UC Santa Barbara, Department of Mechanical Engineering
ME103 Dynamical Systems. Chapter Slides.

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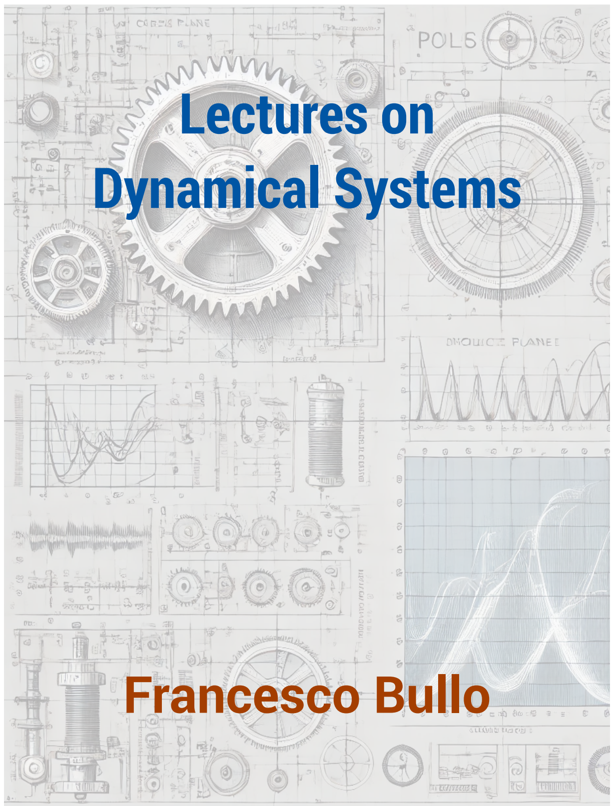
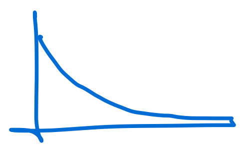
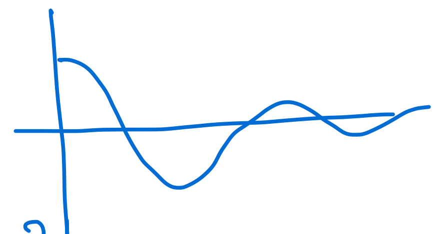
http://motion.me.ucsb.edu/ME103-Fall2025/syllabus.html



Contents

Dynamical Systems

Part I : Models = what are dynamical systems?
where do they arise? ME examples ($F=ma$)

Part II : Analysis = how do dynamical system behave?
First order =  Second order 
How do I understand free response & forced response?
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Part III : Control
PID
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Chapter 3

Thermal and Fluid Systems

This chapter introduces the fundamental aspects of dynamical systems, focusing on heat transfer, fluid dynamics, and the linearization of nonlinear systems.

We begin with an exploration of *conduction* as a primary mechanism of heat transfer, governed by *Fourier's law of heat conduction*. This concept is analogous to Ohm's law and Fick's first law, providing a basis for modeling heat flow in systems such as buildings, where differential equations account for thermal resistance and capacity. The section further discusses temperature control strategies, including *hysteresis control*, to mitigate chattering in on-off control systems. Following this, the principles of *incompressible fluid flow* are examined, emphasizing continuity, force equilibrium, and nonlinear resistance. These principles are illustrated through examples like water tanks and pistons, highlighting how fluid dynamics can lead to stable equilibrium states.

The chapter then focuses on the technique of *linearization*, a method used to simplify the analysis of nonlinear dynamical systems by approximating them with linear models near equilibrium points. This approach is crucial for designing controllers and understanding system behavior, as demonstrated in systems with quadratic drag and pendulum dynamics.

The appendix extends the discussion with basic models of *convective* and *radiative* heat transfer, introducing key parameters and laws such as the *Stefan-Boltzmann law*. Finally, the chapter examines the linearization of systems using *Jacobians* and the analysis of systems with *state-dependent inertia*. Examples such as the double pendulum illustrate how Jacobians are applied in multivariable settings.

3.1 Heat transfer

The field of thermodynamics includes the study of dynamical models for heat flow (also called heat transfer). The three primary mechanisms for heat flow are conduction, convection, and radiation. In these notes we focus on *conduction*, which is the transfer of heat energy through a conducting material from a region of higher temperature to one of lower temperature.

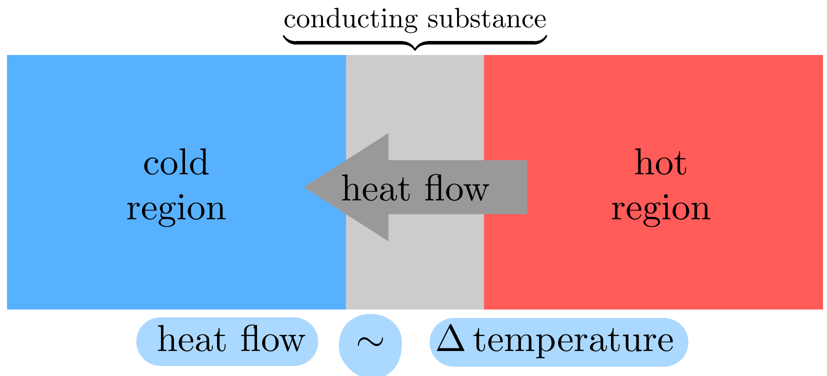


Figure 3.1: Fourier's law states that heat flow is proportional to the temperature differential.

Remark 3.1. *Fourier's law of heat conduction, also known as the resistive heat flow law, is analogous to (i) Ohm's law in electrical circuits, where current is proportional to voltage difference, and (ii) Fick's first law of diffusion in chemistry, where diffusion flux is proportional to concentration gradient.* •

Fourier's law of heat conduction describes the heat flow from region 1 through a substance to region 2 (e.g., from room 1 through a wall to room 2) as:

$$q_{1\to 2} = \frac{1}{r}(T_1 - T_2) \tag{3.1}$$

where

- $q_{1\to 2}$ is the *heat flow rate* from region 1 to region 2, measured in W, i.e., joules per second (J/s);
- T_i is the *temperature* in region $i = 1, 2$, measured in °C, K, or °F;[1]
- r is the *thermal resistance*,[2] measured in °C/W.

Note: In these notes, we allow the heat flow $q_{1\to 2}$ to be both positive (when $T_1 > T_2$) or negative (when $T_2 > T_1$). Other references always assume that $T_1 > T_2$ and talk about only positive heat flow.

Equivalently, one could measure the heat flow in the opposite direction, that is, from room 2 to room 1. Clearly, the heat flow in the opposite direction is equal in magnitude and opposite in sign, that is,

$$q_{2\to 1} = -q_{1\to 2} \qquad \text{so that} \qquad q_{2\to 1} = \frac{1}{r}(T_2 - T_1). \tag{3.2}$$

[1] While the Fahrenheit scale remains the most commonly used in the United States, most of the world uses Celsius. Scientists prefer the Kelvin scale because it is an absolute temperature scale, directly related to absolute zero.

[2] For conduction through a homogeneous material, the thermal resistance is given by $r = \frac{L}{kA}$, where L is the material thickness, k its thermal conductivity, and A the cross-sectional area through which heat flows.

From the *thermal energy balance*, the heat flow from region 1 into region 2 changes the temperature of both regions according to:

$$c_1\dot{T}_1 = q_{2\to1} \quad \text{and} \quad c_2\dot{T}_2 = q_{1\to2} \tag{3.3}$$

where c_i is the *heat capacity* of region i, where $i = 1, 2$, measured in J/°C. These equations express energy conservation for heat transfer between two regions.

3.1.1 A thermometer

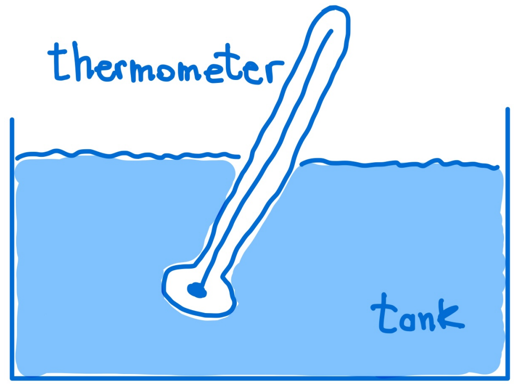


Figure 3.2: Let $T(t)$ denote the thermometer temperature and T_{tank} denote the tank temperature. We assume T_{tank} is constant.

Fourier's law for the heat flow yields

$$q_{\text{tank}\to\text{thermometer}} = \frac{1}{r}\big(T_{\text{tank}} - T(t)\big).$$

Additionally, the thermometer's temperature is governed by

$$c\dot{T}(t) = q_{\text{tank}\to\text{thermometer}}(t).$$

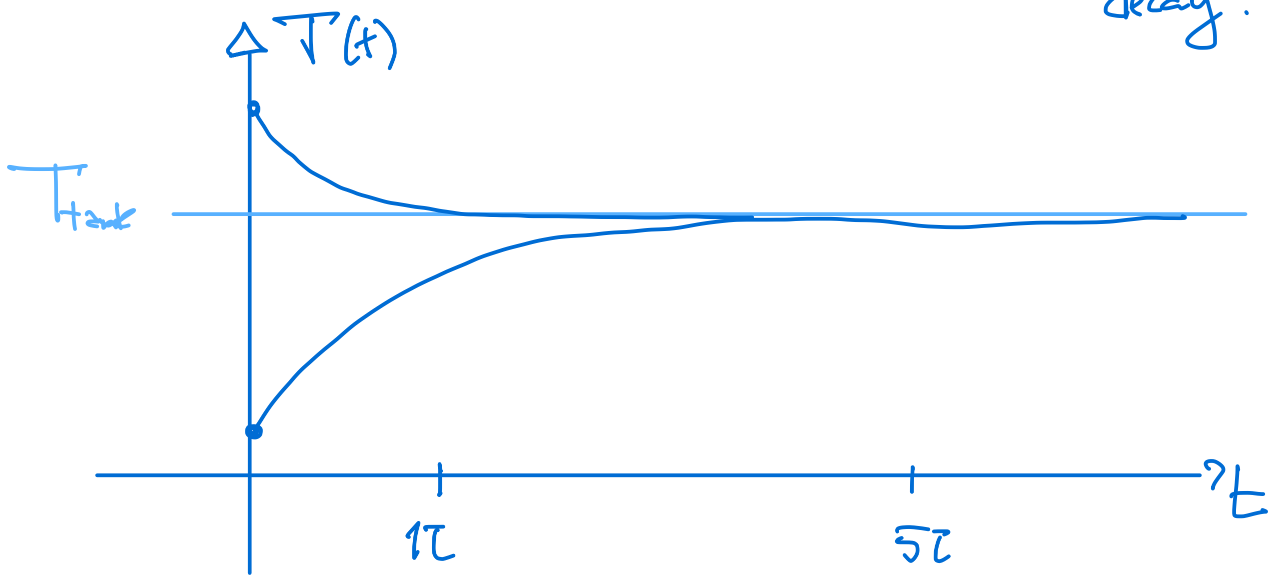
$T(t) =$ thermometer

Combining these two equations gives us the *thermometer dynamics*:

$$\dot{T}(t) = \frac{1}{cr}\big(T_{\text{tank}} - T(t)\big). \tag{3.4}$$

Here $T(t)$ is the variable, while c and r are parameters.

Note: The thermometer dynamics (3.4) is a first order dynamical system, similar to the linear growth/decay model (1.1) in Chapter 1 and the car velocity system (2.4) in Chapter 2. The thermometer time constant is given by $\tau = cr$. These dynamics (3.4) are also referred to as *Newton's law of cooling or heating*.



In class assignment

To measure the tank's temperature we immerse in it a thermometer.
Are we measuring the correct temperature?

3.1.2 A building thermal system

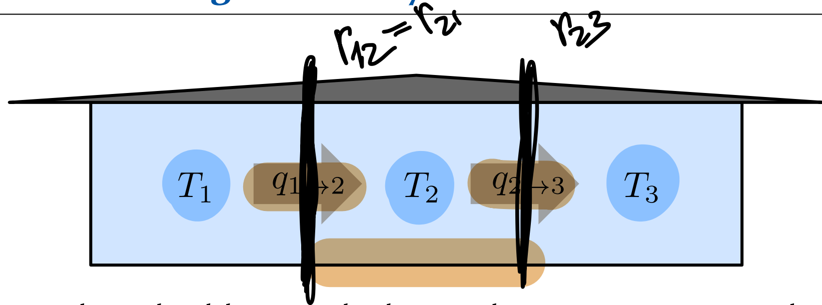


Figure 3.3: Schematic of a three-room building thermal system. Rooms $1, 2,$ and 3 exchange heat through thermal resistances r_{12} and r_{23}. Each room is modeled with a uniform temperature T_i and thermal capacity c_i.

