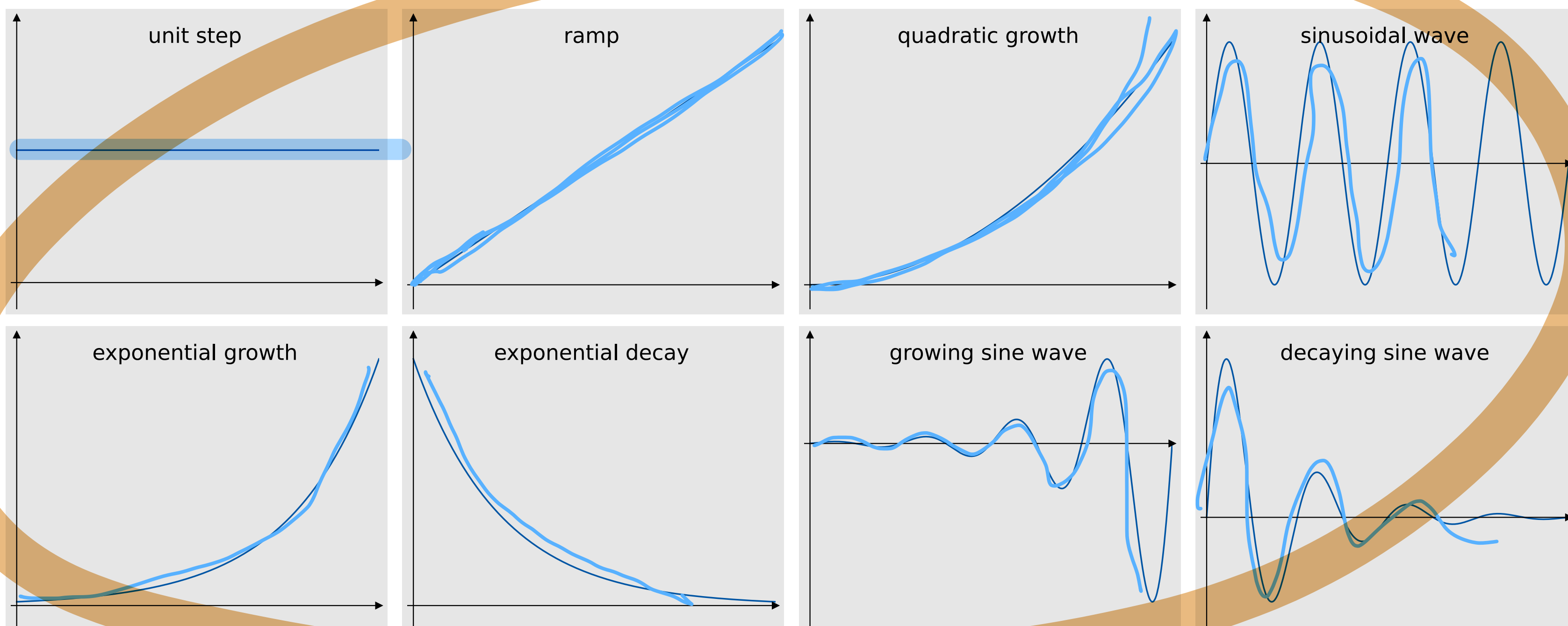


Table 4.1: Examples of exponential signals. Top row: unit step function, ramp function, quadratic growth, and sinusoidal wave. Bottom row: exponential growth and decay, exponentially growing and decaying sinusoids.

First, when $a = 0$, the Laplace transform of the *unit step function* is given by:

$$\mathcal{L}[\mathbf{1}(t)] = \frac{1}{s} \tag{4.14}$$

Next, we employ the integral property ((P3)) to compute

$$\mathcal{L}[t] = \mathcal{L}[t\mathbf{1}(t)] = \mathcal{L}\Big[\int_0^t 1(\tau)\, d\tau\Big] = \frac{1}{s}\mathcal{L}[\mathbf{1}(t)] = \frac{1}{s} \cdot \frac{1}{s},$$

yielding the Laplace transform of the *unit ramp function*:

$$\mathcal{L}[t] = \frac{1}{s^2} \tag{4.15}$$

Third, using the inverse Euler formula for the sinusoidal function and the linearity property ((P1)), we calculate

$$\mathcal{L}[\sin(\omega t)] = \mathcal{L}\Big[\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\Big] = \frac{1}{2i}\Big(\mathcal{L}[e^{i\omega t}] - \mathcal{L}[e^{-i\omega t}]\Big)$$
$$= \frac{1}{2i}\Big(\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\Big) = \frac{1}{2i}\frac{(s + i\omega) - (s - i\omega)}{(s - i\omega)(s + i\omega)}$$

resulting in the Laplace transform of the *sine wave*:

$$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2} \tag{4.16}$$

Fourth, applying the complex translation property ((P4)), we determine the Laplace transform of the *damped sine wave*:

$$\mathcal{L}[e^{-at}\sin(\omega t)] = \frac{\omega}{(s + a)^2 + \omega^2} \tag{4.17}$$

In class assignment

Compute the Laplace transform of $\cos(\omega t)$ using two different methods.

Fifth, the Laplace transforms of the cosine wave and the damped cosine wave are derived similarly using the inverse Euler formula for the cosine. Alternatively, the derivative-with-respect-to-time property ((P2)) (noting that $\frac{d}{dt}\sin(\omega t) = \omega\cos(\omega t)$) can be utilized, confirming that both approaches yield the same results:

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \text{and} \qquad \mathcal{L}[\mathrm{e}^{-at}\cos(\omega t)] = \frac{s+a}{(s+a)^2+\omega^2} \tag{4.18}$$

4.1.5 Unit pulse and impulse functions

Dynamical systems are occasionally influenced by a significant force over a brief time interval. Examples include: a ball rebounding off the floor, a hammer striking a nail, a bullet impacting a wall, an explosion affecting a flexible structure, or a car crash.

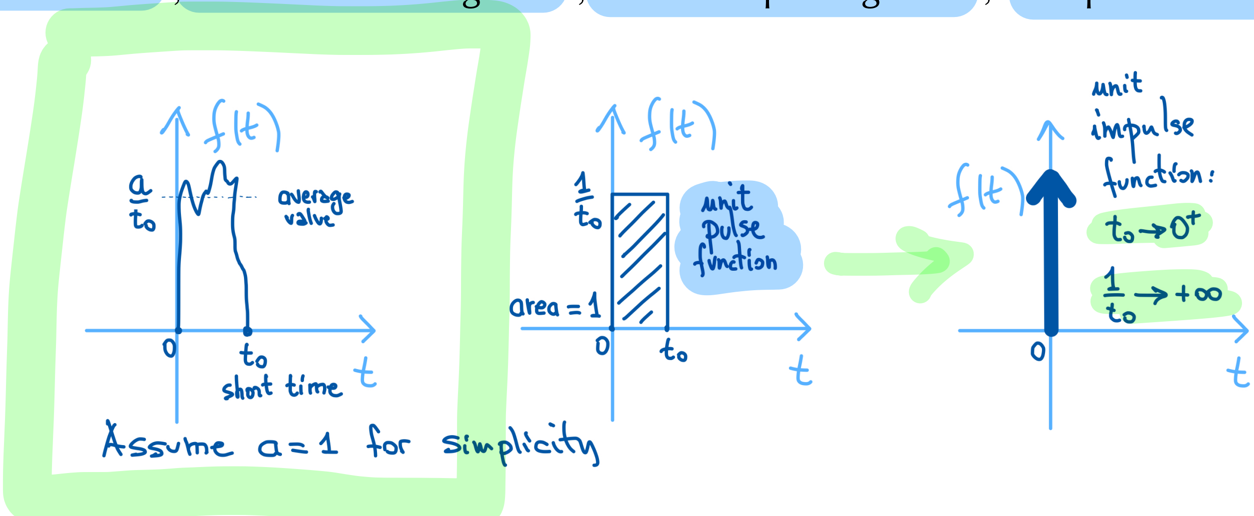


Figure 4.2: Left panel: a realistic pulse with a rapid rise, possible oscillations, and a subsequent drop to zero over a short time interval t_0. The pulse area is a positive amount a; we assume $a = 1$ to define the unit pulse and unit impulse functions.
Center panel: a *unit pulse function* with duration t_0 and amplitude $1/t_0$.
Right panel: in the limit as $t_0 \to 0^+$, we define the *unit impulse function*.

The *unit pulse function* with duration t_0 is defined as

$$\text{pulse}(t) = \begin{cases} \dfrac{1}{t_0} & \text{if } 0 < t < t_0 \\ 0 & \text{if } t < 0 \text{ or } t > t_0 \end{cases} \tag{4.19}$$

Note: The specific shape of a realistic pulse function is not critical; only the area matters. If the pulse has an area $a \neq 1$, then the appropriate function is $a \cdot \text{pulse}(t)$.

We now define an idealized version of the unit pulse function to facilitate simpler calculations.

As depicted in Figure 4.2, the *unit impulse function* is defined to be zero at all times except zero and to have an area equal to 1:

$$\delta(t) = \begin{cases} +\infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases} \quad \text{such that} \quad \int_{-\infty}^{+\infty} \delta(\tau)\, d\tau = 1 \tag{4.20}$$

The impulse function is a mathematical construct that simplifies calculations when the pulse duration t_0 is very short compared to the system's response time.

Even though we do not provide the proof here, it is useful to note:

$$\mathcal{L}[\delta(t)] = 1 \tag{4.21}$$

The relationship among the impulse function, the step function, and the ramp function is illustrated in Figure 4.3.

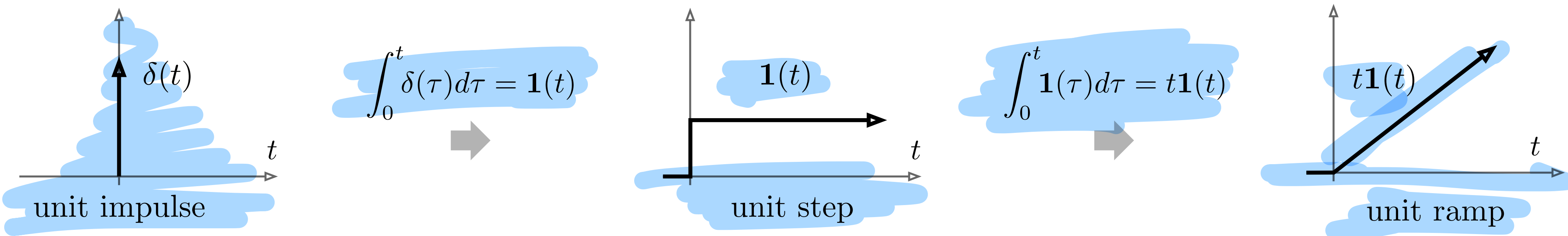
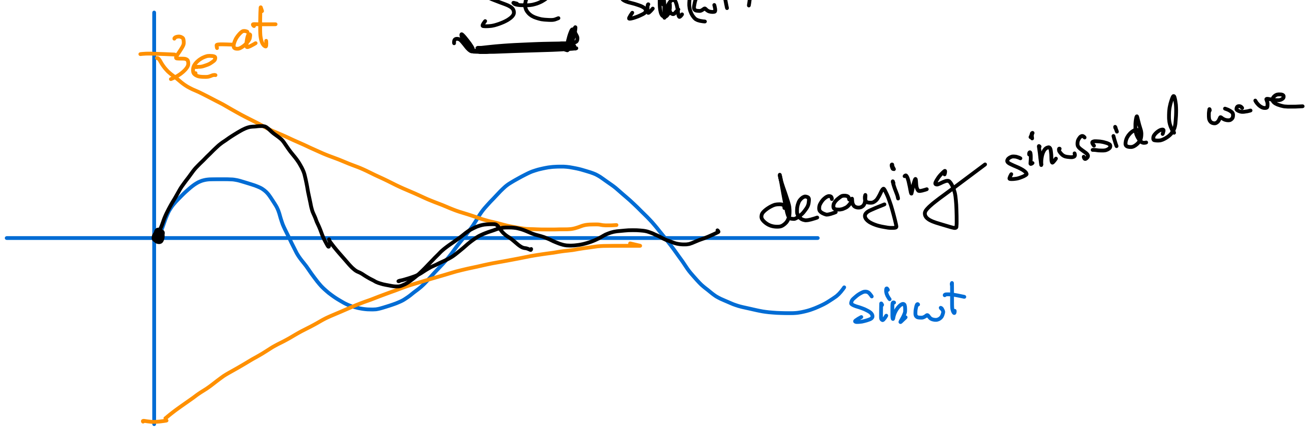


Figure 4.3: From unit impulse to unit step to unit ramp function, via integration with respect to time.