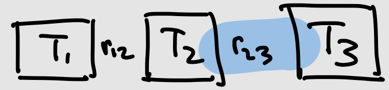
Consider a building with three adjacent rooms, numbered $1, 2, 3$, as illustrated in Figure 3.4. Assume that each room has uniform temperature at all times. Heat can flow between rooms 1 and 2, and between rooms 2 and 3. Considering room 2, we note the additive effect:

$$c_2 \dot{T}_2 = q_{1\to 2} + q_{3\to 2} \qquad \text{where, for example,} \quad q_{1\to 2} = \frac{1}{r_{12}}(T_1 - T_2).$$

Similar equations hold for the other rooms and the other heat flow rates.

$$q_{2\to 3} = - q_{3\to 2}$$

The overall heat flow in this building, modeled by Fourier law (3.1), is described by the *building thermal dynamics*:



$$\begin{aligned}
c_1 \dot{T}_1 &= \frac{1}{r_{12}}(T_2 - T_1) \\
c_2 \dot{T}_2 &= \frac{1}{r_{12}}(T_1 - T_2) + \frac{1}{r_{23}}(T_3 - T_2) \\
c_3 \dot{T}_3 &= \frac{1}{r_{23}}(T_2 - T_3)
\end{aligned} \tag{3.5}$$

where

- T_i is the temperature in room i, for $i = 1, 2, 3$,
- c_i is the thermal capacity of room i, for $i = 1, 2, 3$, and
- $r_{ij} = r_{ji}$ is the thermal resistance between rooms i and j, for $i, j = 1, 2, 3$ and $i \neq j$.

Numerical simulation of building thermal system: convergence to uniform temperature

```python
import numpy as np; from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt
plt.rcParams.update({"text.usetex": True, "font.family": "serif", ...
    "font.serif": ["Computer Modern Roman"] })

def heat_flow_dynamics(t, state, c1, c2, c3, r12, r23):
    T1, T2, T3 = state
    T1_dot = (T2 - T1) / (r12 * c1)
    T2_dot = (T1 - T2) / (r12 * c2) + (T3 - T2) / (r23 * c2)
    T3_dot = (T2 - T3) / (r23 * c3)
    return [T1_dot, T2_dot, T3_dot]

# Parameters for the heat flow system
c1 = 1000  # Heat capacity of room 1 (J/C)
c2 = 1000  # Heat capacity of room 2 (J/C)
c3 = 1000  # Heat capacity of room 3 (J/C)
r12 = 0.2 # Thermal resistance between rooms 1 and 2 (C/W)
r23 = 0.9 # Thermal resistance between rooms 2 and 3 (C/W)

# Time array and initial conditions: [T1, T2, T3]
t = np.linspace(0, 1800, 12000)
initial_conditions = [20.0, 28.0, 35.0]
sol = solve_ivp(heat_flow_dynamics, [t[0], t[-1]], initial_conditions, ...
    t_eval=t, args=(c1, c2, c3, r12, r23), method='RK45')

# Plotting
plt.figure(figsize=(9, 5))
blues = ['#002447', '#003c76', '#0055A4', '#006CD4', '#0085ff', ...
    '#239cff', '#58b1ff']; oranges = ['#471b00', '#752d00', '#a43e00', ...
    '#d35000', '#ff6100', '#ff7f1a', 'ff9b56']
plt.plot(sol.t, sol.y[0], label='Temperature of Room 1 (Celsius)', ...
    color=blues[1])
plt.plot(sol.t, sol.y[1], label='Temperature of Room 2 (Celsius)', ...
    color=blues[3])
plt.plot(sol.t, sol.y[2], label='Temperature of Room 3 (Celsius)', ...
    color=blues[5])

# Add LaTeX labels to the plot
plt.text(300, 24.5, r'$T_1(t)$', fontsize=14, ha='center', va='center', ...
    bbox=dict(facecolor='white', edgecolor='none', pad=5.0))
plt.text(600, 26.6, r'$T_2(t)$', fontsize=14, ha='center', va='center', ...
    bbox=dict(facecolor='white', edgecolor='none', pad=5.0))
plt.text(900, 29.5, r'$T_3(t)$', fontsize=14, ha='center', va='center', ...
    bbox=dict(facecolor='white', edgecolor='none', pad=5.0))

plt.grid(True); plt.xlabel('time $t$'); plt.ylabel('temperature (Celsius)')
plt.xticks(np.arange(0, 1801, 300), ['0', '5 min', '10 min', '15 min', ...
    '20 min', '25 min', '30 min'])
plt.legend(); plt.xlim(0, 1800); plt.title('Temperatures of the rooms')
plt.tight_layout(); plt.savefig("building.pdf", bbox_inches='tight')
```

Listing 3.1: Python script generating Figure 3.4. Available at building.py

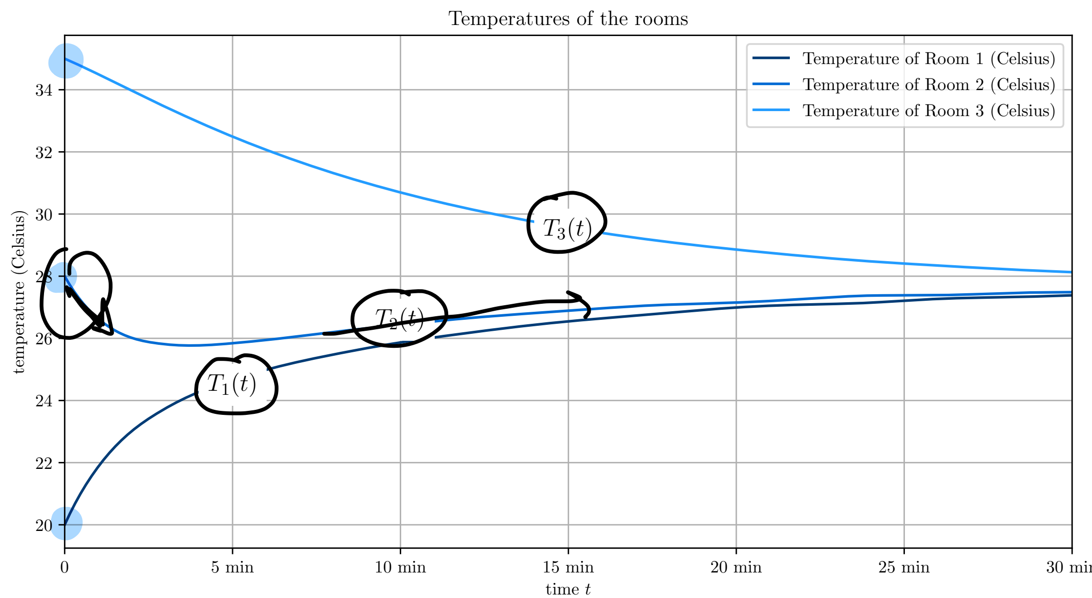


Figure 3.4: Solutions of the building thermal dynamics (3.5). Since the thermal resistance between rooms 2 and 3 is selected much larger than that between rooms 1 and 2, rooms 1 and 2 quickly reach similar temperature. Ultimately, however, all temperatures equalize.

3.1.3 A controlled building thermal system

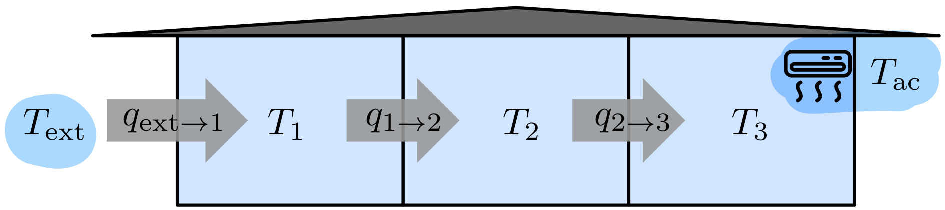


Figure 3.5: Extended building thermal system with control. Room 1 is coupled to an external environment at temperature T_{ext} through resistance $r_{1\text{ext}}$, and room 3 is connected to an air conditioner supplying air at constant temperature T_{ac}. The control signal $u(t)$ switches the air conditioner ON or OFF depending on room 3's temperature.

We now include two additional effects and design a *temperature controller*. We assume that:

(i) room 1 is in direct contact with an *external environment* at a constant temperature T_{ext} through a thermal resistance $r_{1\text{ext}}$. We assume T_{ext} is causing the external environment to heat the building;

(ii) room 3 is equipped with an air conditioning system supplying air at a constant temperature T_{ac} where $T_{\text{ac}} < T_{\text{ext}}$. The air conditioner's effect is modulated by a *control signal $u(t)$*, which is *binary*:

$$u(t) = \begin{cases} 1, & \text{if the air conditioner is ON,} \\ 0, & \text{if the air conditioner is OFF;} \end{cases} \tag{3.6}$$

(iii) as an initial *control design*, we select a desired temperature that is cooler than T_{ext} but warmer than T_{ac}. For example, with $T_{\text{ext}} = 30°\text{C}$ and $T_{\text{ac}} = 20°\text{C}$, we turn ON the air conditioner every time $T_3 > 23°\text{C}$, that is,

$$u(T_3) = \begin{cases} 1 & \text{if } T_3 > 23°\text{C}, \\ 0 & \text{if } T_3 \leq 23°\text{C}. \end{cases} \tag{3.7}$$

This control law is called *on-off control*.

$T_{ext} = 30°C$. $T_{ac} = 20°C$

In summary, the governing differential equations for this system are:

$$
\begin{aligned}
c_1\dot{T}_1 &= \frac{1}{r_{12}}(T_2 - T_1) + \frac{1}{r_{1\text{ext}}}(T_{\text{ext}} - T_1), \\
c_2\dot{T}_2 &= \frac{1}{r_{12}}(T_1 - T_2) + \frac{1}{r_{23}}(T_3 - T_2), \\
c_3\dot{T}_3 &= \frac{1}{r_{23}}(T_2 - T_3) + k u(T_3)(T_{\text{ac}} - T_3).
\end{aligned}
\tag{3.8}
$$

where $k > 0$ is a proportionality constant determining the effectiveness of the air conditioner and related to the airflow into room 3.

$u(T_3) =$ 1
0

Numerical simulation: no control, only the external environment

```python
import numpy as np; import matplotlib.pyplot as plt; import matplotlib
plt.rcParams.update({"text.usetex": True, "font.family": "serif", "font.serif": ...
    ["Computer Modern Roman"] })

def heat_flow_dynamics(T1, T2, T3, c1, c2, c3, r12, r23, k, Tac, Text):
    u = 0 # no control = no air conditioning
    T1_dot = (T2 - T1) / (r12 * c1) + (Text - T1) / (r1ext * c1)
    T2_dot = (T1 - T2) / (r12 * c2) + (T3 - T2) / (r23 * c2)
    T3_dot = (T2 - T3) / (r23 * c3) + k * u * (Tac - T3) / c3
    return T1_dot, T2_dot, T3_dot, u

# Parameters
c1, c2, c3, r12, r23, r1ext, Text, Tac, k = 1000, 1000, 1000, 0.2, 0.9, .5, 30, ...
    20, 3.0
u_prev = [0]; t_end = 5400; dt = t_end / 24000; times = np.arange(0, t_end + dt, dt)

# Initialize values
T1_vals, T2_vals, T3_vals, u_vals = [18.0], [18.0], [27.0], [0]
T1, T2, T3 = 15.0, 18.0, 27.0
for t in times[1:]:
    T1_dot, T2_dot, T3_dot, u = heat_flow_dynamics(T1, T2, T3, c1, c2, c3, r12, ...
        r23, k, Tac, Text)
    T1 += dt * T1_dot; T2 += dt * T2_dot; T3 += dt * T3_dot
    T1_vals.append(T1); T2_vals.append(T2); T3_vals.append(T3); u_vals.append(u)

# Create the figure and the gridspec
fig = plt.figure(figsize=(9, 6)); gs = matplotlib.gridspec.GridSpec(2, 1, ...
    height_ratios=[3, 1])
blues = ['#002447', '#003c76', '#0055A4', '#006CD4', '#0085ff', '#239cff', '#58b1ff']
orngs = ['#471b00', '#752d00', '#a43e00', '#d35000', '#ff6100', '#ff7f1a', '#ff9b56']

# First subplot (Temperatures)
ax1 = plt.subplot(gs[0]); ax1.set_ylim(15, 28)
ax1.plot(times, T1_vals, label='Temperature of Room 1 (C)', color=blues[1])
ax1.plot(times, T2_vals, label='Temperature of Room 2 (C)', color=blues[3])
ax1.plot(times, T3_vals, label='Temperature of Room 3 (C)', color=blues[5])
ax1.grid(True); ax1.legend(); ax1.set_ylabel('Temperature (C)')
ax1.set_title('Temperatures of the rooms and AC control signal over 1.5 hours')
ax1.set_yticks(np.arange(15, max(max(T1_vals), max(T2_vals), max(T3_vals)) + 1, 1))
ax1.set_xlim(0, t_end); ax1.set_xticks([0, 900, 1800, 2700, 3600, 4500, 5400])
ax1.set_xticklabels(['0', '15 min', '30 min', '45 min', '60 min', '75 min', '90 min'])
ax1.text(900, 25.5, r'$T_1(t)$', fontsize=14, ha='center', va='center', ...
    bbox=dict(facecolor='white', edgecolor='none', pad=5.0))
ax1.text(1800, 27, r'$T_2(t)$', fontsize=14, ha='center', va='center', ...
    bbox=dict(facecolor='white', edgecolor='none', pad=5.0))
ax1.text(2700, 26.75, r'$T_3(t)$', fontsize=14, ha='center', va='center', ...
    bbox=dict(facecolor='white', edgecolor='none', pad=5.0))

# Second subplot (Control signal)
ax2 = plt.subplot(gs[1]); ax2.set_xlim(0, t_end); ax2.grid(True)
ax2.plot(times, u_vals, label='Control signal (u)', color='black')
ax2.set_xlabel('time $t$'); ax2.set_ylabel('control signal')
ax2.set_xticks([0, 900, 1800, 2700, 3600, 4500, 5400])
ax2.set_xticklabels(['0', '15 min', '30 min', '45 min', '60 min', '75 min', '90 min'])
ax2.set_yticks([0, 1]); ax2.set_yticklabels(['OFF', 'ON']); ax2.set_ylim(-0.1, 1.1)

# Save the figure
plt.tight_layout(); plt.savefig("building-Text.pdf", bbox_inches='tight')
```