4.1.6 Table of Laplace transforms

Given the example of the exponential function with Laplace transform $\mathcal{L}[\mathrm{e}^{-at}] = \frac{1}{s+a}$ and given the properties of the Laplace transform, we can derive the Table 4.2 of Laplace transforms.

$$3e^{-at}\sin(\omega t)$$



Function of time $f(t)$			Laplace transform $F(s)$ and its poles	
In this table, we consider only *exponential signals*.			Laplace transforms of *exponential signals* are rational functions.	
(1)	Unit impulse	$\delta(t)$	1	none
(2)	Unit step	$\mathbf{1}(t)$	$\frac{1}{s}$	$s = 0$
(3)	Unit ramp	t	$\frac{1}{s^2}$	$s = 0$ repeated
(4)	Exponential function	e^{-at}	$\frac{1}{s+a}$	$s = -a$
(5)	Sine wave	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s = \pm i\omega$
(6)	Cosine wave	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s = \pm i\omega$
(7)	Damped sine wave	$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$s = -a \pm i\omega$
(8)	Damped cosine wave	$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	$s = -a \pm i\omega$

Table 4.2: Table of Laplace transforms. The coefficient a can be either positive or negative, ω is positive, and n is a natural number. Rows (7) and (8) are more general than (5) and (6), as they reduce to (5) and (6) when $a = 0$.

In summary, from the Table 4.2 of Laplace Transforms, we observe that each exponential signal (function of t) transforms into a rational function of s. Each rational function of s may have zero, one, or multiple poles. In Figure 4.4, for each point s^* in the complex plane (indicated with an $\times$ symbol), we illustrate the function of time whose Laplace transform has a pole at s^*.

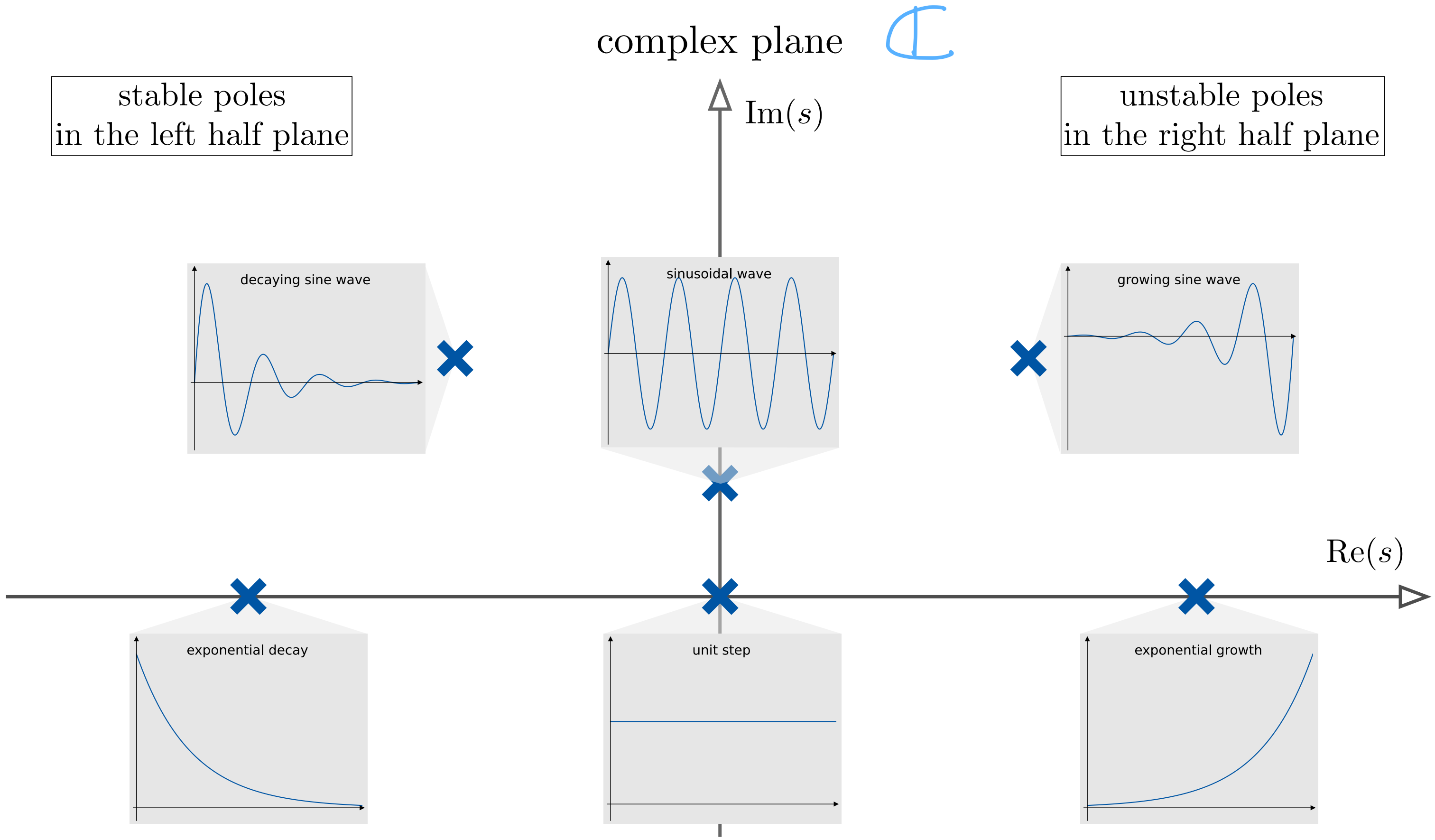


Figure 4.4: Functions of time associated with poles in the complex plane. For each point s^* in the complex plane (indicated with an $\times$ symbol), we illustrate the function of time whose Laplace transform has a pole at s^*. All functions of time in the diagram are exponential signals.

4.2 The inverse Laplace transform

We now discuss how to compute the inverse Laplace transform of rational functions. Specifically, given a rational function $F(s)$, we aim to find the function of time $f(t)$ such that $\mathcal{L}[f(t)] = F(s)$. There are three methods:

(i) Utilizing extensive *lookup tables* with example pairs $(f(t), F(s))$,

(ii) The method of *partial fraction expansion*,

(iii) Employing symbolic manipulation software, such as the SymPy symbolic computing library in Python.

Regarding lookup tables, an additional table of Laplace transforms is available in Section 4.4. However, due to the vast number of possible cases, generating extensive Laplace transform tables is impractical. Therefore, a combination of lookup tables and partial fraction expansions is typically employed.

4.2.1 Partial fraction expansions

Typically, we consider functions $F(s)$ that can be expressed as the sum of simpler functions:

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s) \tag{4.22}$$

Thus,

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[F_1(s)] + \mathcal{L}^{-1}[F_2(s)] + \cdots + \mathcal{L}^{-1}[F_n(s)] \tag{4.23}$$

$$= f_{\mathbf{1}}(t) + f_2(t) + \cdots + f_n(t) \tag{4.24}$$

where $f_{\mathbf{1}}(t)$, $f_2(t)$, ..., $f_n(t)$ are the inverse Laplace transforms of $F_1(s)$, $F_2(s)$, ..., $F_n(s)$, respectively. Therefore, it is useful to have a lookup table with numerous pairs $(f(t), F(s))$.

Problem Setup: We often consider a rational function

$$F(s) = \frac{\mathrm{Num}(s)}{\mathrm{Den}(s)}, \tag{4.25}$$

where the degree of the numerator is less than or equal to the degree of the denominator. Our goal is to compute the inverse Laplace transform of $F(s)$.

Step 1: The first step is to identify the poles of F and factorize the denominator:

$$F(s) = \frac{\mathrm{Num}(s)}{(s+p_1)(s+p_2)\dots(s+p_n)} \tag{4.26}$$

Step 2: The second step is to expand F in a *partial fraction expansion*:

$$F(s) = \frac{r_1}{s+p_1} + \frac{r_2}{s+p_2} + \cdots + \frac{r_n}{s+p_n}.$$

Each coefficient r_i is called a *residue* at the pole $-p_i$. Sometimes the expansion includes more complex terms (e.g., when there are complex conjugate poles, as in the fourth example below, or repeated poles, as in the fifth example below).

Step 3: Compute the residues $r_1, \ldots, r_n$, using one of the following methods:
Matching the numerators: always applicable, though sometimes lengthy,
Single-pole residue formula: a quick shortcut when applicable.
We will illustrate these methods in the examples below.

Step 4: The final step is straightforward:

$$\mathcal{L}^{-1}[F(s)] = r_1\,\mathcal{L}^{-1}\left[\frac{1}{s+p_1}\right] + r_2\,\mathcal{L}^{-1}\left[\frac{1}{s+p_2}\right] + \cdots + r_n\,\mathcal{L}^{-1}\left[\frac{1}{s+p_n}\right] = r_1\,\mathrm{e}^{-p_1 t} + r_2\,\mathrm{e}^{-p_2 t} + \cdots + r_n\,\mathrm{e}^{-p_n t} \tag{4.27}$$

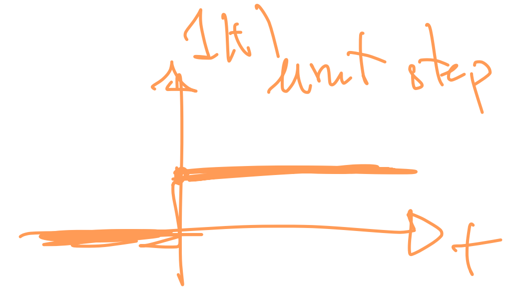
First example: Rational function already written as sum of simple terms

We now consider various examples of partial fraction expansions and corresponding inverse Laplace transforms. As a first example, consider a function with real poles only, already written in partial fraction expansion:

$$F_1(s) = 2 + \frac{3}{s} + \frac{4}{s+5} \tag{4.28}$$

Using rows (1), (2), and (4) of Table 4.2 and the linearity property, we compute

$$f_1(t) = \mathcal{L}^{-1}[F_1(s)] = 2\delta(t) + 3 + 4\,\mathrm{e}^{-5t} \tag{4.29}$$



Second example: Two isolated real poles

As a second example, consider

$$F_2(s) = \frac{s+2}{s^2+7s+12} \tag{4.30}$$

We compute

$$s^2+7s+12=0 \iff s=-3,-4 \iff s^2+7s+12=(s+3)(s+4). \tag{4.31}$$

Thus, we write the partial fraction expansion

$$F_2(s) = \frac{s+2}{(s+3)(s+4)} = r_1\frac{1}{s+3} + r_2\frac{1}{s+4} \tag{4.32}$$

We now describe first the *matching the numerators* method and then the *single-pole residue formula*.

$$\mathcal{L}^{-1}[F_2(s)] = f_2(t) = r_1 e^{-3t} + r_2 e^{-4t}$$

$$s^2+7s+12=0; \quad s_{12} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-7 \pm \sqrt{49-48}}{2} = \frac{-7 \pm 1}{2} = -4, -3$$

$s^2 + 7s + 12 = (s+3)(s+4)$

Matching the numerators We start by rewriting equation (4.32) as

$$\frac{r_1}{s+3} + \frac{r_2}{s+4} = \frac{s+2}{s^2+7s+12} = \frac{r_1(s+4)+r_2(s+3)}{(s+3)(s+4)} \tag{4.33}$$

The denominators are identical due to the partial fraction expansion. We therefore focus on the numerators:

$$s+2 = r_1(s+4) + r_2(s+3) = (r_1+r_2)s + (4r_1+3r_2) \tag{4.34}$$

We now equate each power of s:

$$\begin{cases} 1 &= r_1 + r_2 \\ 2 &= 4r_1 + 3r_2 \end{cases} \tag{4.35}$$

This is a *linear system of 2 equations in 2 variables*. After some calculations, we obtain

$$r_1 = -1 \qquad \text{and} \qquad r_2 = 2. \tag{4.36}$$

$$\mathcal{L}^{-1}[F_2(s)] = f_2(t) = (-1)\cdot\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] + (2)\,\mathcal{L}^{-1}\left[\frac{1}{s+4}\right]$$

$F_2(s) = \frac{s+2}{s^2+7s+12}$

$$= -e^{-3t} + 2e^{-4t}.$$

Single-pole residue formula We now describe the *single-pole residue formula* method. This method applies only to *single real poles*. The residue r_i associated with a pole $-p_i$ is given by

$$r_i = (s + p_i)F(s)\Big|_{s=-p_i} \tag{4.37}$$

Note: this formula is correct only when the pole p is not repeated; if the root is complex, it's typically easier to use the matching method.