Listing 3.2: Python script generating Figure 3.6. Available at building-Text.py

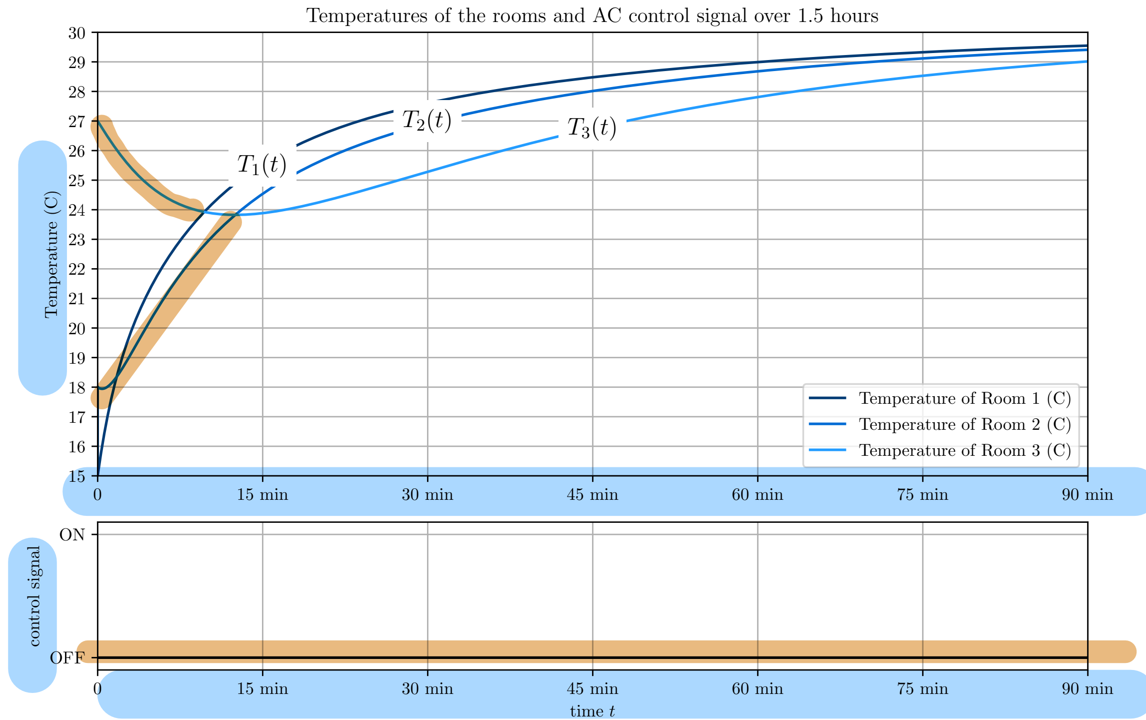


Figure 3.6: Solutions of the building system with heat exchange with the external environment (3.8) and zero control (the air conditioning is off).
The external environment is at $T_{\text{ext}} = 30°\text{C}$ causing the temperature in each room to gradually converge to $30°\text{C}$.

Numerical simulation: chattering in on-off control

```
import numpy as np; from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt; import matplotlib.gridspec as gridspec
plt.rcParams.update({"text.usetex": True, "font.family": "serif", "font.serif": ...
    ["Computer Modern Roman"] })

def heat_flow_dynamics(t, state, c1, c2, c3, r12, r23, k, Tac, Text):
    T1, T2, T3 = state
    u = 1.0 if T3 > 23.0 else 0.0    ## on off control
    air_conditioning = k * u * (Tac - T3)
    T1_dot = (T2 - T1) / (r12 * c1) + (Text - T1) / (r1ext * c1)
    T2_dot = (T1 - T2) / (r12 * c2) + (T3 - T2) / (r23 * c2)
    T3_dot = (T2 - T3) / (r23 * c3) + air_conditioning / c3
    return [T1_dot, T2_dot, T3_dot, u]

# Parameters for the heat flow system
c1, c2, c3, r12, r23, r1ext, k = 1000, 1000, 1000, 0.2, 0.9, .5, 3.0
Text, Tac = 30, 20; initial_conditions = [17.0, 18.0, 27.0]
t = np.linspace(0, 1800, 18000)

def wrap_dynamics(t, y):
    return heat_flow_dynamics(t, y, c1, c2, c3, r12, r23, k, Tac, Text)[:3]

sol = solve_ivp(wrap_dynamics, [t[0], t[-1]], initial_conditions, t_eval=t, ...
    method='LSODA'); u_values = [heat_flow_dynamics(ti, y, c1, c2, c3, r12, r23, ...
    k, Tac, Text)[3] for ti, y in zip(sol.t, sol.y.T)]

# Create the figure and the gridspec
fig = plt.figure(figsize=(9, 6)); gs = gridspec.GridSpec(2, 1, height_ratios=[3, 1])
blues = ['#002447', '#003c76', '#0055A4', '#006CD4', '#0085ff', '#239cff', '#58b1ff']
oranges = ['#471b00', '#752d00', '#a43e00', '#d35000', '#ff6100', '#ff7f1a', 'ff9b56']

# First subplot (Temperatures)
ax1 = plt.subplot(gs[0])
ax1.plot(sol.t, sol.y[0], label='Temperature of Room 1 (C)', color=blues[1])
ax1.plot(sol.t, sol.y[1], label='Temperature of Room 2 (C)', color=blues[3])
ax1.plot(sol.t, sol.y[2], label='Temperature of Room 3 (C)', color=blues[5])
ax1.axhline(y=23, color=oranges[4], linestyle='--', linewidth=1.5,
            label='Target Temperature (23C)')
ax1.grid(True); ax1.set_ylabel('Temperature (C)'); ax1.legend()
ax1.set_title('Temperatures of the rooms and AC control signal over 30 minutes')

# Add LaTeX labels to the plot
ax1.text(150, 20.5, r'$T_1(t)$', fontsize=14, ha='center', va='center', ...
    bbox=dict(facecolor='white', edgecolor='none', pad=5.0))
ax1.text(300, 20.5, r'$T_2(t)$', fontsize=14, ha='center', va='center', ...
    bbox=dict(facecolor='white', edgecolor='none', pad=5.0))
ax1.text(800, 22.5, r'$T_3(t)$', fontsize=14, ha='center', va='center', ...
    bbox=dict(facecolor='white', edgecolor='none', pad=5.0))

ax1.set_yticks(np.arange(17, max(max(sol.y[0]), max(sol.y[1]), max(sol.y[2])) + ...
    1, 1))
ax1.set_xlim(0, 1800); ax1.set_xticks([0, 600, 1200, 1800])
ax1.set_xticklabels(['0', '10 min', '20 min', '30 min'])

# Second subplot (Control signal)
ax2 = plt.subplot(gs[1])
ax2.plot(sol.t, u_values, label='Control signal (u)', color='black')
ax2.grid(True); ax2.set_xlabel('time $t$'); ax2.set_ylabel('control signal')
ax2.set_xticks([0, 600, 1200, 1800]); ax2.set_xlim(0, 1800)
ax2.set_xticklabels(['0', '10 min', '20 min', '30 min'])
ax2.set_yticks([0, 1]); ax2.set_yticklabels(['OFF', 'ON']); ax2.set_ylim(-0.1, 1.1)
plt.tight_layout(); plt.savefig("building-onoff.pdf", bbox_inches='tight')
```

Listing 3.3: Python script generating Figure 3.7. Available at
building-onoff.py

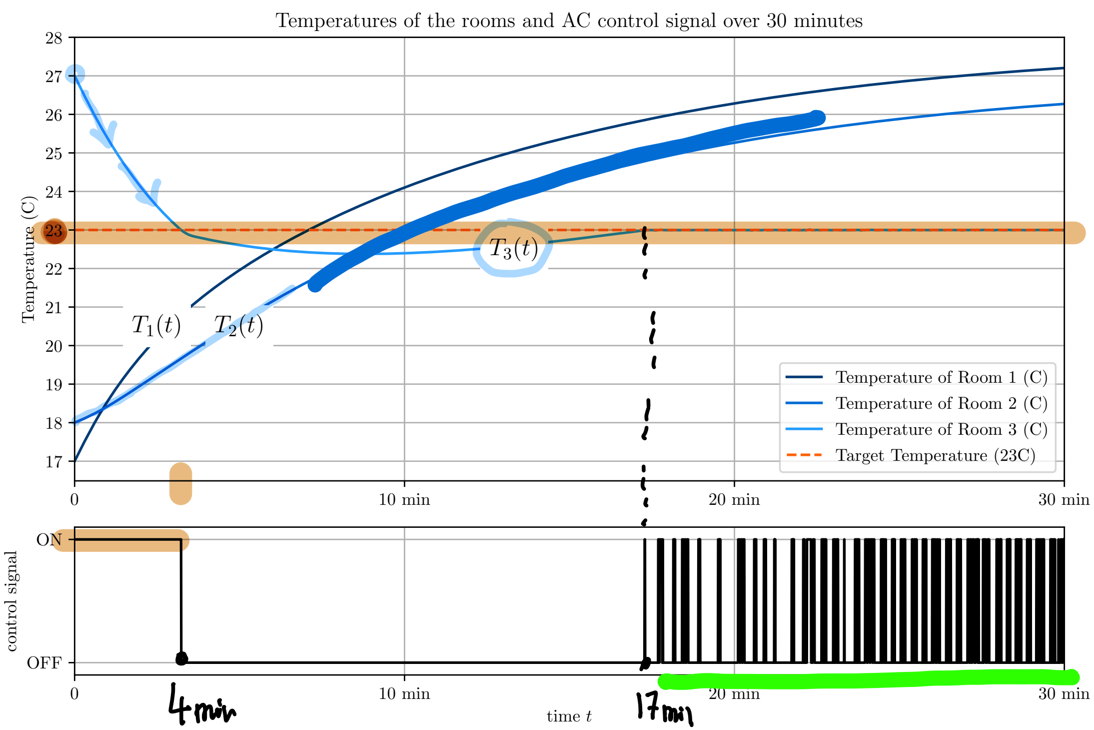
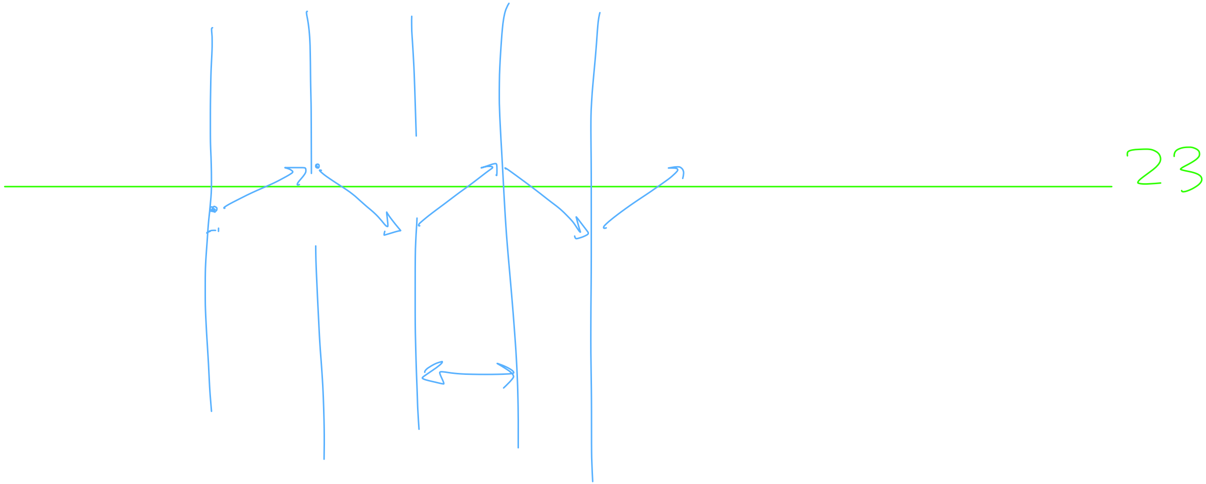


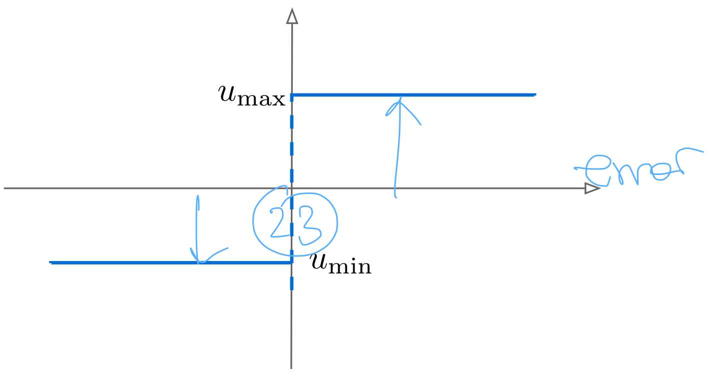
Figure 3.7: Solutions of the building system with air conditioning and heat exchange with the external environment (3.8).
Room 1 starts at a cold $10°$C, but as it is directly exposed to a warm external environment at $30°$C, it ends up as the warmest room.
Room 3 starts at $27°$C and therefore the air conditioning is initially ON.
As long as $T_2 < T_3$, room 3 is cooled by room 2. Once room 2 becomes hot, room 1 starts to warm up and then the air conditioning exhibit a *chattering behavior* about the desired temperature of $23°$C. Frequent fast switching is undesirable as it can damage the air conditioner.



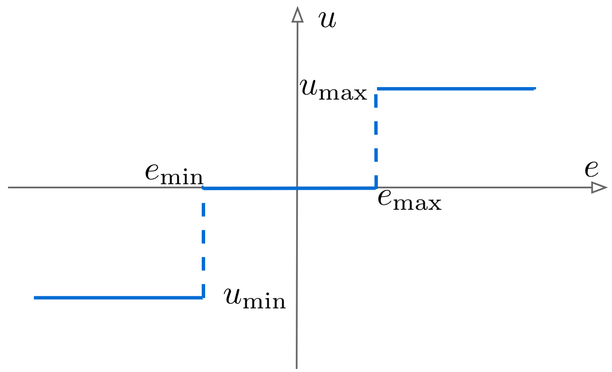
23

The *on-off control* is perhaps the simplest control strategy one can design: when the error is positive, the control is on; when the error is negative, the control is off.

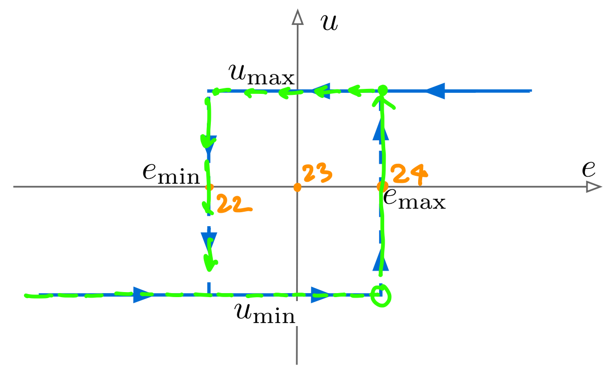
In Figure 3.8, we introduce two variations with different switching and thresholding behaviors: *dead-zone control* and *hysteresis control*.



(a) on-off control



(b) dead zone control



(c) hysteresis control

Figure 3.8: Illustration of binary control laws relating the control input u to the error signal e: *on–off control* (left), *dead-zone control* (center), and *hysteresis control* (right).

Numerical simulation: hysteresis control

```python
import numpy as np; import matplotlib.pyplot as plt; import matplotlib
plt.rcParams.update({"text.usetex": True, "font.family": "serif", "font.serif": ...
    ["Computer Modern Roman"] })

def heat_flow_dynamics(T1, T2, T3, c1, c2, c3, r12, r23, k, Tac, Text):
    u = 1.0 if T3 > 24.0 else (0.0 if T3 < 22.0 else u_prev[0])
    u_prev[0] = u          ## hysteresis control
    T1_dot = (T2 - T1) / (r12 * c1) + (Text - T1) / (r1ext * c1)
    T2_dot = (T1 - T2) / (r12 * c2) + (T3 - T2) / (r23 * c2)
    T3_dot = (T2 - T3) / (r23 * c3) + k * u * (Tac - T3) / c3
    return T1_dot, T2_dot, T3_dot, u

# Parameters
c1, c2, c3, r12, r23, r1ext, Text, Tac, k = 1000, 1000, 1000, 0.2, 0.9, .5, 30, ...
    20, 3.0;
u_prev = [0]; t_end = 5400; dt = t_end / 24000; times = np.arange(0, t_end + dt, dt)

# Initialize values
T1_vals, T2_vals, T3_vals, u_vals = [18.0], [18.0], [27.0], [0]
T1, T2, T3 = 15.0, 18.0, 27.0
for t in times[1:]:
    T1_dot, T2_dot, T3_dot, u = heat_flow_dynamics(T1, T2, T3, c1, c2, c3, r12, ...
        r23, k, Tac, Text);  T1 += dt * T1_dot; T2 += dt * T2_dot; T3 += dt * T3_dot
    T1_vals.append(T1); T2_vals.append(T2); T3_vals.append(T3); u_vals.append(u)

# Create the figure and the gridspec
fig = plt.figure(figsize=(9, 6)); gs = matplotlib.gridspec.GridSpec(2, 1, ...
    height_ratios=[3, 1])
blues = ['#002447', '#003c76', '#0055A4', '#006CD4', '#0085ff', '#239cff', '#58b1ff']
orngs = ['#471b00', '#752d00', '#a43e00', '#d35000', '#ff6100', '#ff7f1a', '#ff9b56']

# First subplot (Temperatures)
ax1 = plt.subplot(gs[0]); ax1.set_ylim(15, 28)
ax1.plot(times, T1_vals, label='Temperature of Room 1 (C)', color=blues[1])
ax1.plot(times, T2_vals, label='Temperature of Room 2 (C)', color=blues[3])
ax1.plot(times, T3_vals, label='Temperature of Room 3 (C)', color=blues[5])
ax1.axhline(y=22, color=orngs[4], linestyle='--', linewidth=1.5, label='Lower ...
    Bound (22C)')
ax1.axhline(y=24, color=orngs[4], linestyle='--', linewidth=1.5, label='Upper ...
    Bound (24C)')
ax1.grid(True); ax1.legend(); ax1.set_ylabel('Temperature (C)')
ax1.set_title('Temperatures of the rooms and AC control signal over 1.5 hours')
ax1.set_yticks(np.arange(15, max(max(T1_vals), max(T2_vals), max(T3_vals)) + 1, 1))
ax1.set_xlim(0, t_end); ax1.set_xticks([0, 900, 1800, 2700, 3600, 4500, 5400])
ax1.set_xticklabels(['0', '15 min', '30 min', '45 min', '60 min', '75 min', '90 min'])
ax1.text(900, 25, r'$T_1(t)$', fontsize=14, ha='center', va='center', ...
    bbox=dict(facecolor='white', edgecolor='none', pad=5.0))
ax1.text(1800, 26.25, r'$T_2(t)$', fontsize=14, ha='center', va='center', ...
    bbox=dict(facecolor='white', edgecolor='none', pad=5.0))
ax1.text(2700, 23, r'$T_3(t)$', fontsize=14, ha='center', va='center', ...
    bbox=dict(facecolor='white', edgecolor='none', pad=5.0))

# Second subplot (Control signal)
ax2 = plt.subplot(gs[1]); ax2.set_xlim(0, t_end); ax2.grid(True)
ax2.plot(times, u_vals, label='Control signal (u)', color='black')
ax2.set_xlabel('time $t$'); ax2.set_ylabel('control signal')
ax2.set_xticks([0, 900, 1800, 2700, 3600, 4500, 5400])
ax2.set_xticklabels(['0', '15 min', '30 min', '45 min', '60 min', '75 min', '90 min'])
ax2.set_yticks([0, 1]); ax2.set_yticklabels(['OFF', 'ON']); ax2.set_ylim(-0.1, 1.1)

# Save the figure
plt.tight_layout(); plt.savefig("building-hysteresis.pdf", bbox_inches='tight')
```

Listing 3.4: Python script generating Figure 3.9. Available at building-hysteresis.py

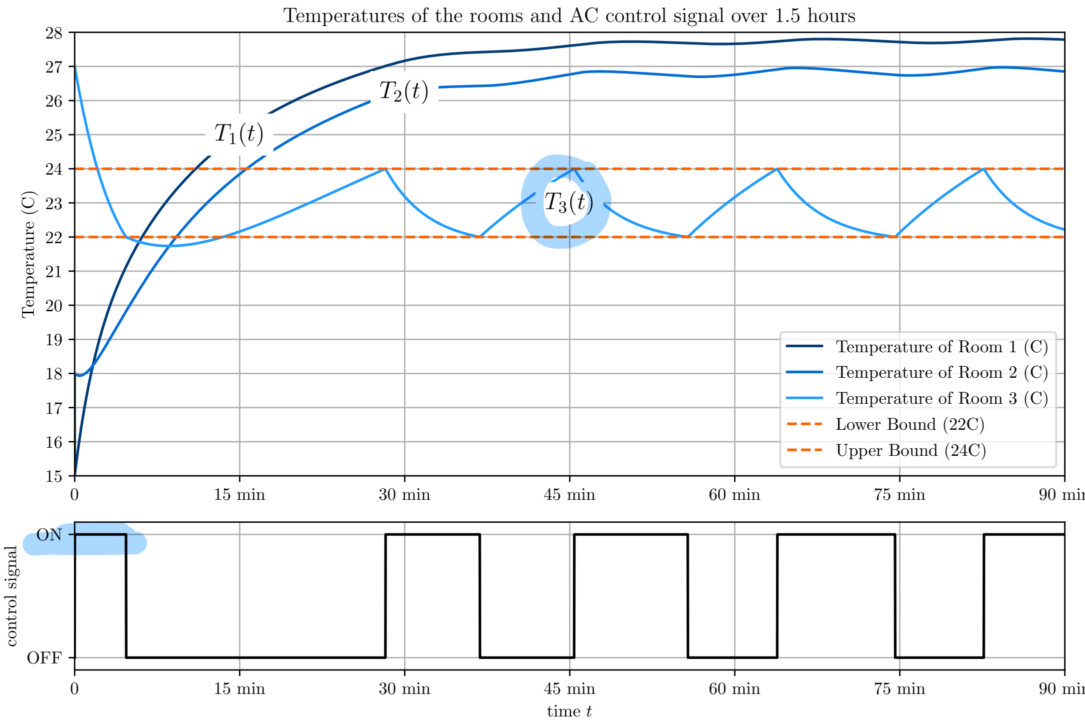


Figure 3.9: Temperature control in a building system with air conditioning and external heat exchange as described in (3.8).

We now design a controller with *hysteresis*:

- The air conditioner turns ON when the temperature exceeds 24°C.
- The air conditioner turns OFF when the temperature falls below 22°C.
- No action is taken if the temperature remains between 22°Cand 24°C.

3.2 Fluid dynamics

Fluid flows are widely studied and of major importance in mechanical engineering. In what follows we focus on incompressible flows. The fundamental physical principle governing fluid flow is *continuity* or *conservation of mass*.

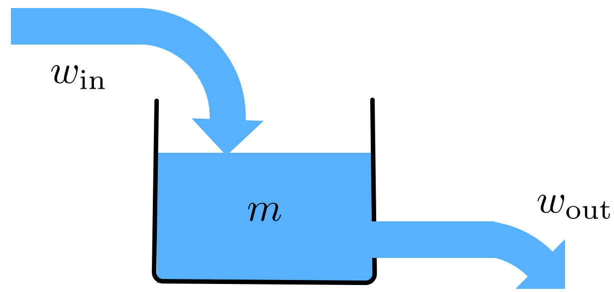


Figure 3.10: Water tank example

For the water tank example in Figure 3.10, the conservation of mass is expressed as:

$$\dot{m} = w_{\text{in}} - w_{\text{out}} \tag{3.9}$$

where

- m is the *fluid mass* inside the tank, measured in kg,
- w_{in} is the *incoming mass flow rate* into the tank, measured in kg/sec, and
- w_{out} is the *outgoing mass flow rate* from the tank.

A second physical principle in incompressible fluid flow is *force equilibrium*.

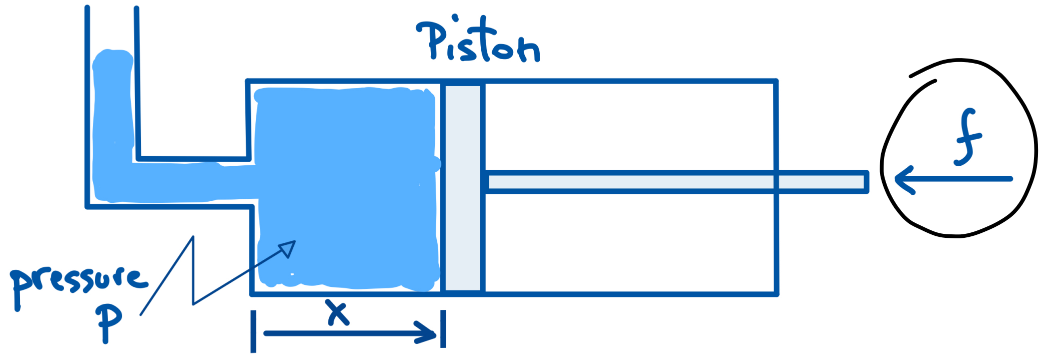


Figure 3.11: Piston example with incompressible fluid in chamber

In the piston example shown in Figure 3.11, the fluid system counteracts a force applied to the piston. According to Newton's second law, the force balance is given by

$$M\ddot{x} = Ap - f \tag{3.10}$$

where

- M is the mass of the piston,
- x is the position of the piston,
- A is the cross-sectional area of the piston,
- p is the *pressure* in the piston chamber, and
- f is the force applied to the piston.

A third and final physical principle governing fluid flow is the law of *nonlinear resistance*.