For $F_2(s) = \frac{s+2}{(s+3)(s+4)} = r_1 \frac{1}{s+3} + r_2 \frac{1}{s+4}$, the single-pole residue formula (4.37) gives:

$$r_1 = (s+3)F_2(s)\Big|_{s=-3} = \frac{s+2}{s+4}\Big|_{s=-3} = \frac{-1}{+1} = -1 \tag{4.38}$$

$$r_2 = (s+4)F_2(s)\Big|_{s=-4} = \frac{s+2}{s+3}\Big|_{s=-4} = \frac{-2}{-1} = +2 \tag{4.39}$$

In summary, both methods yield

$$f_2(t) \; = \; \mathcal{L}^{-1}[F_2(s)] = \mathcal{L}^{-1}\left[(-1)\frac{1}{s+3} + (+2)\frac{1}{s+4}\right] \; = \; -\,\mathrm{e}^{-3t} + 2\,\mathrm{e}^{-4t} \tag{4.40}$$

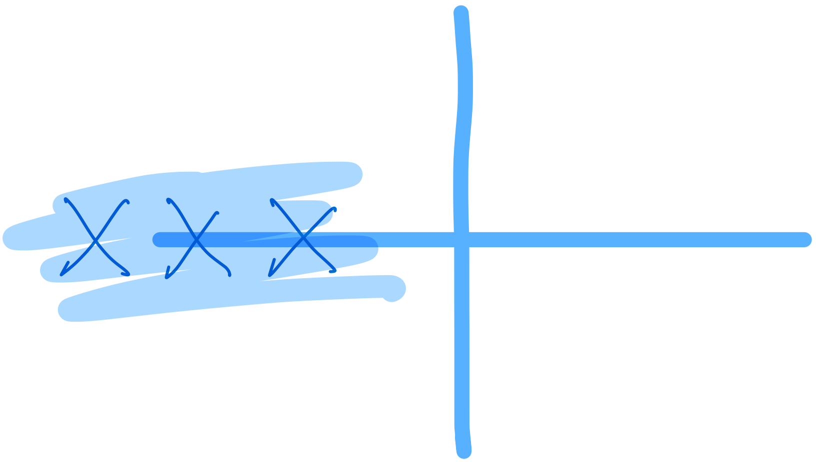
Third example: Three isolated real poles

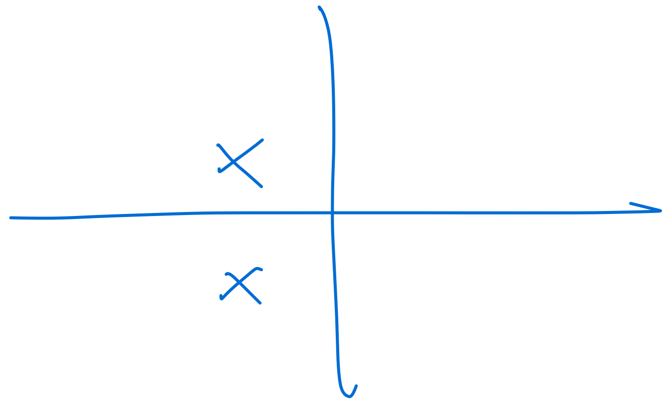
In this third example, we examine the scenario of multiple isolated poles by considering the following rational function:

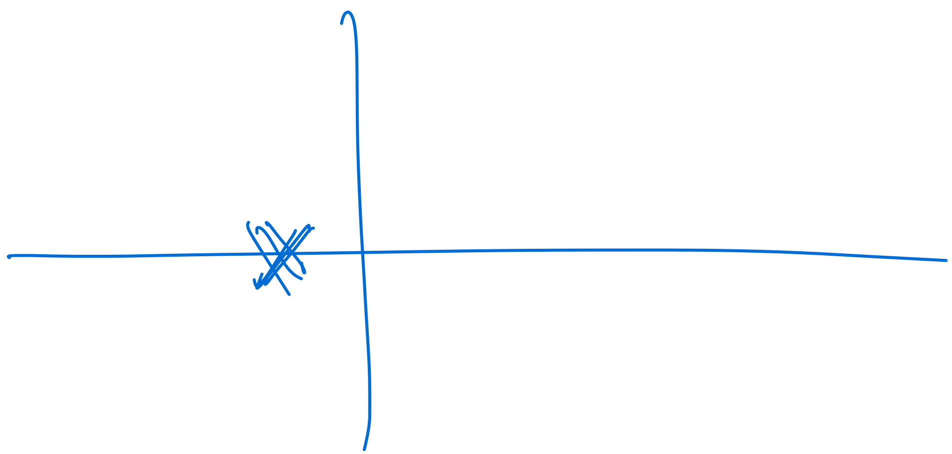
$$F_3(s) = \frac{2}{s^3 + 6s^2 + 11s + 6} \tag{4.41}$$

At this point we would like to write the denominator as a multiplication of terms.

$s^3 + 6s^2 + 11s + 6$







In general, finding the roots of a high-order polynomial can be challenging. However, the rational root theorem simplifies this task for polynomials with integer coefficients: the only possible integer roots are the divisors of the constant term.

For the function $F_3(s)$, the denominator has integer coefficients, and the constant term is 6. Thus, the possible integer roots are ± 1, ± 2, ± 3, and ± 6. By substituting these values into the denominator, we determine that $s = -1$, $s = -2$, and $s = -3$ are indeed roots. Consequently, we can factor the denominator as

6: 1, 2, 3

$$s^3 + 6s^2 + 11s + 6 = (s+1)(s+2)(s+3) \tag{4.42}$$

±1, ±2, ±3

in fact, −1, −2, −3

and express the partial fraction expansion as

$$F_3(s) = \frac{2}{(s+1)(s+2)(s+3)} = r_1 \frac{1}{s+1} + r_2 \frac{1}{s+2} + r_3 \frac{1}{s+3} \tag{4.43}$$

Alternatively, one can use mathematical software.

```python
# Python code to compute symbolically the roots of a polynomial
from sympy import symbols, solve
s = symbols('s')
roots = solve(s**3 + 6*s**2 + 11*s + 6, s)
print("Roots:", roots)
```

$$f_3(t) = r_1 e^{-t} + r_2 e^{-2t} + r_3 e^{-3t}$$

$$\mathcal{L}[e^{-at}] = \frac{1}{s+a}$$

Given $F_3(s) = \frac{2}{(s+1)(s+2)(s+3)} = r_1 \frac{1}{s+1} + r_2 \frac{1}{s+2} + r_3 \frac{1}{s+3}$, the single-pole residue formula (4.37) yields

$$r_1 = (s+1)F_3(s)\Big|_{s=-1} = \frac{2}{(s+2)(s+3)}\Big|_{s=-1} = +1 \tag{4.44}$$

$$r_2 = (s+2)F_3(s)\Big|_{s=-2} = \frac{2}{(s+1)(s+3)}\Big|_{s=-2} = -2 \tag{4.45}$$

$$r_3 = (s+3)F_3(s)\Big|_{s=-3} = \frac{2}{(s+1)(s+2)}\Big|_{s=-3} = +1 \tag{4.46}$$

In summary,

$$f_3(t) \; = \; \mathcal{L}^{-1}\left[(+1)\frac{1}{s+1} + (-2)\frac{1}{s+2} + (+1)\frac{1}{s+3}\right] \; = \; \mathrm{e}^{-t} - 2\,\mathrm{e}^{-2t} + \mathrm{e}^{-3t} \tag{4.47}$$

Fourth example: One pair of complex conjugate poles

In this fourth example, we consider a rational function with a complex conjugate pair of roots:

$$F_4(s) = \frac{8s + 12}{s^2 + 6s + 25} \tag{4.48}$$

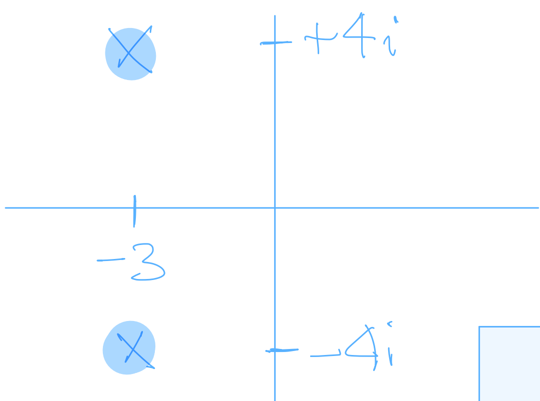
We compute[2]

$$s^2 + 6s + 25 = 0 \iff s = -3 \pm 4\mathrm{i} \iff s^2 + 6s + 25 = (s+3)^2 + 4^2 \tag{4.49}$$

Thus, the denominator is of the form $(s+a)^2 + \omega^2$ where $a = 3$ and $\omega = 4$. In this case, the single-pole residue formula (4.37) is inconvenient to apply. Therefore, we proceed by *matching the numerators*.

We recall rows (7) and (8) for damped sine and cosine waves in Table 4.2 of Laplace transforms, and seek coefficients α, β such that:

$$\frac{8s + 12}{s^2 + 6s + 25} = \left(\alpha \frac{\omega}{(s+a)^2 + \omega^2} + \beta \frac{s+a}{(s+a)^2 + \omega^2}\right)_{a=3,\omega=4} = \alpha \frac{4}{s^2 + 6s + 25} + \beta \frac{s+3}{s^2 + 6s + 25} \tag{4.50}$$



[2] For the quadratic equation $az^2 + bz + c = 0$, recall the classic formula for the roots $z_{1,2} = (-b \pm \sqrt{b^2 - 4ac})/(2a)$.