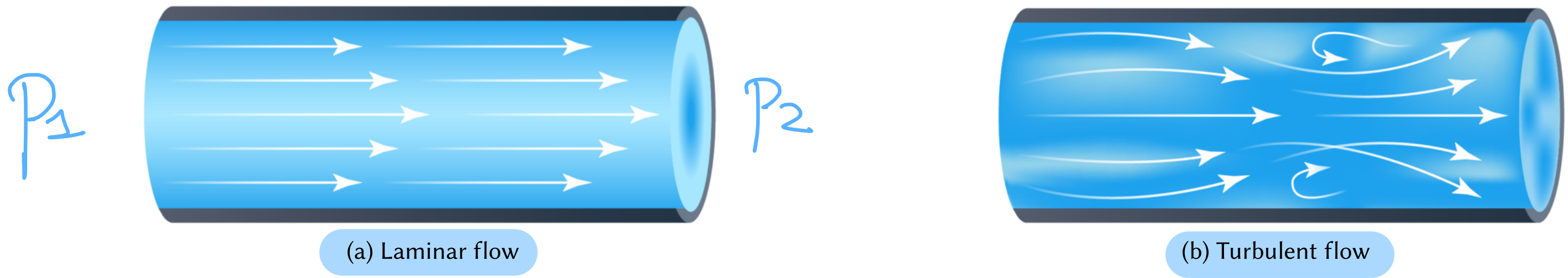


Figure 3.12: Fluid flow in a pipe: laminar versus turbulent. Sourced from `https://engineerexcel.com/laminar-flow-versus-turbulent-flow` without permission

Consider fluid flowing through a pipe from a location with high pressure p_1 to a location with low pressure p_2, where $p_1 > p_2$. The resistance effect is modeled approximately by

$$w = \frac{1}{r}(p_1 - p_2)^{1/\alpha} \tag{3.11}$$

where

- w is the *mass flow rate*,
- $p_1 - p_2 > 0$ is the *pressure difference*,
- r is a *flow resistance*, determined by the pipe's roughness, diameter, length, and the fluid's viscosity, and
- $1 < \alpha < 2$ is a *flow behavior parameter*, which does not directly correlate to laminar or turbulent flow as in Figure 3.12. Generally:
  - $\alpha \approx 2$ is an appropriate value for high flow rates through pipes or nozzles,
  - $\alpha \approx 1$ is an appropriate value for very slow flows through long pipes,
  - $1 < \alpha < 2$ corresponds to intermediate regimes.

(A comprehensive analysis of flow behavior is beyond the scope of these notes.)

An example water tank: height dynamics with pressure-dependent outflow

Starting from the water tank model $\dot{m} = w_{\text{in}} - w_{\text{out}}$, we write $m(t) = A\rho h(t)$, where $h(t) \geq 0$ is the *water height*, A is the cross-sectional area[3] of the tank, and ρ is the density of water. The dynamics of the water height can be expressed as

$$\dot{h} = \frac{1}{A\rho}(w_{\text{in}} - w_{\text{out}}). \tag{3.12}$$

We assume a constant incoming mass flow rate w_{in}. For the outgoing mass flow rate, using the flow resistance model (3.11) with $\alpha = 1/2$, we have

$$w_{\text{out}}(h) = \frac{1}{r}\big(p_{\text{hydrostatic}}(h) - p_{\text{ambient}}\big)^{1/2}, \tag{3.13}$$

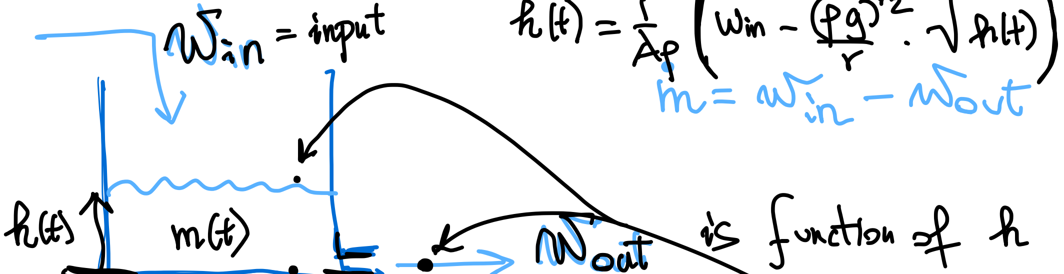
where p_{ambient} is the ambient pressure and $p_{\text{hydrostatic}}(h)$ is the *hydrostatic pressure*, that is, the pressure exerted by a fluid at equilibrium due to the force of gravity.

We now recall *Pascal's principle*: the hydrostatic pressure at the bottom of the tank is the sum of the ambient pressure and the pressure due to the water column. Therefore

$$p_{\text{hydrostatic}}(h) = p_{\text{ambient}} + \rho g h. \tag{3.14}$$

In summary, the tank height dynamics are

$$\dot{h} = \frac{1}{A\rho}\left(w_{\text{in}} - \frac{1}{r}(\rho g h)^{1/2}\right) = -\frac{\sqrt{\rho g}}{rA\rho}\sqrt{h} + \frac{1}{A\rho} w_{\text{in}}. \tag{3.15}$$



[3] The cross-sectional area of a tank is the area of the shape formed by intersecting the tank with a plane perpendicular to its axis at a given level.

For simplicity, we introduce the parameter $a = \frac{\sqrt{\rho g}}{rA\rho} > 0$ and $b = \frac{1}{A\rho} > 0$ so that we can rewrite the water tank equations as

$$\dot{h} = -a\sqrt{h} + bw_{\text{in}} \tag{3.16}$$

Next, we compute the equilibria of the water tank system by setting $\dot{h} = 0$. Note that there is only one equilibrium:

$$a\sqrt{h^*} = bw_{\text{in}} \quad \iff \quad h^* = (b/a)^2 w_{\text{in}}^2. \tag{3.17}$$

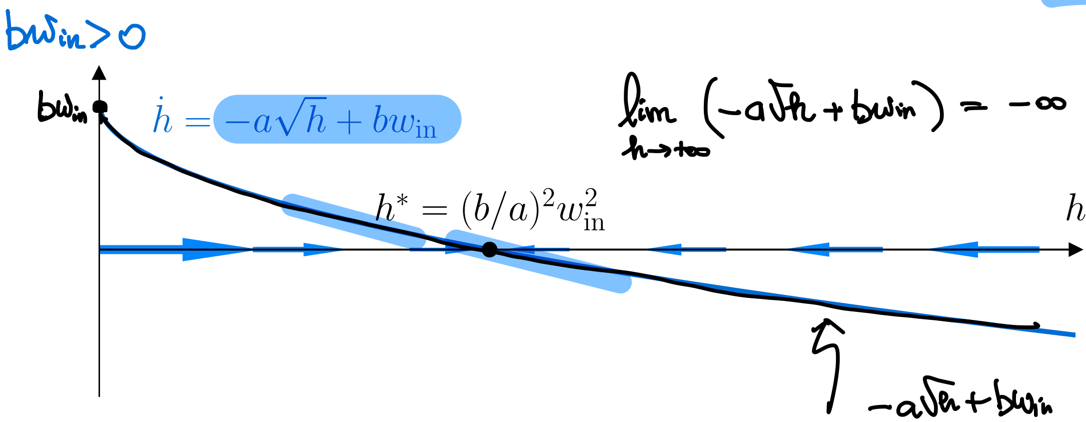
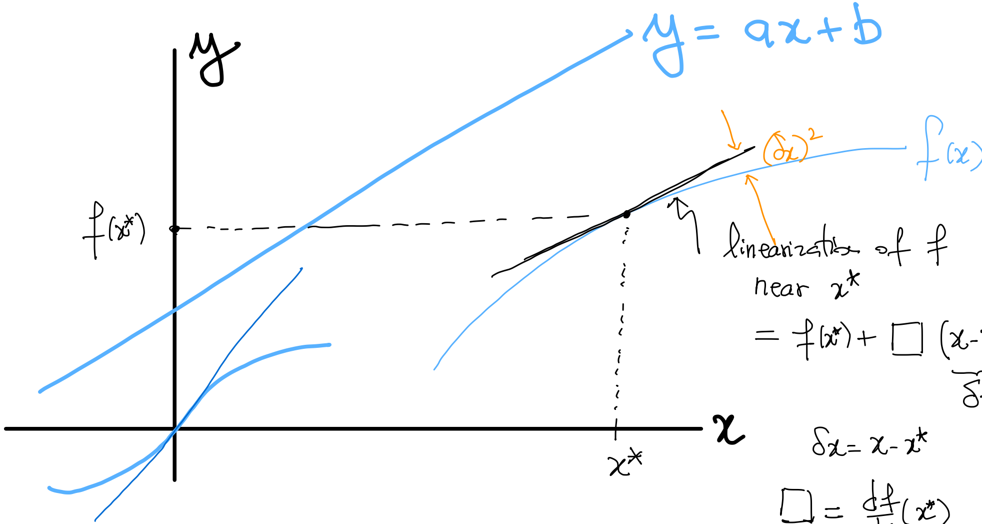


Figure 3.13: Phase portrait on the line for the water tank system with input mass flow rate w_{in}.
We observe that there exists only one equilibrium point and that it is stable.

$$\dot{h} = -a\sqrt{h} + b w_{in}$$



$$= f(x^*) + \square \, \underbrace{(x - x^*)}_{\delta x}$$

$$\delta x = x - x^*$$

$$\square = \frac{df}{dx}(x^*)$$

$f(x) \simeq$ its linearization near x^*

$$\simeq f(x^*) + \frac{df}{dx}(x^*) \cdot \delta x$$

3.3 Linearization of nonlinear physical systems

Many dynamical systems encountered in engineering and science are nonlinear, making them difficult to analyze and control. A widely used approach to simplify the analysis is linearization, which constructs a linear approximation of a nonlinear system around a specific equilibrium point.

This section introduces linearization by starting with one-dimensional systems (without input). Each nonlinear term is expanded in a Taylor series about the equilibrium point. The zeroth-order terms determine the equilibrium and vanish from the linearized model; the first-order terms define the dynamics of the linear approximation.

Linearization transforms a nonlinear system into a linear one that is easier to study, simulate, and design controllers for. However, the accuracy of the approximation depends on how close the trajectory remains to the expansion point. Multiple equilibrium points may yield multiple distinct linearized systems, each capturing local behavior near a different region of the state space.

3.3.1 Linearization of one-dimensional dynamical systems and of nonlinearities of one variable

We start by reviewing the *Taylor expansion for a scalar function of a scalar variable at an expansion point*. The *first-order Taylor expansion* of a function $f(x)$ differentiable at a point x^* is

$$f(x) \approx f(x^*) + \frac{df}{dx}(x^*) \cdot (x - x^*) = f(x^*) + \frac{df}{dx}(x^*) \cdot \delta x,$$

where:

- $f(x^*)$ is the function at x^*,
- $\frac{df}{dx}(x^*)$ is the derivative of the function, evaluated at x^*, and
- $\delta x = x - x^*$ is the difference between the point of interest x and the expansion point x^*.

MADE

$$f(x) \not\approx f(x^*) + \frac{df}{dx}(x)\,(x - x^*)$$

$$\delta f = f(x) - f(x^*)$$

$$\delta f(x) = \frac{\partial f}{\partial x}(x^*) \cdot \delta x$$

When linearizing a dynamical system, we will always select the point x^* to be an equilibrium point. In other words, given

$$\dot{x} = f(x) \tag{3.18}$$

we select x^* to satisfy the equilibrium equation $f(x^*) = 0$. Using the first-order Taylor expansion of f, we write the linearization as:

$$\dot{\delta x} = \frac{df}{dx}(x^*)\delta x \tag{3.19}$$

where we used the following fact: since x^* is constant, $\frac{d}{dt}x = \frac{d}{dt}\delta x$.

Note: if the dynamics is $\dot{x} = a(x - x^*) + f(x)$, then the term $a(x - x^*)$ is already linear so that the linearization about x^* is: $\dot{\delta x} = a\delta x + \frac{df}{dx}(x^*)\delta x$.

Yes but not natural

$f(x) \simeq f(x^*) + \frac{df}{dx}(x^*) \cdot \delta x$, $\quad \delta x = x - x^* \ldots \quad \delta x(t) = x(t) - x^*$.

x^* is equilibrium : $f(x^*) = 0$.

$\frac{d}{dt}\delta x(t) = \dot{\delta x}(t) = \dot{x}(t) = f(x) \simeq \frac{df}{dx}(x^*)\,\delta x(t)$

x^* eq of f

Example #1: Velocity dynamics with quadratic drag. We now revisit the modeling of dampers first studied in Section 2.1.1. The car

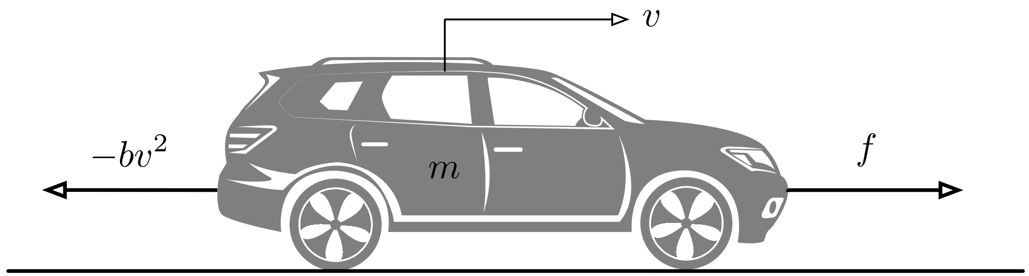


Figure 3.14: Moving car subject to an engine propulsion force f and a *quadratic drag force* equal to $-bv^2$.

velocity system subject to a quadratic aerodynamic drag is

$$m\dot{v} = -bv^2 + f, \tag{3.20}$$

where $m > 0$ is the mass, $b > 0$ is the drag coefficient, and f is a constant thrust force; see Figure 3.14. The velocity v^* is an equilibrium if

$$0 = -b(v^*)^2 + f \quad \Longleftrightarrow \quad (v^*)^2 = \frac{f}{b}.$$

Let $\delta v = v - v^*$ and Taylor expand v^2 about v^*:

$$v^2 \approx v^2\big|_{v=v^*} + \frac{d}{dv}v^2\big|_{v=v^*} \cdot \delta v = (v^*)^2 + 2v^*\delta v.$$

Substituting into the dynamics (3.20), we obtain

$$m\dot{\delta v} = -b(v^*)^2 - 2bv^*\delta v + f = \big(-b(v^*)^2 + f\big) - 2bv^*\delta v.$$

By the equilibrium condition, the constant term cancels and the linearization of (3.20) about v^* is

$$\dot{\delta v} = -\frac{2bv^*}{m}\delta v. \tag{3.21}$$

Example #2: Mass–spring-damper dynamics with a nonlinear spring. Consider a mass-damper system with a saturating spring

$$m\ddot{x} + b\dot{x} + f_{\text{spring}}(x) = 0, \qquad \text{where} \quad f_{\text{spring}}(x) = a\tanh(kx), \tag{3.22}$$

with constants $m, b, a, k > 0$. See Figure 3.15 for a plot of the scaled hyperbolic function.

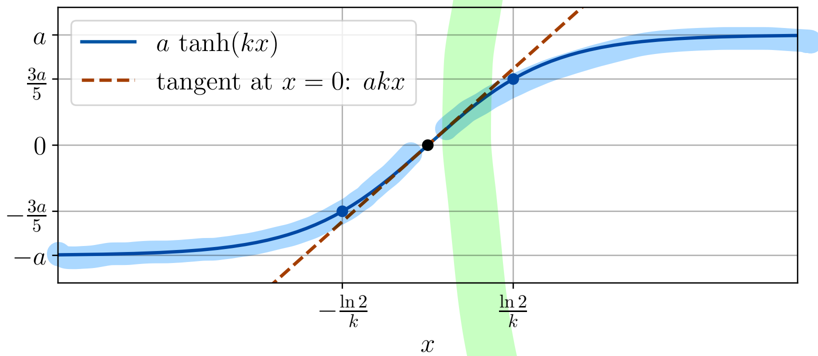


Figure 3.15: Plot of the scaled hyperbolic function and its linearization at the origin.
The function $f_{\text{spring}}(x) = a\tanh(kx)$ is odd, vanishes at $x = 0$, and saturates to $\pm a$ as $x \to \pm\infty$.
$f_{\text{spring}}(x) = a\tanh(kx) = a\dfrac{e^{kx} - e^{-kx}}{e^{kx} + e^{-kx}}$.

Let $v = \dot{x}$ and note that $x^* = 0$, $v^* = 0$ is an equilibrium point. Define the deviation variable $\delta x = x - x^*$ and Taylor expand $f_{\text{spring}}(x)$ about $x^* = 0$:

$$f_{\text{spring}}(x) \approx f_{\text{spring}}(0) + f'_{\text{spring}}(0)\delta x.$$

We note:

$$f_{\text{spring}}(0) = 0; \quad f'_{\text{spring}}(x) = \frac{4ak\,e^{-2kx}}{(1 + e^{-2kx})^2} \implies f'_{\text{spring}}(0) = ak.$$

In summary, substituting into the dynamics (3.22), the mass dynamics with linearized spring near $x^* = 0$ is

$$m\ddot{\delta x} + b\dot{\delta x} + ak\delta x = 0. \tag{3.23}$$

$$\dot{x} = f(x, u)$$

1. select equilibrium point (x^*, u^*) st $f(x^*, u^*) = 0$
2. define variation variables & their derivatives

$$\delta x = x - x^* \quad , \quad \delta\dot{x} = \dot{x} \quad , \quad \delta\ddot{x} = \ddot{x} \quad \ldots$$
$$\delta u = u - u^*$$

3. compute 1^{st} order Taylor expansion for nonlinear term

eg: $x \cdot u \simeq x^* u^* + u^* \delta x + x^* \delta u$

4. substitute T expansions into ODE and express every term in δx, δu

5. isolate the 0^{th} order term, if any, and realize they are $f(x^*, u^*) = 0$!

Example #3: The pendulum dynamics and the small-angle approximation. We consider the pendulum dynamics introduced in Section 2.3.1.

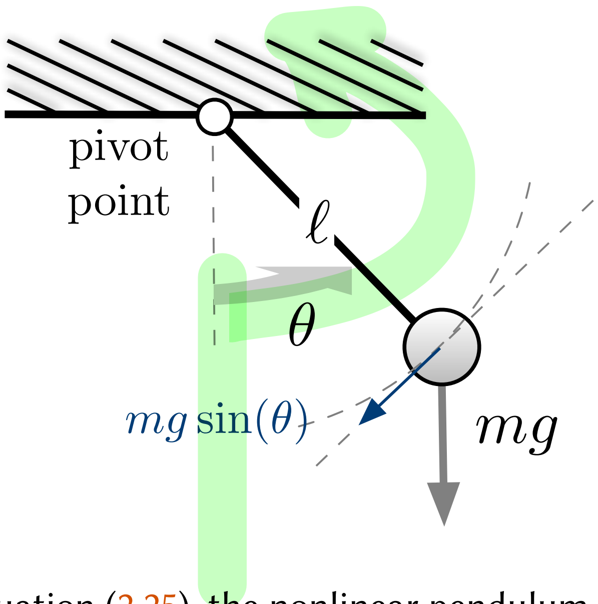
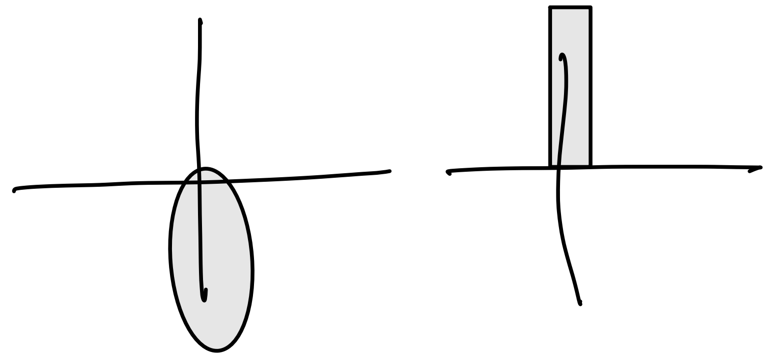


Figure 3.16: A pendulum of length ℓ with mass m concentrated at its end.
The variable is the angle θ, measured counterclockwise from the zero value when the pendulum is in its vertical rest state.
The moment of inertia of the pendulum about the pivot point is $I = m\ell^2$.
The pendulum is subject to the gravity force of magnitude mg, which translates into a restoring torque of magnitude $m\ell g\sin(\theta)$. Additionally, friction (due to air or due to the mechanical rotation at the pivot point) is described by a damping coefficient b.



As in equation (2.25), the nonlinear pendulum dynamics is

$$m\ell^2\ddot{\theta} + b\dot{\theta} + m\ell g\sin(\theta) = 0. \tag{3.24}$$

Defining $\omega = \dot{\theta}$ and noting that the equilibrium condition is $\sin\theta^* = 0$, we identify two equilibrium points:

- the equilibrium point $(\theta^*_{\text{down}}, \omega^*) = (0, 0)$ corresponding to the pendulum in its *down position*, and
- the equilibrium point $(\theta^*_{\text{up}}, \omega^*) = (\pi, 0)$ corresponding to the pendulum in its *up position*.

First, we compute the first-order Taylor expansion of $\sin(\theta)$ about $\theta^*_{\text{down}} = 0$, obtaining the famous *small-angle approximation*:

$$\sin\theta \approx \sin(0) + \cos(0)\,(\theta - 0) = \theta.$$

We rewrite this approximation as $\sin\theta \approx \theta - \theta^*_{\text{down}} = \delta\theta$. The linearization of system (3.24) about the down position is therefore

$$+m\ell^2\ddot{\delta\theta} + b\dot{\delta\theta} + m\ell g\delta\theta = 0. \tag{3.25}$$

Second, we compute the first-order Taylor expansion of $\sin(\theta)$ about $\theta^*_{\text{up}} = \pi$, noting that $\cos(\pi) = -1$, and obtain

$$\sin\theta \approx \sin(\pi) + \cos(\pi)\,(\theta - \pi) = -(\theta - \pi).$$

We rewrite this approximation as $\sin\theta \approx -(\theta - \theta^*_{\text{up}}) = -\delta\theta$. The linearization of system (3.24) about the up position is therefore

$$+m\ell^2\ddot{\delta\theta} + b\dot{\delta\theta} - m\ell g\delta\theta = 0. \tag{3.26}$$

$f(x) \approx f(x^*) + \frac{df}{dx}(x^*)(x - x^*)$

$f(\theta) = \sin\theta$

$\theta^*_{\text{down}} = 0$. $\quad \sin\theta \simeq \sin(0) + \cos(0)\cdot(\theta - 0) = \theta$

$\theta^*_{\text{up}} = \pi$. $\quad \sin\theta \simeq \sin(\pi) + \cos(\pi)\cdot(\theta - \pi) = -(\theta - \pi)$

when θ is small (near 0), $\sin\theta \simeq \theta$

" " " near π, $\sin\theta \simeq -(\theta - \pi)$

- $$m\ell^2\ddot{\theta} + b\dot{\theta} + m\ell g\sin(\theta) = 0. \quad (1)$$

- $$\sin\theta \approx \sin(\pi) + \cos(\pi)(\theta - \pi) = -(\theta - \pi). \quad (2)$$

$$= -\delta\theta$$

$$\delta\theta = \theta - \pi \quad (\theta_{relative})$$

$$\delta\dot{\theta} = \dot{\theta}$$

$$\delta\ddot{\theta} = \ddot{\theta}$$

$$m\ell^2\delta\ddot{\theta} = m\ell^2\ddot{\theta} = -b\dot{\theta} - m\ell g\sin\theta$$

$$= -b\delta\dot{\theta} - m\ell g(-\delta\theta)$$

$$m\ell^2\delta\ddot{\theta} + b\delta\dot{\theta} - m\ell g\delta\theta = 0$$

Linearization

$f(x)$, $f: \mathbb{R} \to \mathbb{R}$ scalar function of a scalar

$f(\theta) = \sin\theta$

$f(x,y) = xy$

Taylor Expansion of a scalar function of two variables

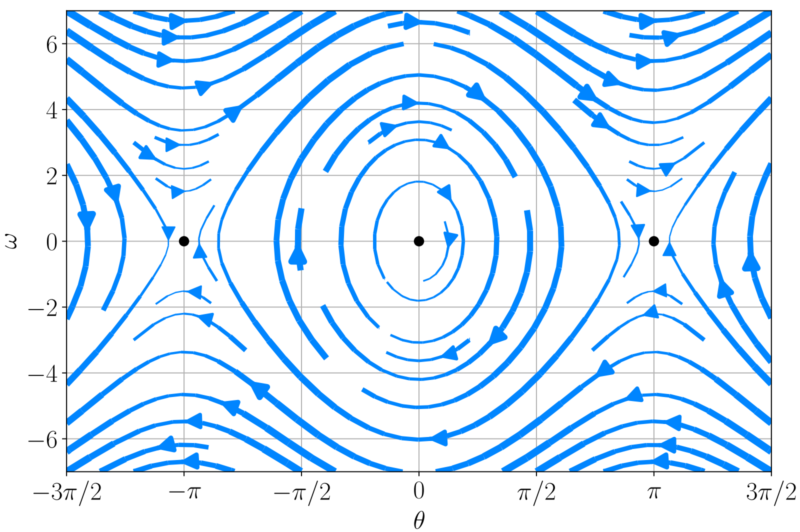
$$f(x,y) \cong f(x^*,y^*) + \frac{\partial f}{\partial x}(x^*,y^*)\,\delta x + \frac{\partial f}{\partial y}(x^*,y^*)\,\delta y \quad \cancel{+ \text{ higher order terms}}$$

at (x^*, y^*)

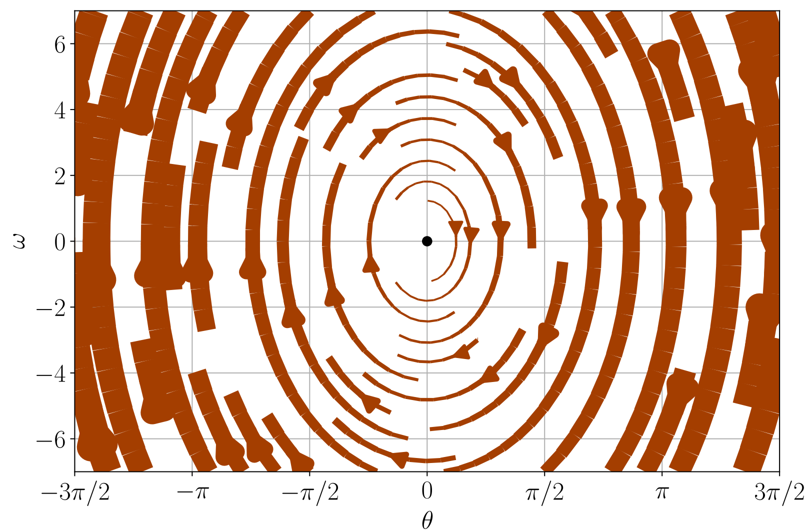
For example: $f(x,y) = xy$. $(x^*, y^*) = (1, 2)$

$$f(x,y) \cong 2 + y\Big|_{y=y^*=2} \delta x + x\Big|_{x=x^*=1} \delta y = 2 + 2\,\delta x + \delta y$$

The pendulum down case corresponds to approximating $\sin(\theta)$ by θ. This approximation is reasonable for small θ and because the point $(\theta, \dot{\theta}) = (0, 0)$ is a stable equilibrium of the pendulum dynamics. We note that this approximated model is identical to a linear mass-spring-damper system. Let us now verify empirically the impact of this approximation. We compare the phase portrait for the exact dynamics (2.26) with $\sin\theta$ and for the approximated dynamics, where $\sin\theta$ is replaced by θ.