By matching the numerators and each power of s, we obtain

$$8s + 12 = 4\alpha + \beta(s+3) \quad \Longrightarrow \quad \begin{cases} 8 = \beta \\ 12 = 4\alpha + 3\beta \end{cases} \tag{4.51}$$

so that $\beta = +8$ and $12 = 4\alpha + 24$, that is, $\alpha = -3$. In summary,

$$f_4(t) = \mathcal{L}^{-1}\left[(-3)\frac{4}{s^2+6s+25} + (+8)\frac{s+3}{s^2+6s+25}\right] = -3\,\mathrm{e}^{-3t}\sin(4t) + 8\,\mathrm{e}^{-3t}\cos(4t) \tag{4.52}$$

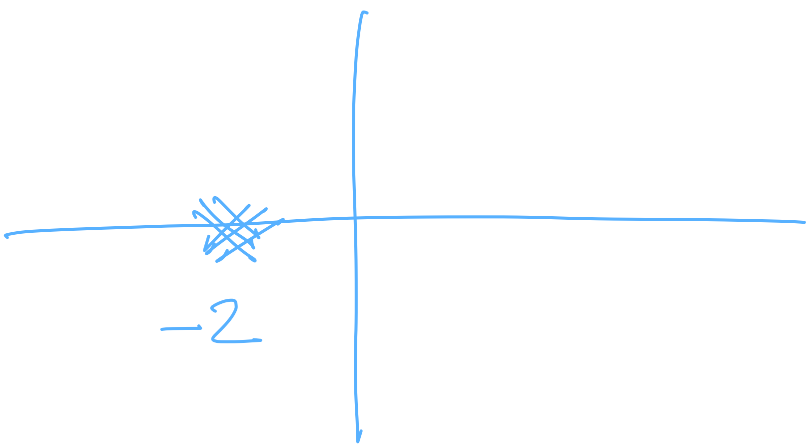
Fifth example: Repeated real poles

In this fifth and final example, we consider the case of a repeated pole. We examine

$$F_5(s) = \frac{s^2 + 3s + 3}{(s+2)^3} \tag{4.53}$$

where the pole $s = -2$ is repeated three times. In this case, the appropriate partial fraction expansion includes three terms (corresponding to the multiplicity of the pole):

$$F_5(s) = \frac{s^2 + 3s + 3}{(s+2)^3} = \alpha \frac{1}{s+2} + \beta \frac{1}{(s+2)^2} + \gamma \frac{1}{(s+2)^3} \tag{4.54}$$



$$\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+2)^3}\right]$$

Since the single-pole residue formula (4.37) does not apply, we proceed by *matching the numerators*:

$$\frac{s^2+3s+3}{(s+2)^3} = \alpha\frac{1}{s+2} + \beta\frac{1}{(s+2)^2} + \gamma\frac{1}{(s+2)^3} = \frac{\alpha(s+2)^2+\beta(s+2)+\gamma}{(s+2)^3} \tag{4.55}$$

$$\implies \quad s^2+3s+3 = \alpha(s^2+4s+4)+\beta(s+2)+\gamma \tag{4.56}$$

$$\implies \begin{cases} 1=\alpha \\ 3=4\alpha+\beta \\ 3=4\alpha+2\beta+\gamma \end{cases} \implies \begin{cases} \alpha=+1 \\ \beta=3-4=-1 \\ \gamma=3-4-2\cdot(-1)=1. \end{cases} \tag{4.57}$$

In summary,

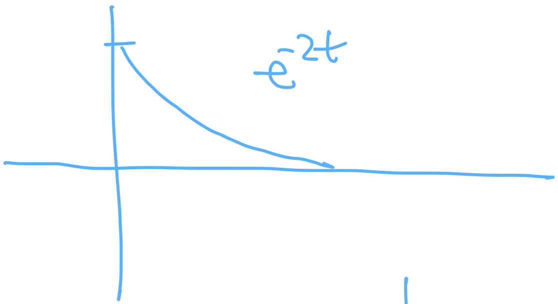
$$f_5(t) = \mathcal{L}^{-1}\left[\frac{s^2+3s+3}{(s+2)^3}\right] = \mathcal{L}^{-1}\left[(+1)\frac{1}{s+2}+(-1)\frac{1}{(s+2)^2}+(+1)\frac{1}{(s+2)^3}\right] \tag{4.58}$$

$$= \mathrm{e}^{-2t} - t\,\mathrm{e}^{-2t} + \frac{1}{2!}t^2\,\mathrm{e}^{-2t} = \left(1-t+\tfrac{1}{2}t^2\right)\mathrm{e}^{-2t} \tag{4.59}$$

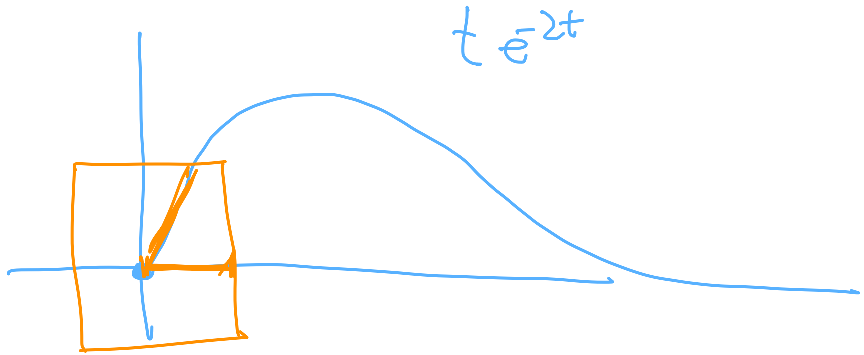
where we have used row (11) from Table 4.3 of additional Laplace transforms to compute

$$\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = t\,\mathrm{e}^{-2t} \qquad \text{and} \qquad \mathcal{L}^{-1}\left[\frac{1}{(s+2)^3}\right] = \frac{1}{2}t^2\,\mathrm{e}^{-2t}\,. \tag{4.60}$$

$$f_5(t) = e^{-2t} - t e^{-2t} + \frac{1}{2} t^2 e^{-2t} = e^{-2t}\left(1 - t + t^2/2\right)$$



e^{2t}



$t\,e^{2t}$

4.2.2 Symbolic mathematics software: The Python SymPy library

Programming notes

It is useful to briefly compare leading software libraries for symbolic mathematics, also known as computer algebra systems.

- Leading commercial software systems include Maple, Mathematica, and Maxima. These systems offer advanced tools for calculus, linear algebra, number theory, and differential equations.
- SymPy is an open-source Python library for symbolic mathematics. It provides basic functionality in calculus, algebra, discrete math, and geometry, and it can compute Laplace transforms and inverse Laplace transforms, as detailed in (Meurer et al, 2017). SymPy is ideal for those who prioritize open-source flexibility and Python integration, while commercial systems may offer more extensive features and optimizations.

```python
# Import SymPy for symbolic math operations
from sympy import symbols, diff, integrate, Eq, solve, sin, cos, series

# Define a symbol x
x = symbols('x')

# 1. Differentiate a symbolic expression
expr = sin(x) * cos(x)
derivative = diff(expr, x)
print("Example calculations performed by SymPy:\n")
print(f"Derivative of sin(x) * cos(x): {derivative}")

# 2. Integrate a symbolic expression
integral = integrate(expr, x)
print(f"Integral of sin(x) * cos(x): {integral}")

# 3. Solve a symbolic equation
equation = Eq(x**2 + 2*x - 8, 0)
solutions = solve(equation, x)
print(f"Solutions to x^2 + 2x - 8 = 0: {solutions}")

# 4. Perform a Taylor series expansion
taylor_series = series(sin(x), x, 0, 6)
print(f"Taylor series of sin(x): {taylor_series}")
```

Listing 4.1: Python script illustrating SymPy's capabilities, see Figure 4.5. Available at sympy-demo.py

```
Example calculations performed by SymPy:

Derivative of sin(x) * cos(x): -sin(x)**2 + cos(x)**2
Integral of sin(x) * cos(x): sin(x)**2/2
Solutions to x^2 + 2x - 8 = 0: [-4, 2]
Taylor series of sin(x): x - x**3/6 + x**5/120 + O(x**6)
```

Figure 4.5: Output of the sympy-demo.py program.

Symbolic computation of inverse Laplace transforms

```python
from sympy import symbols, inverse_laplace_transform, latex
from sympy.abc import s, t

# Define symbols
a, b, c, omega = symbols('a b c omega', real=True)

# Define the rational functions for which we want the inverse ...
    Laplace transform
functions = [
    2 + 3/s + 4/(s + 5),            # ex1: a single pole
    (s + 2) / (s**2 + 7*s + 12),    # ex2: two real poles
    2 / (s**3 + 6*s**2 + 11*s + 6),# ex3: multiple isolated poles
    (8*s + 12) / (s**2 + 6*s + 25),# ex4: complex conjugate poles
    (s**2 + 3*s + 3) / (s + 2)**3, # ex5: a repeated pole
    # Symbolic examples
    (a*s + b) / (s**2 + 7*s + 12), # ex6: two poles
    1 / ((s + a) * (s + b) * (s + c)), # ex7: three poles
    1 / ((s + a) * (s**2 + omega**2))  # ex8: a real, two ...
        conjugate poles
]

# Prepare the LaTeX content
latex_content = "Examples of inverse Laplace ...
    transforms:\n\\begin{align}\n"

# Loop through the functions
for i, F in enumerate(functions):
    # Compute the inverse Laplace transform
    f = inverse_laplace_transform(F, s, t)

    # Generate the LaTeX line for the current function
    latex_line = f"\\mathcal{{L}}^{{-1}} \\left[ {latex(F)} ...
        \\right] &= {latex(f)} \\label{{eq:example{i+1}}}"

    # Add a line break after each equation, except the last one
    if i < len(functions) - 1:
        latex_line += " \\\\ \n"

    # Add vertical space after the first five examples
    if i == 4:
        latex_line += "\\nonumber\\\\ \n"

    # Add the current line to the LaTeX content
    latex_content += latex_line

# Close the LaTeX content with align
latex_content += "\\end{align}\n"

# Write the LaTeX to a file
with open("inverseLaplace.tex", "w") as file:
    file.write(latex_content)
```

Listing 4.2: Python script generating the LaTeX output in Figure 4.6.
Available at inverseLaplace.py

Examples of inverse Laplace transforms:

$$\mathcal{L}^{-1}\left[2+\frac{4}{s+5}+\frac{3}{s}\right] = 2\delta(t)+3\theta(t)+4e^{-5t}\theta(t) \tag{4.61}$$

$$\mathcal{L}^{-1}\left[\frac{s+2}{s^2+7s+12}\right] = \left(2-e^{t}\right)e^{-4t}\theta(t) \tag{4.62}$$

$$\mathcal{L}^{-1}\left[\frac{2}{s^3+6s^2+11s+6}\right] = \left(e^{2t}-2e^{t}+1\right)e^{-3t}\theta(t) \tag{4.63}$$

$$\mathcal{L}^{-1}\left[\frac{8s+12}{s^2+6s+25}\right] = -\left(3\sin(4t)-8\cos(4t)\right)e^{-3t}\theta(t) \tag{4.64}$$

$$\mathcal{L}^{-1}\left[\frac{s^2+3s+3}{(s+2)^3}\right] = \frac{\left(t^2-2t+2\right)e^{-2t}\theta(t)}{2} \tag{4.65}$$

$$\mathcal{L}^{-1}\left[\frac{as+b}{s^2+7s+12}\right] = \left(4a-b-(3a-b)e^{t}\right)e^{-4t}\theta(t) \tag{4.66}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(a+s)(b+s)(c+s)}\right] = \frac{\left((a-b)e^{t(a+b)}-(a-c)e^{t(a+c)}+(b-c)e^{t(b+c)}\right)e^{-t(a+b+c)}\theta(t)}{(a-b)(a-c)(b-c)} \tag{4.67}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(a+s)(\omega^2+s^2)}\right] = \frac{\left(\omega+(a\sin(\omega t)-\omega\cos(\omega t))e^{at}\right)e^{-at}\theta(t)}{\omega(a^2+\omega^2)} \tag{4.68}$$

Figure 4.6: Examples of inverse Laplace transforms of rational functions, via the SymPy symbolic computing library (Meurer et al, 2017). The first five examples are numerical and the last three examples are parametric.
In SymPy, the function $\theta(t)$ is the unit step function $\mathbf{1}(t)$.
The first five examples are the same inverse Laplace transforms that we computed in the previous section.

4.3 Solving linear differential equations

As before, let $F(s) = \mathcal{L}[f(t)]$. We now recall the derivative-with-respect-to-time property (P2) and extend it to higher-order derivatives by applying it repeatedly. We have

$$\mathcal{L}\Big[\frac{d}{dt}f(t)\Big] = sF(s) - f(0) \tag{4.69}$$

so that

$$\mathcal{L}\Big[\frac{d^2}{dt^2}f(t)\Big] = s\,\mathcal{L}\Big[\frac{d}{dt}f(t)\Big] - \frac{df}{dt}(0) = s\big(sF(s) - f(0)\big) - \frac{df}{dt}(0) \tag{4.70}$$

$$= s^2F(s) - sf(0) - \frac{df}{dt}(0) \tag{4.71}$$

Applying the property repeatedly, we can compute higher-order time derivatives:

$$\mathcal{L}\Big[\frac{d^n f}{dt^n}(t)\Big] = s^nF(s) - s^{n-1}f(0) - s^{n-2}\frac{df}{dt}(0) - \ldots - \frac{d^{n-1}f}{dt^{n-1}}(0) \tag{4.72}$$

$$= s^nF(s) - \sum_{k=0}^{n-1} s^{n-1-k}\frac{d^k f}{dt^k}(0) \tag{4.73}$$

We use these properties to *transform differential equations into algebraic equations*. In what follows, let

$$X(s) = \mathcal{L}[x(t)], \qquad Y(s) = \mathcal{L}[y(t)], \qquad \text{and} \qquad U(s) = \mathcal{L}[u(t)].$$

Example Calculations

$$\tau \dot{x} + x = k u \quad \ldots \quad \tau > 0, \quad k > 0$$

$$\begin{cases} \tau \dot{x}(t) + x(t) = k\, u(t). \\ x(0) = x_0 \end{cases}$$

forced first order system.

if $x_0 = 0$ and $u(t) = 0$, then

$\tau \dot{x}(t) + x(t) = 0$, $x(0) = 0$.