(a) Exact pendulum dynamics



(b) Pendulum dynamics linearized at the pendulum down configuration

Figure 3.17: Phase portrait for the undamped pendulum dynamics and its linearization about the "pendulum down" configuration. Note that the phase portraits are reasonably similar for small θ.

3.3.2 Linearization of two-dimensional dynamical systems and of nonlinearities of two variables

We recall the Taylor expansion of a smooth scalar function $f(y_1, y_2)$ of two real variables about the point $(0,0)$. Keeping only the first–order terms, we have

$$f(y_1, y_2) \approx f(0,0) + \frac{\partial f}{\partial y_1}(0,0)y_1 + \frac{\partial f}{\partial y_2}(0,0)y_2. \tag{3.27}$$

Here $f(0,0)$ is the function value at the expansion point, while $\frac{\partial f}{\partial y_1}(0,0)$ and $\frac{\partial f}{\partial y_2}(0,0)$ are the partial derivatives evaluated at $(0,0)$, giving the local linear sensitivity of f to each variable.

Example #4: A static nonlinear example. As example, we consider the function

$$g(y_1, y_2) = y_1 \sin(y_2), \tag{3.28}$$

where y_1 and y_2 are variables. We compute the first-order Taylor expansion of g about the point $(y_1, y_2) = (a, \theta)$, for some given amplitude a and angle θ. We define the perturbations $\delta y_1 = y_1 - a$ and $\delta y_2 = y_2 - \theta$. The zero-th order term is

$$g(A, \theta) = A\sin(\theta). \tag{3.29}$$

The partial derivatives of g at (a, θ) are

$$\frac{\partial g}{\partial y_1}(a,\theta) = \sin(\theta), \qquad \frac{\partial g}{\partial y_2}(a,\theta) = a\cos(\theta). \tag{3.30}$$

so that the first-order Taylor approximation is

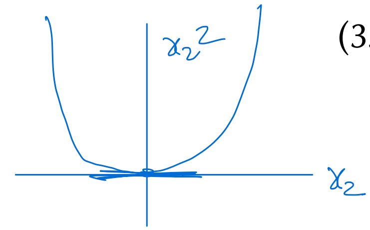
$$g(y_1, y_2) \approx a\sin(\theta) + \sin(\theta)\delta y_1 + a\cos(\theta)\delta y_2. \tag{3.31}$$

Example #5: Two dimensional nonlinear dynamical system. Consider the nonlinear system

$$\dot{x}_1 = -\sin(x_1 + x_2) + x_2^2, \tag{3.32a}$$
$$\dot{x}_2 = e^{x_1 x_2} - \cos(x_2). \tag{3.32b}$$

Note that the point $(x_1^*, x_2^*) = (0, 0)$ is an equilibrium. Define

$$\delta x_1 = x_1 - x_1^*, \quad \delta x_2 = x_2 - x_2^*. \tag{3.33}$$



For the $\dot{x}_1$ equation (3.32a), we expand each term about $(0, 0)$:

$$-\sin(x_1 + x_2) \approx -(x_1 + x_2) = -\delta x_1 - \delta x_2,$$
$$x_2^2 \approx 0 + 0 \cdot \delta x_2 = 0. \qquad \text{(no first-order linear term)}$$

The linearized expression for $\dot{\delta x}_1$ is

$$\dot{\delta x}_1 \approx -\delta x_1 - \delta x_2. \tag{3.34}$$

For the $\dot{x}_2$ equation (3.32b), we expand each term about $(0, 0)$:

$$e^{x_1 x_2} \approx 1 + x_1 x_2 \approx 1, \qquad \text{(no first-order linear term because } x_1 x_2 \text{ is second order)}$$
$$-\cos(x_2) \approx -(1 - \delta x_2^2/2) \approx -1. \qquad \text{(also no first-order linear term)}$$

The constant terms $+1$ and -1 cancel due to the equilibrium condition, so

$$\dot{\delta x}_2 \approx 0. \tag{3.35}$$

In summary, the linearized system about $(0, 0)$ is

$$\dot{\delta x}_1 \approx -\delta x_1 - \delta x_2, \tag{3.36a}$$
$$\dot{\delta x}_2 \approx 0. \tag{3.36b}$$

3.3.3 Linearization of a control system

A *dynamical system with a control input* or a *control system* with n state variables x and m state variables u is a system of the form

$$\dot{x} = f(x, u) \tag{3.37}$$

For example, first order systems include:

- the car velocity system (2.4) $\dot{v} = -(b/m)v + f/m$ has $n = 1$ state v and $m = 1$ input f;
- the water tank system (3.16) $\dot{h} = -a\sqrt{h} + bw_{\text{in}}$ has 1 state h and 1 input w_{in};

and second order systems include:

- the forced mass-spring-damper system (2.12) $m\ddot{x} + b\dot{x} + kx = f$ has 2 states $(x, \dot{x})$ and 1 input f;
- the air-conditioned building system (3.8) has three states (T_1, T_2, T_3) and 2 inputs: the air conditioning control u and the external temperature T_{ext}.

For a control system (3.37), an *equilibrium* is a pair (x^*, u^*) such that

$$f(x^*, u^*) = 0 \tag{3.38}$$

implying the constant trajectory $\big(x(t), u(t)\big) = \big(x^*, u^*\big)$ is a solution for all time. For example

- for the water tank system $\big(h^*, w^*_{\text{in}}\big) = \big((b/a)^2(w^*)^2, w^*\big)$ is an equilibrium,
- for the forced mass-spring-damper system $(x, \dot{x}, f) = (f_0/k, 0, f_0)$ is an equilibrium.

Example #6: The controlled building model as a nonlinear system with an input. From the controlled building thermal system in Section 3.1.3 consider only the dynamics of the third room. With simple parameter values the model is

$$\dot{T}_3 = (T_2 - T_3) + u(T_{ac} - T_3). \tag{3.39}$$

Given constant values of T_2 and T_{ac}, we assume that (T_3^*, u^*) is an equilibrium point. We then define the difference variables about the expansion values:

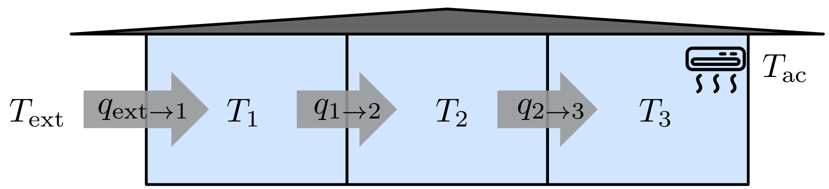


Figure 3.18: We report Figure 3.5 for convenience.

$$\delta T_3 = T_3 - T_3^*, \quad \delta u = u - u^*.$$

The first term in equation (3.39) expands as

$$T_2 - T_3 \approx (T_2 - T_3^*) - \delta T_3.$$

The second term in equation (3.39) expands as

$$\begin{aligned} u(T_{ac} - T_3) &= (u^* + \delta u)\big((T_{ac} - T_3^*) - \delta T_3\big) \\ &\approx u^*(T_{ac} - T_3^*) - u^*\delta T_3 + (T_{ac} - T_3^*)\delta u. \end{aligned}$$

(dropping the higher order term $-\delta u \delta T_3$)

The equilibrium condition $(T_2 - T_3^*) + u^*(T_{ac} - T_3^*) = 0$ eliminates the constant terms and the remaining first-order terms give the linearization of the control system (3.39) about (T_3^*, u^*):

$$\delta \dot{T}_3 \approx (-1 - u^*)\delta T_3 + (T_{ac} - T_3^*)\delta u. \tag{3.40}$$

For example, we might set $T_2 = 20$, $T_{ac} = 10$, and consider the equilibrium point $(T_3^*, u^*) = (15, 1)$. The linearized equation read:

$$\delta \dot{T}_3 = (-1 - 1)\delta T_3 + (10 - 15)\delta u = -2\delta T_3 - 5\delta u.$$

$$\dot{T}_3 = (T_2 - T_3) + \underbrace{(T_{ac} - T_3)u}$$

(1) $f(T_3, u) \approx f(T_3^*, u^*) + \underbrace{\frac{\partial f}{\partial T_3}(T_3^*, u^*)}_{-u^*} \cdot \delta T_3 + \underbrace{\frac{\partial f}{\partial u}(T_3^*, u^*)}_{T_{ac} - T_3^*} \delta u$

$$= (T_2 - T_3) + (T_{ac} - T_3^*)u^* + (-u^*)\delta T_3 + (T_{ac} - T_3^*)\delta u$$

Want to write full system in δT_3 & δu. $\delta T_3 = T_3 - T_3^*$

$\Rightarrow T_3 = T_3^* + \delta T_3$

$\dot{T}_3 = \dot{\delta T_3}$

Then

(3) 0th order term, 1st order, 0th order, 1st order, 1st order

$$\dot{\delta T_3} = \underbrace{(T_2 - T_3^*) - \delta T_3}_{(2)} + (T_{ac} - T_3^*)u^* + (-u^*)\delta T_3 + (T_{ac} - T_3^*)\delta u$$

(4) At (T_3^*, u^*), $0 = \dot{T}_3 = (T_2 - T_3^*) + (T_{ac} - T_3^*)u^*$

(5) $\dot{\delta T_3} = (-1 - u^*)\delta T_3 + (T_{ac} - T_3^*)\delta u$ (only 1st order terms in $\delta T_3, \delta u$)

3.4 Historical notes and further resources

The study of heat transfer and fluid dynamics has a rich history, with significant contributions from early scientists and engineers. Joseph Fourier's work in 1822 laid the groundwork for understanding heat conduction, introducing Fourier's law, which remains a cornerstone in the study of thermodynamics and heat transfer. Fourier's insights have influenced countless applications, from industrial processes to climate control systems. In fluid dynamics, the contributions of Daniel Bernoulli in 1738 provided a fundamental understanding of fluid behavior with Bernoulli's principle, which describes the relationship between fluid velocity and pressure. This principle has been instrumental in fields ranging from aerodynamics to hydraulics.

Additional example systems of thermal and fluids dynamics systems can be found in many references, including for example (Ogata, 2003).

For comprehensive details on heat flow models, including the differential form of Fourier's law, we refer for example to Wikipedia: Thermal conduction and Wikipedia: Thermal conductivity.

It is instructive to consider how a binary ON/OFF controller is implemented in a circuit. A *relay circuit* is an electrical control device that typically uses an electromagnet to mechanically switch an electrical load on or off. It consists of a coil, a set of contacts, and a spring-loaded mechanism. When an electrical current is applied to the coil, a magnetic field attracts or repels the contacts, causing them to make or break an electrical connection. Relays are commonly used in automotive systems, industrial automation, and electronics. They are especially useful in high-voltage or high-current devices for electrical isolation and protection.

3.5 Appendix: Basic models in convective and radiative heat transfer

In this appendix, we present simple low-dimensional examples of convective and radiative heat transfer.

3.5.1 Convective heat transfer

This model describes how an object cools or heats over time when placed in an environment where heat is exchanged between the object's surface and the surrounding fluid. This model is commonly used to describe transient heat conduction in simple geometries with convective boundary conditions. The *convective heat transfer dynamics* is:

$$\frac{dT}{dt} = -\frac{hA}{mc}(T - T_\infty), \tag{3.41}$$

where:

- $T(t)$ is the temperature of the object at time t,
- T_∞ is the ambient temperature (assumed constant),
- h is the convective heat transfer coefficient,
- A is the surface area of the object,
- m is the mass of the object, and
- c is the specific heat capacity.

This first-order linear ODE provides insights into cooling rates in real-world applications like engines, heat exchangers, or electronic devices.

3.5.2 Radiative cooling of a blackbody

The *Stefan-Boltzmann law* describes radiative heat transfer from an object to its surroundings. For a body emitting thermal radiation, the *radiative cooling dynamics of a blackbody* is

$$\frac{dT}{dt} = -\frac{\sigma \varepsilon A}{mc} \left(T^4 - T_\infty^4 \right), \tag{3.42}$$

where:

- $T(t)$ is the temperature of the object at time t,
- T_∞ is the ambient temperature (assumed constant),
- σ is the Stefan-Boltzmann constant,
- ε is the emissivity of the object,
- A is the surface area,
- m is the mass, and
- c is the specific heat capacity.

This nonlinear ODE models is especially relevant for high-temperature systems like engines, furnaces, or even spacecraft, where radiation is a dominant heat transfer mechanism. A key feature of this model is that radiation depends on T^4 and so the model is nonlinear.