$x(t) = 0$. $X(s) = 0$.

Take Laplace transform:

(remember $\mathcal{L}[\dot{x}(t)] = sX(s) - \underbrace{x(0)}_{x_0}$)

$$\tau \left(sX(s) - x_0 \right) + X(s) = k\, U(s)$$

$$\mathcal{L}[e^{-at}] = \frac{1}{s+a}$$

$$\underbrace{(\tau s + 1)}_{\text{characteristic polynomial}} \underbrace{X(s)}_{\text{state}} = \underbrace{k\, U(s)}_{\text{due to input}} + \underbrace{\tau x_0}_{\text{due to initial condition}}$$

$$X(s) = \underbrace{\frac{k\, U(s)}{\tau s + 1}}_{X_{forced}(s)} + \underbrace{\frac{\tau x_0}{\tau s + 1}}_{X_{free}(s)}$$

Forced response   Free response

$$x_{free}(t) = \mathcal{L}^{-1}\left[\frac{\tau x_0}{\tau s + 1} \right] = \mathcal{L}^{-1}\left[\frac{x_0}{s + \underbrace{1/\tau}_{a}} \right] = x_0\, e^{-t/\tau}$$

4.3.1 A differential equation with non-zero initial conditions and zero input

Consider the differential equation

$\ddot{y}(t) + 7\dot{y}(t) + 12y(t) = 0$

FREE RESPONSE

$$\ddot{y} + 7\dot{y} + 12y = 0, \qquad y(0) = y_0, \quad \dot{y}(0) = v_0 \tag{4.74}$$

Our objective is to determine the solution $y(t)$ as a function of the initial conditions y_0 and v_0.

To achieve this objective, we apply the Laplace transform to both sides, resulting in

$$(s^2Y(s) - sy_0 - v_0) + 7(sY(s) - y_0) + 12Y(s) = 0$$

where $Y(s) = \mathcal{L}[y(t)]$ and the derivative properties (4.69) and (4.70) are utilized. Collecting terms involving $Y(s)$, we have:

$$\begin{aligned} (s^2 + 7s + 12)Y(s) - sy_0 - v_0 - 7y_0 &= 0 \\ \iff \qquad Y(s) = \frac{sy_0 + (v_0 + 7y_0)}{s^2 + 7s + 12} &= \frac{sy_0 + (v_0 + 7y_0)}{(s+3)(s+4)}. \end{aligned} \tag{4.75}$$

Equation (4.66) from the previous section on partial fraction expansions states:

$$\mathcal{L}^{-1}\left[\frac{sa + b}{s^2 + 7s + 12}\right] = (4a - b)\,\mathrm{e}^{-4t} - (3a - b)\,\mathrm{e}^{-3t}.$$

Here, with $a = y_0$ and $b = v_0 + 7y_0$, the solution to the differential equation (4.74) is

$$y(t) = (4y_0 - v_0 - 7y_0)\,\mathrm{e}^{-4t} - (3y_0 - v_0 - 7y_0)\,\mathrm{e}^{-3t} = -(3y_0 + v_0)\,\mathrm{e}^{-4t} + (4y_0 + v_0)\,\mathrm{e}^{-3t} \tag{4.76}$$

In summary, by employing the Laplace transform, we have converted the differential equation (4.74) into the algebraic equation (4.75). We then solved the algebraic equation and derived the solution to the differential equation using a partial fraction expansion.

mass-spring damper with zero input force and nonzero initial conditions

4.3.2 A differential equation with zero initial conditions and non-zero input

Given a constant scalar f, consider the differential equation

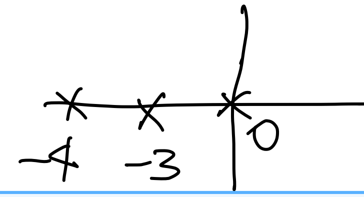
$$\ddot{y} + 7\dot{y} + 12y = f, \qquad y(0) = 0,\ \dot{y}(0) = 0 \tag{4.77}$$

We apply the Laplace transform to both sides to obtain

$$s^2 Y(s) + 7sY(s) + 12Y(s) = \frac{f}{s} \tag{4.78}$$

where $Y(s) = \mathcal{L}[y(t)]$, using the derivative properties (4.69) and (4.70), and the equality $\mathcal{L}[f] = \mathcal{L}[f \cdot \mathbf{1}(t)] = f/s$. Thus,

$$Y(s) = \frac{f}{s(s^2 + 7s + 12)} = \frac{f}{s(s+3)(s+4)} \tag{4.79}$$



Equation (4.67) from the previous section on partial fraction expansions states:

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)\,(s+b)\,(s+c)}\right] = \frac{1}{(a-b)(a-c)(b-c)}\Big((b-c)\,\mathrm{e}^{-at} + (a-b)\,\mathrm{e}^{-ct} + (c-a)\,\mathrm{e}^{-bt}\Big) \tag{4.80}$$

Here, with $a = 0$, $b = 3$, and $c = 4$, the solution to the differential equation (4.77) is

$$y(t) = \frac{f}{(-3)(-4)(3-4)}\Big((3-4) + (-3)\,\mathrm{e}^{-4t} + (4)\,\mathrm{e}^{-3t}\Big) \tag{4.81}$$

$$y(t) = \frac{f}{12}\Big(1 + 3\,\mathrm{e}^{-4t} - 4\,\mathrm{e}^{-3t}\Big) \tag{4.82}$$

$Y(s) = r_1 \frac{1}{s} + r_2 \frac{1}{s+3} + r_3 \frac{1}{s+4} \ldots$

FORCED RESPONSE

4.3.3 Combining the previous two examples

Consider the differential equation

$$\ddot{y} + 7\dot{y} + 12y = f, \qquad y(0) = y_0,\ \dot{y}(0) = v_0 \tag{4.83}$$

Note: this problem involves the same differential equation as in Sections 4.3.1 and 4.3.2, but here we have both non-zero initial conditions y_0 and v_0, and a constant scalar input f.

We assert that the solution is the sum of the solutions from the previous cases. Adding the solution from equation (4.76) to the solution from equation (4.81), we obtain

$$y(t) = -(3y_0 + v_0)\,\mathrm{e}^{-4t} + (4y_0 + v_0)\,\mathrm{e}^{-3t} + \frac{f}{12}\left(1 + 3\,\mathrm{e}^{-4t} - 4\,\mathrm{e}^{-3t}\right) \tag{4.84}$$

The justification for this formula is provided in the subsequent section.

4.3.4 General case

Suppose we have a dynamical system with an input of the form:

$$a_0 y(t) + a_1 \frac{dy}{dt}(t) + \cdots + a_n \frac{d^n y}{dt^n}(t) = u(t) \tag{4.85}$$

where

- y is the output and u is the control input,
- $a_0, \ldots, a_n$ are constant coefficients.

We assume the initial conditions are given by:

$$\frac{d^i y}{dt^i}(0) = y_0^i \qquad \text{for } i = 0, 1, \ldots, n-1 \tag{4.86}$$

$$\mathcal{L}[\dot{y}(t)] = sY(s) - y(0)s^0$$

$$\mathcal{L}[\ddot{y}(t)] = s^2 Y(s) - s\dot{y}(0) - y(0)s^0$$

$$\mathcal{L}[\dddot{y}(t)] = s^3 Y(s) - s^2 \ddot{y}(0) - s\dot{y}(0) - y(0)s^0$$

$$\vdots$$

$$\mathcal{L}[y^{(i)}(t)] = s^i Y(s) - \sum_{k=0}^{i-1} s^{i-1-k} y^{k}(0)$$

The Laplace transform of both sides of the differential equation (4.85) yields:

$$\sum_{i=0}^{n} a_i \left(s^i Y(s) - \sum_{k=0}^{i-1} s^{i-1-k} y_0^k \right) = U(s) \tag{4.87}$$

After reorganizing, we have:

$$Y(s) = \frac{U(s)}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} + \frac{\sum_{i=0}^{n} \sum_{k=0}^{i-1} a_i s^{i-1-k} y_0^k}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} =: Y_{\text{forced}}(s) + Y_{\text{free}}(s)$$

which implies

$$y(t) = \mathcal{L}^{-1}\left[Y_{\text{forced}}(s)\right] + \mathcal{L}^{-1}\left[Y_{\text{free}}(s)\right] = y_{\text{forced}}(t) + y_{\text{free}}(t)$$

We derive several important insights:

(i) The *forced response* $y_{\text{forced}}(t)$ is due to a non-zero input $u(t)$, with zero initial conditions.

(ii) The *free response* $y_{\text{free}}(t)$ is due to non-zero initial conditions, with zero input $u(t) = 0$.

(iii) *The response $y(t)$ is the sum of the forced and free responses*. In a *linear control system, the response is determined by two causes: the initial conditions and the external input. The overall response is the sum of the effects produced by each cause.*

(iv) The *characteristic polynomial* is the denominator in the Laplace transform of both forced and free responses:

$$\sum_{i=0}^{n} a_i s^i = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \tag{4.88}$$

(v) The roots of the characteristic polynomial determine the exponential signals in the free response.

4.4 Additional Laplace transform properties and Laplace transform pairs

The Laplace transform satisfies numerous properties beyond (P1)–(P4) listed in Section 4.1.3. Here are some additional properties that can be useful for analyzing dynamical systems:

(P5) **Final Value Theorem:** $\lim_{t\to+\infty} f(t) = \lim_{s\to 0} sF(s)$, if the limit of f exists finite[3]

(P6) **Initial Value Theorem:** $\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$

(P7) **Time delay:** $\mathcal{L}[f(t-T)] = \mathrm{e}^{-sT} F(s)$ (recall $f(t-T) = 0$ for all $t < T$)

(P8) **Convolution:** $\mathcal{L}\Big[\int_0^t f(\tau)g(t-\tau)d\tau\Big] = F(s)G(s)$, where $\int_0^t f(\tau)g(t-\tau)d\tau$ is the *convolution integral*.

(P9) **Time scaling:** $\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

(P10) **Complex derivative:** $\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$

Higher-order derivatives: $\mathcal{L}[(-1)^n t^n f(t)] = (-1)^s \frac{d}{ds}F(s)$

Additionally, we present in this section some Laplace transform pairs that complement those presented in the first Table 4.2 of Laplace transform pairs in Section 4.1.6.

[3] The limit of f exists finite if all the poles of the rational function $sF(s)$ are on the left half plane, as we will study in the next chapter.

	Function of time $f(t)$	*Laplace transform $F(s)$ and its poles*	
	In this table, we consider only exponential signals.	Laplace transforms of exponential signals are rational functions.	
(9)	t^n (for any $n = 1, 2, 3, \dots$)	$\frac{n!}{s^{n+1}}$	$s = 0$, repeated $n+1$ times
(10)	$t\,\mathrm{e}^{-at}$	$\frac{1}{(s+a)^2}$	$s = -a$, repeated
(11)	$t^n\,\mathrm{e}^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$s = -a$, repeated $n+1$ times
(7)	$\mathrm{e}^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$s = -a \pm \mathrm{i}\omega$
(8)	$\mathrm{e}^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	$s = -a \pm \mathrm{i}\omega$
(12)	$\frac{1}{b-a}(\mathrm{e}^{-at} - \mathrm{e}^{-bt})$	$\frac{1}{(s+a)(s+b)}$	$s = -a, -b$, $a \neq b$
(13)	$\frac{1}{b-a}(b\,\mathrm{e}^{-bt} - a\,\mathrm{e}^{-at})$	$\frac{s}{(s+a)(s+b)}$	$s = -a, -b$, $a \neq b$
(14)	$\frac{1}{(a-b)(a-c)(b-c)}\Big((c-a)\,\mathrm{e}^{-bt} + (a-b)\,\mathrm{e}^{-ct} + (b-c)\,\mathrm{e}^{-at}\Big)$	$\frac{1}{(s+a)(s+b)(s+c)}$	$s = -a, -b, -c$, $a \neq b \neq c$
(14)	$\frac{1}{\omega(a^2+\omega^2)}\Big(\omega\,\mathrm{e}^{-at} + a\sin(\omega t) - \omega\cos(\omega t)\Big)$	$\frac{1}{(s+a)(s^2+\omega^2)}$	$s = -a, \pm\mathrm{i}\omega$

Table 4.3: Row (11) is more general than rows (9) and (10) (as well as rows (1), (2), and (4) in Table 4.2) and focuses on the case of a single real pole, possibly repeated. Recall $n!$ is the factorial of n: $0! = 1, 1! = 1, 2! = 2, 3! = 6, \dots$.
Rows (7), (8), (12), and (13) capture all possible cases of two poles, not repeated. Either both poles are real or the two poles are complex conjugate. Rows (7), (8) are repeated here for convenience.
Rows (14) and (15) are only two examples of a rational function with three distinct poles.

4.5 Appendix: A brief review of complex numbers

We let i denote the *imaginary unit*, that is, $\mathrm{i}^2 = -1$. Any *complex number* z is of the form

$$z = x + \mathrm{i}y, \tag{4.89}$$

where x and $\mathrm{i}y$ are the real and imaginary parts. When useful, we denote the set of complex numbers by $\mathbb{C}$, also known as the *complex plane*. The *conjugate* of z is

$$\bar{z} = x - \mathrm{i}y. \tag{4.90}$$

The *magnitude* (or *modulus*) of a complex number z is

$$|z| = \sqrt{x^2 + y^2}. \tag{4.91}$$

Recall that $|z_1 z_2| = |z_1| \cdot |z_2|$ and $|1/z| = 1/|z|$. The *argument* of a complex number z is the angle θ formed by the line representing the complex number in the complex plane with the positive real axis, measured counterclockwise. The argument[4] is denoted by

$$\arg(z) = \theta. \tag{4.92}$$

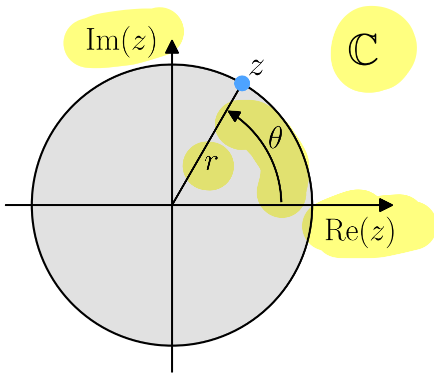


Figure 4.7: The *Euler formula* is $\mathrm{e}^{\mathrm{i}\theta} = \cos(\theta) + \mathrm{i}\sin(\theta)$. A complex number can be represented in *polar form* as $r(\cos\theta + \mathrm{i}\sin\theta)$, where r is the magnitude of the complex number and θ is its argument:

$$z = r\big(\cos(\theta) + \mathrm{i}\sin(\theta)\big).$$

The inverse Euler formulas are:

$$\sin(\theta) = \frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2\mathrm{i}} \quad \text{and} \quad \cos(\theta) = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}.$$

[4] The argument of a complex number $z = a + bi$ can be computed using the two-argument arctangent function, also known as atan2(b, a), which accounts for the signs of a and b to return the correct quadrant for the angle.

4.6 Exercises

Section 4.1: The Laplace transform

E4.1 **Laplace transforms #1.** Given a signal $x(t)$, let $X(s)$ denote its Laplace transform. Using Laplace transform properties and tables, compute

(i) $\mathcal{L}\left[\dot{x}(t) + 4\,\mathrm{e}^{-2t} - 3\right]$,

(ii) $\mathcal{L}\left[\int_0^t x(\tau)d\tau + \cos(5t)\right]$,

(iii) $\mathcal{L}\left[\ddot{x}(t) - t^2\,\mathrm{e}^{-t} + \mathrm{e}^t \sin(7t)\right]$.

Hint: Use row (11) in Table 4.3.

Answer:

(i) From the derivative property (P2), $\mathcal{L}[\dot{x}(t)] = sX(s) - x(0)$. The term $4\,\mathrm{e}^{-2t}$ transforms to $\frac{4}{s+2}$ and the constant -3 transforms to $\frac{-3}{s}$. Using linearity,

$$\mathcal{L}\left[\dot{x}(t) + 4\,\mathrm{e}^{-2t} - 3\right] = sX(s) - x(0) + \frac{4}{s+2} - \frac{3}{s} \tag{4.93}$$

(ii) From the integral property (P3),

$$\mathcal{L}\left[\int_0^t x(\tau)d\tau\right] = \frac{1}{s}X(s). \tag{4.94}$$

The transform of $\cos(5t)$ is $\frac{s}{s^2+25}$. Using linearity,

$$\mathcal{L}\left[\int_0^t x(\tau)d\tau + \cos(5t)\right] = \frac{1}{s}X(s) + \frac{s}{s^2+25} \tag{4.95}$$

(iii) From the second derivative property (4.70),

$$\mathcal{L}[\ddot{x}(t)] = s^2X(s) - sx(0) - \dot{x}(0). \tag{4.96}$$

From row (11) in Table 4.3,

$$\mathcal{L}[t^2\,\mathrm{e}^{-t}] = \frac{2}{(s+1)^3}. \tag{4.97}$$

The transform of $\mathrm{e}^t \sin(7t)$ is $\frac{7}{(s-1)^2+49}$. Combining terms,

$$\mathcal{L}\left[\ddot{x}(t) - t^2\,\mathrm{e}^{-t} + \mathrm{e}^t \sin(7t)\right] = s^2X(s) - sx(0) - \dot{x}(0) - \frac{2}{(s+1)^3} + \frac{7}{(s-1)^2+49} \tag{4.98}$$

▼

E4.2 **Laplace transforms #2.** Given a signal $x(t)$, let $X(s)$ denote its Laplace transform. Using Laplace transform properties and tables, compute

(i) $\mathcal{L}\left[\ddot{x}(t)+3\,\mathrm{e}^{-2t}-5\right]$,
(ii) $\mathcal{L}\left[t\,\mathrm{e}^{-2t}-t^2\,\mathrm{e}^{-t}+\mathrm{e}^{3t}\sin(5t)\right]$,
(iii) $\mathcal{L}\left[\sin(\omega t+\theta)\right]$, where ω and θ are constant parameters.

Answer:

(i) From the second derivative property,

$$\mathcal{L}[\ddot{x}(t)] = s^2X(s) - sx(0) - \dot{x}(0). \tag{4.99}$$

The transform of $3\,\mathrm{e}^{-2t}$ is $\frac{3}{s+2}$ and the transform of -5 is $\frac{-5}{s}$. Using linearity,

$$\mathcal{L}\left[\ddot{x}(t)+3\,\mathrm{e}^{-2t}-5\right] = s^2X(s) - sx(0) - \dot{x}(0) + \frac{3}{s+2} - \frac{5}{s} \tag{4.100}$$

(ii) From row (11) in Table 4.3,

$$\mathcal{L}[t\,\mathrm{e}^{-2t}] = \frac{1}{(s+2)^2}, \quad \mathcal{L}[t^2\,\mathrm{e}^{-t}] = \frac{2}{(s+1)^3}. \tag{4.101}$$

The transform of $\mathrm{e}^{3t}\sin(5t)$ is $\frac{5}{(s-3)^2+25}$. Combining,

$$\mathcal{L}\left[t\,\mathrm{e}^{-2t}-t^2\,\mathrm{e}^{-t}+\mathrm{e}^{3t}\sin(5t)\right] = \frac{1}{(s+2)^2} - \frac{2}{(s+1)^3} + \frac{5}{(s-3)^2+25} \tag{4.102}$$

(iii) Using $\sin(\omega t+\theta) = \sin(\omega t)\cos(\theta) + \cos(\omega t)\sin(\theta)$,

$$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}, \quad \mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}. \tag{4.103}$$

Therefore,

$$\mathcal{L}\left[\sin(\omega t+\theta)\right] = \frac{\omega\cos(\theta)+s\sin(\theta)}{s^2+\omega^2} \tag{4.104}$$

▼

E4.3 **Laplace transforms #3.** Given a signal $x(t)$, let $X(s)$ denote its Laplace transform. Using Laplace transform properties and tables, compute

(i) $\mathcal{L}\left[\dot{x}(t) + \mathrm{e}^{-3t} + 2\right]$

(ii) $\mathcal{L}\left[\int_0^t x(\tau)d\tau + \sin(3t)\right]$

(iii) $\mathcal{L}\left[\ddot{x}(t) + t\,\mathrm{e}^{-t} - \mathrm{e}^{2t}\cos(6t)\right]$

Section 4.2: The Inverse Laplace transform

E4.4 **Inverse Laplace transforms #1.** Using the Tables 4.2 and 4.3 of Laplace transforms and the partial fraction expansion method, verify:

$$\mathcal{L}^{-1}\left[\frac{1}{s\tau+1}\right] = \frac{1}{\tau}\,\mathrm{e}^{-t/\tau}, \tag{4.105}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s\,(s\tau+1)}\right] = 1-\mathrm{e}^{-t/\tau}, \tag{4.106}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2\,(s\tau+1)}\right] = t-\tau(1-\mathrm{e}^{-t/\tau}). \tag{4.107}$$

E4.5 **Inverse Laplace transforms #2.** Compute the inverse Laplace transforms of:

(i) $\dfrac{s+3}{s(s+1)}$,

(ii) $\dfrac{s+3}{s^2+2s+10}$,

(iii) $\dfrac{s+3}{(s+1)^2(s+2)}$, and

(iv) $\dfrac{s+3}{s(s^2+\omega^2)}$.

Section 4.3: Solving linear differential equations

E4.6 **Solving a differential equation via the Laplace transform #1.** Consider the differential equation (without inputs)

$$\ddot{x} + 4\dot{x} + 5x = 0, \qquad x(0) = \dot{x}(0) = 1. \tag{4.108}$$

(i) Compute the solution in the Laplace domains $X(s) = \mathcal{L}[x(t)]$.
(ii) Expand the solution in a partial fraction.
(iii) Compute the inverse Laplace transform to obtain $x(t)$.

Answer:

(i) We take the Laplace transform of left and right hand side:

$$s^2X(s) - sx(0) - \dot{x}(0) + 4sX(s) - 4x(0) + 5X(s) = 0 \tag{4.109}$$

$$\iff \quad s^2X(s) - s - 1 + 4sX(s) - 4 + 5X(s) = 0 \tag{4.110}$$

We then compute $X(s)$ and write it in partial fraction expansion

$$X(s) = \frac{s+5}{s^2+4s+5} \tag{4.111}$$

(ii) We then write $X(s)$ in partial fraction expansion. First, we note that $s^2 + 4s + 5 = (s + 2)^2 + 1$, that is, we have a pair of complex conjugate roots. Therefore, as in the fourth example in Section 4.2.1, we know that damped sine and cosine waves will appear. After solving the linear equations to match the numerators we obtain:

$$X(s) = \frac{s+5}{s^2+4s+5} = \frac{s+2}{(s+2)^2+1} + \frac{3}{(s+2)^2+1} \tag{4.112}$$

where, in the last equality, we wrote the denominator as the sum of two positive numbers.

(iii) Finally, using the inverse Laplace transform of damped sine and cosine from the Table 4.2 of Laplace transforms:

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathrm{e}^{-2t}\cos(t) + 3\,\mathrm{e}^{-2t}\sin(t) \tag{4.113}$$

▼

E4.7 **Solving a differential equation via the Laplace transform #2.** Consider the differential equation

$$\ddot{y}(t) + 4\dot{y}(t) + 3y(t) = 6,$$

with initial conditions $y(0) = 2$ and $\dot{y}(0) = 0$.

(i) Take the Laplace transform of both sides and solve for $Y(s)$.
(ii) Perform partial fraction expansion on $Y(s)$.
(iii) Find $y(t)$ by taking the inverse Laplace transform.

Answer:

(i) Using $\mathcal{L}[\ddot{y}(t)] = s^2Y(s) - sy(0) - \dot{y}(0)$ and $\mathcal{L}[\dot{y}(t)] = sY(s) - y(0)$, we obtain:

$$s^2Y(s) - s \cdot 2 - 0 \; + \; 4\big[sY(s) - 2\big] \; + \; 3Y(s) = 6 \cdot \frac{1}{s}.$$

Simplify:

$$(s^2 + 4s + 3)Y(s) - 2s - 8 = \frac{6}{s}.$$

Hence

$$(s^2 + 4s + 3)Y(s) = 2s + 8 + \frac{6}{s}.$$

So

$$\boxed{Y(s) = \frac{2s^2 + 8s + 6}{s(s^2 + 4s + 3)}.}$$

(ii) Factor the denominator $s^2 + 4s + 3 = (s + 1)(s + 3)$. We decompose:

$$Y(s) = \frac{2s^2 + 8s + 6}{s(s+1)(s+3)}.$$

We seek constants A, B, C such that

$$\frac{2s^2 + 8s + 6}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}.$$

Multiply both sides by $s(s + 1)(s + 3)$ and compare coefficients to find $A = 2$, $B = 3$, $C = 1$. Thus

$$\boxed{Y(s) = \frac{2}{s} + \frac{3}{s+1} + \frac{1}{s+3}.}$$

(iii) Using $\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = \mathrm{e}^{-at}$ and $\mathcal{L}^{-1}\left[\frac{1}{s}\right] = \mathbf{1}(t)$, the solution is

$$y(t) = 2 \cdot \mathbf{1}(t) + 3\,\mathrm{e}^{-t} + \mathrm{e}^{-3t}\,.$$

▼

E4.8 **Solving a differential equation via the Laplace transform #3.** Consider the differential equation (without inputs):

$$\dddot{x} + 6\ddot{x} + 5\dot{x} - 12x = 0, \qquad x(0) = 1, \quad \dot{x}(0) = 0, \quad \ddot{x}(0) = -1. \tag{4.114}$$

(i) Compute the solution in the Laplace domain $X(s) = \mathcal{L}[x(t)]$.

(ii) Compute the roots of the denominator of the rational function $X(s)$.

(iii) Expand $X(s)$ in partial fractions and compute the unknown coefficients.

(iv) Compute the inverse Laplace transform to obtain $x(t)$.

E4.9 **Solving a differential equation via the Laplace transform #4.** Consider the differential equation with an input

$$\ddot{y} - y = t, \qquad y(0) = \dot{y}(0) = 1. \tag{4.115}$$

(i) Compute the solution in the Laplace domain $Y(s) = \mathcal{L}[y(t)]$.

(ii) Compute the inverse Laplace transform of $Y(s)$ to obtain $y(t)$.

Hint: There are at least two distinct ways to solve this problem. One possible way is to use these additional Laplace transform pairs: $\mathcal{L}[\cosh(t)] = \dfrac{s}{s^2-1}$ and $\mathcal{L}[\sinh(t)] = \dfrac{1}{s^2-1}$.

E4.10 **Programming exercise.** Verify the solutions to ordinary differential equations in Section 4.3 by modifying the following Python SymPy code.

```
# Python code to solve a differential equation
from sympy import Function, dsolve, Eq, symbols, init_printing
from sympy.abc import t

# Define the symbols
y0, v0 = symbols('y0 v0')

# Define the function which represents y(t)
y = Function('y')

# Define the differential equation as in Section 4.3
diffeq = Eq(y(t).diff(t, t) + 7*y(t).diff(t) + 12*y(t), 0)

# Solve the differential equation with initial conditions
solution = dsolve(diffeq, y(t), ics={y(0): y0, y(t).diff(t).subs(t, 0): v0})

# Extract and print the right-hand side
print("The ode solution is:")
rhs = solution.rhs; print("y(t) =", rhs)
```

Listing 4.3: Python script illustrating SymPy's abilities, see Figure 4.8.
Available at sympy-demo-ode.py

```
The ode solution is:
y(t) = (v0 + 4*y0 + (-v0 - 3*y0)*exp(-t))*exp(-3*t)
```

Figure 4.8: Output of the sympy-demo-ode.py program. Simple calculations show that this solution is the same as in equation (4.76).

Answer: No solution is available at this time. ▼

E4.11 **Free response of undamped harmonic oscillator.** Consider the undamped harmonic oscillator $m\ddot{x} + kx = 0$ without any input, where $m > 0$ and $k > 0$.

(i) Compute the free response in the Laplace domain $X(s)$ from initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$

(ii) Compute the free response $x(t)$ by performing the inverse Laplace transform of $X(s)$.

Note: *We studied the undamped harmonic oscillator in Section 2.1.2. The results in this exercises are consistent with that previous analysis.*

E4.12 **Padé approximants for systems with time delayed inputs.** Consider a first-order system with state variable x and time constant $\tau > 0$. Assume that the initial condition is zero and that the input to the system is a *delayed unit impulse*, i.e., a unit impulse that occurs at time $T > 0$.

(i) Write the differential equation that describes this system and the input. Use the unit impulse function δ.
Hint: In other words, the input is zero at each time except $t = T$.

(ii) Compute the solution to the delayed impulse by simply delaying the solution to the unit impulse. In other words,

(a) write down the solution $x(t)$ to the same first-order system with an unit impulse applied at time $t = 0$, and then
(b) modify the solution by delaying by time T.

(iii) Take the Laplace transform of the differential equation from part (i) and compute an expression for $X(s)$. Is $X(s)$ a rational function?
Hint: The time-delay property (P7) in Section 4.4 may be useful.

(iv) For analyzing systems with time delays, the so-called Padé approximants are a useful tool. The *first-order Padé approximant* of the exponential function is

$$e^y \approx \frac{2+y}{2-y}. \tag{4.116}$$

Use the first-order Padé approximant and the inverse Laplace transform to obtain an approximate solution $x_{\text{approx}}(t)$ in the time domain as a function of the parameters τ and T.

(v) Show that, at fixed time constant $\tau > 0$ and time $t > 0$, we have $\lim_{T\to 0}\big(x(t) - x_{\text{approx}}(t)\big) = 0$.

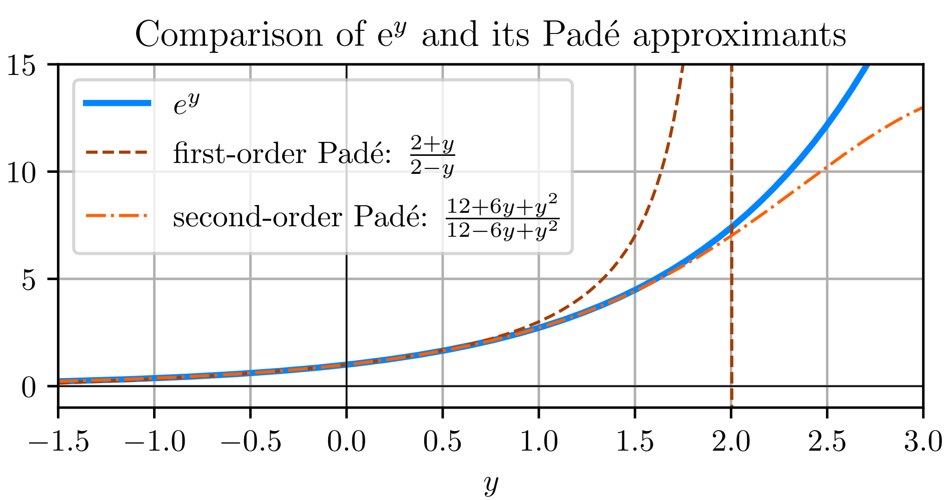


Figure 4.9: Comparison of the exponential function e^y with its first-order Padé approximant $(2+y)/(2-y)$.
The first-order Padé approximant provides a close approximation around $y = 0$, but diverges with a vertical asymptote at $y = 2$.
For completeness, we display also the second-oder Padé approximant $(12+6y+y^2)/(12-6y+y^2)$.
Image generated by pade-approximant.py .

Answer:

(i) The differential equation is

$$\tau\dot{x}(t) + x(t) = \delta(t - T).$$

(ii) The response of a first-order system subject to a unit impulse at time zero is

$$x(t) = \frac{1}{\tau}\,\mathrm{e}^{-t/\tau}, \quad t \geq 0.$$

Therefore, the response to a delayed unit impulse at $t = T$ is

$$x(t) = \begin{cases} 0, & 0 \leq t < T \\ \frac{1}{\tau}\,\mathrm{e}^{-(t-T)/\tau}, & t \geq T. \end{cases}$$

(iii) Taking the Laplace transform using the time-delay property (P7) and rearranging yields

$$X(s) = \frac{\mathrm{e}^{-sT}}{1 + \tau s}$$

No, $X(s)$ is not a rational function because it is not the ratio of polynomials. Hence, the usual methods to compute inverse Laplace transform do not apply.

(iv) Using the Padé approximation yields

$$X_{\text{approx}}(s) = \frac{2 - Ts}{(2 + Ts)(1 + \tau s)}.$$

We can compute the partial fraction expansion as

$$X_{\text{approx}}(s) = \frac{2 - Ts}{(2 + Ts)(1 + \tau s)} = \frac{\alpha}{2 + Ts} + \frac{\beta}{1 + \tau s}.$$

The matching method yields the expressions $\alpha = \frac{4T}{T - 2\tau}$ and $\beta = -\frac{T + 2\tau}{T - 2\tau}$. Substitution and the inverse Laplace transform yields

$$x_{\text{approx}}(t) = \frac{4}{T - 2\tau}\,\mathrm{e}^{-2t/T} - \frac{T + 2\tau}{\tau(T - 2\tau)}\,\mathrm{e}^{-t/\tau}$$

(v) We compute

$$\lim_{T\to 0^+} x_{\text{approx}}(t) = \lim_{T\to 0^+} \frac{4}{T - 2\tau}\,\mathrm{e}^{-2t/T} - \lim_{T\to 0^+} \frac{T + 2\tau}{\tau(T - 2\tau)}\,\mathrm{e}^{-t/\tau} = 0 - \frac{2\tau}{\tau(-2\tau)}\,\mathrm{e}^{-t/\tau} = \frac{1}{\tau}\,\mathrm{e}^{-t/\tau} = \lim_{T\to 0^+} x(t)$$

▼

Section 4.4: Additional Laplace transform properties

E4.13 **Dynamical behavior of a muscle.** In E2.3, we modeled a muscle using a spring damper system and obtained its equations of motion[5] by applying Newton's law.

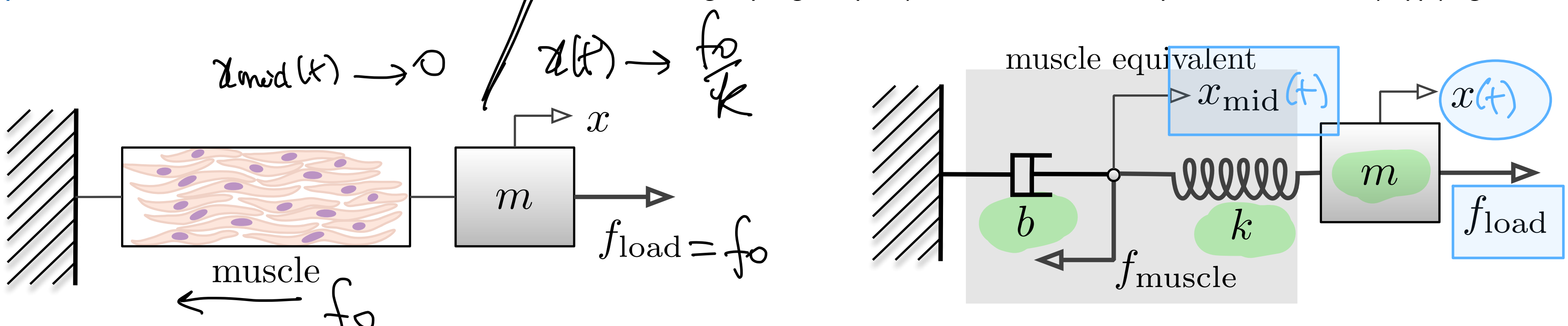


Figure 4.10: Left image: A muscle connected to a fixed point and subject to a load force. Right image: Equivalent mechanical system for the muscle excitation.

The system variables are $x(t)$, $\dot{x}(t)$, and $x_{\text{mid}}(t)$. The system parameters are the positive constants m, k, and b. The system inputs are $f_{\text{load}}(t)$ and $f_{\text{muscle}}(t)$. Assume that the equations of motion for this system are:

$$m\ddot{x} = -k(x - x_{\text{mid}}) + f_{\text{load}}(t),$$
$$b\dot{x}_{\text{mid}} = -k(x_{\text{mid}} - x) - f_{\text{muscle}}(t).$$

(i) Assuming that $x(0) = \dot{x}(0) = x_{\text{mid}}(0) = 0$, convert these differential equations to the Laplace domain.

(ii) Obtain expressions for $X(s)$ and $X_{\text{mid}}(s)$ in terms of the parameters m, b, k, and the Laplace transform of the input forces $F_{\text{load}}(s) = \mathcal{L}[f_{\text{load}}]$ and $F_{\text{muscle}}(s) = \mathcal{L}[f_{\text{muscle}}]$.

(iii) Assume that the load force $f_{\text{load}}(t)$ and the muscle force $f_{\text{muscle}}(t)$ are equal to a constant value f_0 that begins acting at time $t = 0$. Substitute the Laplace transform of these forces into the expressions for $X(s)$ and $X_{\text{mid}}(s)$ computed in part (ii).

(iv) Use the Final Value Theorem (P5) in Section 4.4 (assume the limit exists) in order to find the final positions of $x(t)$ and $x_{\text{mid}}(t)$.

Answer:

when it exists, $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

[5] As a reminder, a muscle connected to a fixed point and subject to a load force can be modeled by the equivalent mechanical system shown in the Figure 2.27. The key elements of the system are: (1) The muscle connects the fixed point to a mass m at position x. (2) The muscle is represented by the interconnection of two components, with the intermediate point at coordinate x_{mid}. (3) The muscle exerts a force $f_{\text{muscle}}(t)$ at the intermediate point. (4) A damper with damping coefficient b connects the intermediate point to the stationary point. (5) A spring with stiffness k and zero rest length connects the intermediate point to the mass. (6) The mass m is subject to a load force $f_{\text{load}}(t)$.

(i) Using the properties of the Laplace transform, we may convert the system to the Laplace domain.

$$ms^2X(s) = -k(X(s) - X_{\text{mid}}(s)) + f_{\text{load}}(s)$$
$$bsX_{\text{mid}}(s) = -k(X_{\text{mid}}(s) - X(s)) - f_{\text{muscle}}(s)$$

(ii) After some algebra and rearranging the equations, we obtain:

$$(ms^2 + k)X(s) - kX_{\text{mid}}(s) = f_{\text{load}}(s),$$
$$-kX(s) + (bs + k)X_{\text{mid}}(s) = -f_{\text{muscle}}(s).$$

We now have a system of two linear equations in two variables.
Step 1: Express $X_{\text{mid}}(s)$ as a function of $X(s)$. From the second equation, solve for $X_{\text{mid}}(s)$ in terms of $X(s)$:

$$(bs + k)X_{\text{mid}}(s) = kX(s) - f_{\text{muscle}}(s),$$
$$X_{\text{mid}}(s) = \frac{kX(s) - f_{\text{muscle}}(s)}{bs + k}. \tag{4.117}$$

Step 2: Substitute into the first equation to compute $X(s)$. Substitute the expression for $X_{\text{mid}}(s)$ into the first equation:

$$(ms^2 + k)X(s) - k\left(\frac{kX(s) - f_{\text{muscle}}(s)}{bs + k}\right) = f_{\text{load}}(s),$$

Multiply both sides by $(bs + k)$ to eliminate the fraction:

$$(bs + k)(ms^2 + k)X(s) - (k^2X(s) - kf_{\text{muscle}}(s)) = f_{\text{load}}(s)(bs + k).$$

Collecting all the terms involving $X(s)$, we obtain

$$\left[(bs + k)(ms^2 + k) - k^2\right] X(s) = f_{\text{load}}(s)(bs + k) + kf_{\text{muscle}}(s).$$

so that

$$X(s) = \frac{f_{\text{load}}(s)(bs + k) + kf_{\text{muscle}}(s)}{(bs + k)(ms^2 + k) - k^2}.$$

Step 3: Solve for $X_{\text{mid}}(s)$. Substitute the expression for $X(s)$ into the equation (4.117)

$$X_{\text{mid}}(s) = \frac{1}{bs + k}\left(k\frac{f_{\text{load}}(s)(bs + k) + kf_{\text{muscle}}(s)}{(bs + k)(ms^2 + k) - k^2} - f_{\text{muscle}}(s)\right).$$

In summary, after some manipulations

$$X(s) = \frac{F_{\text{load}}(s)(bs+k) - kF_{\text{muscle}}(s)}{mbs^3 + mks^2 + kbs}$$

$$X_{\text{mid}}(s) = \frac{-(ms^2+k)F_{\text{muscle}}(s) + kF_{\text{load}}(s)}{mbs^3 + mks^2 + kbs}$$

One can show that the following expressions are equivalent:

$$X(s) = \left(\frac{1}{k} f_{\text{load}}(s) - \frac{1}{bs+k} f_{\text{muscle}}(s)\right) \frac{bks + k^2}{mbs^3 + (mk+bk)s^2 + bks}$$

$$X_{\text{mid}}(s) = \left(\frac{1}{ms^2+k} f_{\text{load}}(s) - \frac{1}{k} f_{\text{muscle}}(s)\right) \frac{mks^2 + k^2}{mbs^3 + (mk+bk)s^2 + bks}$$

(iii) Using the properties of Laplace transforms, we compute $f_{\text{load}}(s) = f_{\text{muscle}}(s) = \frac{f_0}{s}$ and, plugging in,

$$X(s) = \frac{bf_0}{(mbs^2 + mks + bk)s}$$

$$X_{\text{mid}}(s) = \frac{-mf_0}{(mbs^2 + mks + bk)}$$

(iv) Using the final value theorem, we obtain

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \frac{f_0}{k} \qquad \text{and} \qquad \lim_{t\to\infty} x_{\text{mid}}(t) = \lim_{s\to 0} sX_{\text{mid}}(s) = 0$$

▼

E4.14 **Laplace transforms based upon additional properties.** Perform the following calculations using the properties in Section 4.4 on the additional Laplace transform properties.

(i) Compute $\mathcal{L}[z(t-2)\mathbf{1}(t-2)]$, using the time-delay property (P7).

(ii) Compute $\mathcal{L}\left[\int_0^t (t-\tau)x(\tau)d\tau\right]$, using the convolution integral property (P8).

(iii) Compute the final value of $y(t)$ where $Y(s) = \frac{5}{s(s+2)(s+4)}$ using the final value theorem (P5) and assuming the limit exists.

Answer: The solutions are given as follows:

(i) From the time-delay property (P7), the Laplace transform of $z(t-2)\mathbf{1}(t-2)$ is given by $\mathrm{e}^{-2s}Z(s)$, since the signal is delayed by 2 units. Thus, the solution is:

$$\mathcal{L}[z(t-2)\mathbf{1}(t-2)] = \mathrm{e}^{-2s}Z(s)$$

(ii) The convolution theorem states that $\mathcal{L}\left[\int_0^t (t-\tau)x(\tau)d\tau\right] = \frac{X(s)}{s^2}$. Thus, the solution is:

$$\mathcal{L}\left[\int_0^t (t-\tau)x(\tau)d\tau\right] = \frac{X(s)}{s^2}$$

(iii) The final value theorem states that

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s).$$

We first compute the expression for $sY(s)$:

$$sY(s) = s \cdot \frac{5}{s(s+2)(s+4)} = \frac{5}{(s+2)(s+4)}.$$

Now, we take the limit as $s \to 0$:

$$\lim_{s\to 0} \frac{5}{(s+2)(s-4)} = \frac{5}{(0+2)(0+4)} = \frac{5}{2\cdot 4} = \frac{5}{8}.$$

Thus, the final value of $y(t)$ is:

$$\lim_{t\to\infty} y(t) = \frac{5}{8}$$

▼

E4.15 **Conceptual check on the convolution property.** In this exercise, we review the convolution property (P8) in Section 4.4. Consider the signal $f(t) = t^2$ and $g(t) = \mathrm{e}^{-t}\,\mathbf{1}(t)$. Define the convolution

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

(i) Write down the Laplace transforms $F(s)$ and $G(s)$.

(ii) Explain how the convolution theorem gives $H(s)$ in the s-domain.

(iii) Use these results to find $h(t)$ by inverse Laplace transform, without directly evaluating the time convolution integral.

Answer:

(i) Using standard Laplace tables, we get

$$F(s) = \mathcal{L}[t^2] = \frac{2}{s^3}, \quad G(s) = \mathcal{L}[\mathrm{e}^{-t}] = \frac{1}{s+1}.$$

(ii) By the convolution theorem,

$$H(s) = F(s)G(s) = \frac{2}{s^3} \cdot \frac{1}{s+1} = \frac{2}{s^3(s+1)}.$$

(iii) We decompose in partial fractions:

$$\frac{2}{s^3(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1}.$$

Multiplying both sides by $s^3(s+1)$, and matching coefficients, yields $A = 2$, $B = -2$, $C = 2$, $D = -2$. Hence

$$H(s) = \frac{2}{s} - \frac{2}{s^2} + \frac{2}{s^3} - \frac{2}{s+1}.$$

Taking inverse Laplace transforms:

$$h(t) = 2 - 2t + t^2 - 2\,\mathrm{e}^{-t}\,.$$

Thus, by using the convolution theorem, we avoid doing the integral in the time domain explicitly.

▼

E4.16 **An RLC circuit with step input and the final value theorem.** Consider the Final Value Theorem (P5) from Section 4.4. An RLC circuit has $v(t)$ as the voltage across the capacitor and is driven by a step input of amplitude v_0 at $t = 0$. The governing equation is

$$\ell c\ddot{v}(t) + rc\dot{v}(t) + v(t) = v_0, \quad v(0) = 0, \quad \dot{v}(0) = 0, \tag{4.118}$$

where $\ell > 0$, $r > 0$, and $c > 0$ are inductance, resistance, and capacitance.

(i) Take the Laplace transform and solve for $V(s)$.

(ii) Apply the final value theorem to compute $\lim_{t\to\infty} v(t)$.

Answer:

(i) The Laplace transform formulas are

$$\mathcal{L}[\ddot{v}(t)] = s^2V(s), \quad \mathcal{L}[\dot{v}(t)] = sV(s), \quad \mathcal{L}[v(t)] = V(s), \tag{4.119}$$

and for the step input $\mathcal{L}[\mathbf{1}(t)] = \frac{1}{s}$. Substituting into the ODE,

$$\ell cs^2V(s) + rcsV(s) + V(s) = \frac{v_0}{s}. \tag{4.120}$$

Factoring $V(s)$ gives

$$V(s)\left(\ell cs^2 + rcs + 1\right) = \frac{v_0}{s}. \tag{4.121}$$

Therefore

$$V(s) = \frac{v_0}{s\,(\ell cs^2 + rcs + 1)}. \tag{4.122}$$

(ii) By the final value theorem,

$$\lim_{t\to\infty} v(t) = \lim_{s\to 0} sV(s) = \frac{v_0}{\ell c(0)^2 + rc(0) + 1} = v_0. \tag{4.123}$$

The capacitor voltage converges to the input amplitude v_0.

▼

Bibliography

A. Meurer et al. SymPy: symbolic computing in Python. *PeerJ Computer Science*, 3:e103, 2017. doi.