3.6 Appendix: Additional concepts on linearization of nonlinear physical systems

Linearization with a state-dependent inertia

Consider the nonlinear system

$$g(x)\dot{x} = f(x), \tag{3.43}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are differentiable. The matrix function g is referred to as a state-dependent inertia. Let $x^* \in \mathbb{R}^n$ satisfy $f(x^*) = 0$ and define the perturbation $\delta x = x - x^*$. Then

$$g(x^* + \delta x)\dot{\delta x} = f(x^* + \delta x). \tag{3.44}$$

Using first-order Taylor expansions,

$$g(x^* + \delta x) \approx g(x^*) + Dg(x^*)\,\delta x, \quad f(x^* + \delta x) \approx Df(x^*)\,\delta x. \tag{3.45}$$

The term $(Dg(x^*)\,\delta x)\,\dot{\delta x}$ is quadratic in the perturbations, since both δx and $\dot{\delta x}$ are small, and is neglected in the first-order approximation. This yields

$$g(x^*)\dot{\delta x} = Df(x^*)\,\delta x. \tag{3.46}$$

If $g(x^*)$ is invertible, the linearized dynamics are

$$\dot{\delta x} = g(x^*)^{-1} Df(x^*)\,\delta x. \tag{3.47}$$

Next, we consider an example. A double pendulum consists of two identical simple pendulums attached in series. Let θ_1 and θ_2 be the angular displacements from the vertical, measured counterclockwise. Each rod has length ℓ and each mass is m. Neglect friction. From Newton's or Lagrange's equations, the exact dynamics are

$$\begin{cases} m\ell^2\big(\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1-\theta_2)\big) + m\ell^2\dot{\theta}_2^2\sin(\theta_1-\theta_2) + 2mg\ell\sin\theta_1 = 0, \\ m\ell^2\big(\ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1-\theta_2)\big) - m\ell^2\dot{\theta}_1^2\sin(\theta_1-\theta_2) + mg\ell\sin\theta_2 = 0. \end{cases} \tag{3.48}$$

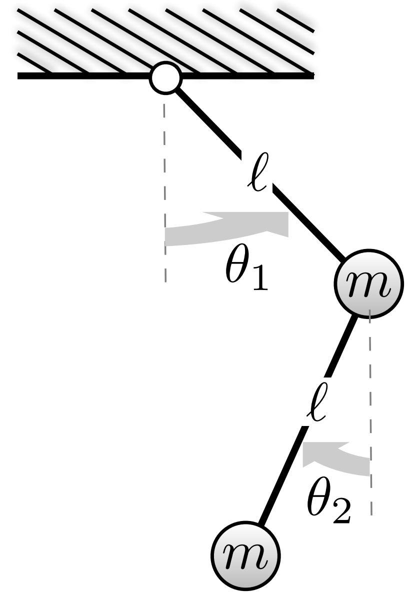


Figure 3.19: Double pendulum

We linearize about the equilibrium $\theta_1 = \theta_2 = 0$ by assuming small angles and keeping only first-order terms. In other words: (i) products of small quantities such as $\theta_1\theta_2$, θ_1^2, $\dot{\theta}_1^2$ are neglected, and (ii) we approximate

$$\sin(\theta_1-\theta_2) \approx \theta_1-\theta_2, \quad \cos(\theta_1-\theta_2) \approx 1, \quad \sin\theta_1 \approx \theta_1, \quad \sin\theta_2 \approx \theta_2. \tag{3.49}$$

Applying these approximations to the first equation in (3.48):

$$m\ell^2\big(\ddot{\theta}_1 + \ddot{\theta}_2\cdot 1\big) + m\ell^2(0)\cdot(\theta_1-\theta_2) + 2mg\ell\theta_1 = 0, \tag{3.50}$$

that is

$$m\ell^2(\ddot{\theta}_1 + \ddot{\theta}_2) + 2mg\ell\theta_1 = 0. \tag{3.51}$$

Similarly, the second equation in (3.48) becomes

$$m\ell^2(\ddot{\theta}_2 + \ddot{\theta}_1) + mg\ell\theta_2 = 0. \tag{3.52}$$

The resulting linearized system is

$$\begin{cases} m\ell^2(\ddot{\theta}_1 + \ddot{\theta}_2) + 2mg\ell\theta_1 = 0, \\ m\ell^2(\ddot{\theta}_1 + \ddot{\theta}_2) + mg\ell\theta_2 = 0. \end{cases} \tag{3.53}$$

3.6.1 Linearization of multivariable dynamical systems via Jacobians

We start by recalling the general expression for the first-order Taylor expansion of a vector-valued function. We consider a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \quad \text{and} \quad f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \in \mathbb{R}^2.$$

The *first-order Taylor expansion of f about a point* $x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \in \mathbb{R}^2$ is:

$$f(x) \approx f(x^*) + J(x^*) \cdot (x - x^*)$$

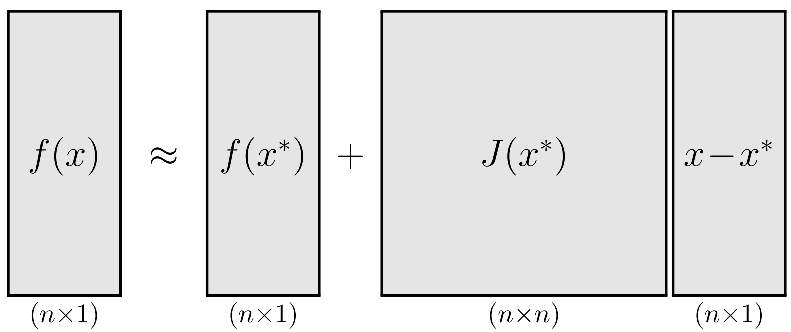
where:

- $f(x^*)$ is the vector function evaluated at x^*,
- $J(x^*)$ is the *Jacobian matrix* of f evaluated at x^*, defined by

$$J(x^*) = \frac{\partial f}{\partial x}(x^*) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x=x^*}$$

- $x - x^*$ is the difference vector between the point of interest x and the expansion point x^*.

Equivalently, emphasizing the dimensions of vectors and matrices:



(3.54)

Next, we focus on a dynamical system with input. For a general system with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$:

$$\dot{x} = f(x, u), \tag{3.55}$$

let (x^*, u^*) be an equilibrium pair: $f(x^*, u^*) = 0$. Define $\delta x = x - x^*$, $\delta u = u - u^*$.

Near the equilibrium pair (x^*, u^*), we consider small variations and write

$$\begin{aligned} x(t) &= x^* + \delta x(t), \\ u(t) &= u^* + \delta u(t), \end{aligned}$$

and we plug into both sides of $\dot{x} = f(x, u)$ to obtain

$$\begin{aligned} \dot{x}(t) = \frac{d}{dt}\big(x^* + \delta x(t)\big) = \frac{d}{dt}\delta x(t) &= f\big(x^* + \delta x(t), u^* + \delta u(t)\big) \\ &\approx f(x^*, u^*) \; + \; \underbrace{\frac{\partial f}{\partial x}(x^*, u^*)}_{=F} \delta x(t) \; + \; \underbrace{\frac{\partial f}{\partial u}(x^*, u^*)}_{=G} \delta u(t), \end{aligned}$$

where we have used the *first-order Taylor expansion about the equilibrium point* (x^*, u^*).

In summary, the linearization is

$$\dot{\delta x} \approx F\delta x + G\delta u, \tag{3.56}$$

where F and G are the Jacobians of f with respect to x and u, evaluated at the equilibrium:

$$F = \frac{\partial f}{\partial x}(x^*, u^*), \quad G = \frac{\partial f}{\partial u}(x^*, u^*). \tag{3.57}$$

As example, we revisit the pendulum equations. As in equation (2.28), the equations of motion are:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{b}{m\ell^2}\omega - \frac{g}{\ell}\sin(\theta) \end{bmatrix} \tag{3.58}$$

with the two equilibria:

- the equilibrium point $\begin{bmatrix} \theta^*_{\text{down}} \\ \omega^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, corresponding to the pendulum in its *down position*,
- the equilibrium point $\begin{bmatrix} \theta^*_{\text{up}} \\ \omega^* \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$, corresponding to the pendulum in its *up position*.

We compute the Jacobian matrix of (3.58) at an arbitrary point (θ, ω):

$$J(\theta, \omega) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos\theta & -\frac{b}{m\ell^2} \end{bmatrix} \tag{3.59}$$

and evaluate it at the two equilibria $\theta^*_{\text{down}} = 0$ and $\theta^*_{\text{up}} = \pi$ (where $\omega^*_{\text{down}} = \omega^*_{\text{up}} = 0$):

$$J_{\text{down}}(\theta^*_{\text{down}} = 0, \omega^*_{\text{down}} = 0) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix} \quad \text{and} \quad J_{\text{up}}(\theta^*_{\text{up}} = \pi, \omega^*_{\text{up}} = 0) = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix} \tag{3.60}$$

Therefore, the two linearizations of the pendulum dynamics are:

$$\begin{bmatrix} \dot{\delta\theta} \\ \dot{\delta\omega} \end{bmatrix} = J_{\text{down}} \begin{bmatrix} \delta\theta \\ \delta\omega \end{bmatrix} \tag{3.61}$$

where $\delta\theta = \theta - \theta^*_{\text{down}} = \theta$ and $\delta\omega = \omega - \omega^*_{\text{down}} = \omega$, and

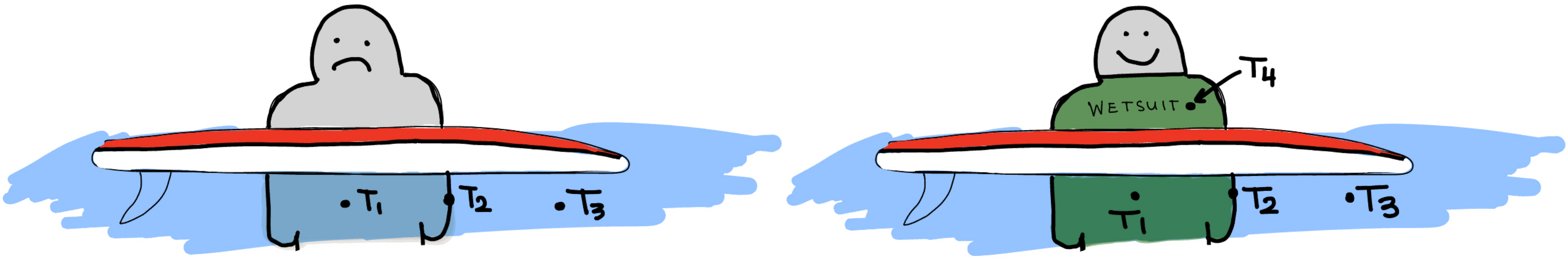
$$\begin{bmatrix} \dot{\delta\theta} \\ \dot{\delta\omega} \end{bmatrix} = J_{\text{up}} \begin{bmatrix} \delta\theta \\ \delta\omega \end{bmatrix} \tag{3.62}$$

where $\delta\theta = \theta - \theta^*_{\text{up}} = \theta - \pi$ and $\delta\omega = \omega - \omega^*_{\text{up}} = \omega$,

3.7 Exercises

Section 3.1: Heat transfer

E3.1 **Heat transfer problems in surfing.** In this problem, we will investigate how the presence or absence of a wetsuit affects the body temperature of a person while surfing in cold water. Assume that all heat transfer occurs through conduction. The system can be analyzed in two scenarios: (a) without a wetsuit and (b) with a wetsuit. These configurations are illustrated in Figure 3.20.



(a) System with no wetsuit

(b) System with a wetsuit

Figure 3.20: Person in the water (a) without a wetsuit and (b) with a wetsuit.

No wetsuit Denote the core body temperature by T_1 and thermal capacity by c_1, and the surface temperature of the body by T_2 and its thermal capacity by c_2. Assume that the body generates heat at a rate q_{body}. Assume the water temperature T_3 is constant.

(i) Write down the dynamics for the system in Figure 3.20a.

(ii) Derive an expression for the equilibrium temperature of the body. Express the result in terms of T_1, q_{body}, and the relevant thermal resistances.

With wetsuit In the second scenario, a wetsuit is introduced into the system. Let T_4 represent the temperature at the surface of the wetsuit. The rest of the setup remains the same.

(iii) Write down the new dynamics for the system in Figure 3.20b.

(iv) Derive a new expression for the equilibrium temperature of the body. Express the result in terms of T_1, T_3 and the appropriate thermal resistances.

(v) Assume the thermal resistances are related by $r_{12} = r_{34} = r_{23} = \frac{1}{10} r_{24}$. How much greater is the difference $T_1 - T_3$ with a wetsuit than without?

E3.2 **Combining thermal resistances.** Thermal resistances can be combined in analogy to electrical resistances.

(i) When heat flows sequentially through multiple materials with thermal resistances $r_1, r_2, \dots$, write the expression for the total resistance.

(ii) When heat flows simultaneously through multiple paths with thermal resistances $r_1, r_2, \dots$, write the expression for the total resistance.

Answer:

(i) Series combination: when heat flows sequentially, the same heat rate passes through each material, and the total temperature drop is the sum of the drops across each layer. Since each drop equals (heat rate) $\times\, r_i$, the total resistance is

$$r_{\text{total}} = r_1 + r_2 + \dots \tag{3.63}$$

(ii) Parallel combination: when heat flows in parallel paths, the temperature drop across each path is the same, and the total heat rate is the sum of the heat rates through each path. Since the heat rate in each path equals (temperature drop)$/r_i$, the total thermal conductance (1/resistance) is the sum of individual thermal conductances, giving

$$\frac{1}{r_{\text{total}}} = \frac{1}{r_1} + \frac{1}{r_2} + \dots \tag{3.64}$$

▼

Section 3.2: Fluid dynamics

E3.3 **Positive and negative fluid flow in a pipe with positive and negative pressure difference.** Given a flow resistance r and a flow behavior parameter α, recall that equation (3.11) assumes $p_1 > p_2$ and provides a mass flow rate w that is always positive. When the pressure difference $p_1 - p_2$ can be both positive and negative, the resulting mass flow rate from 1 to 2 can be both positive and negative.

Let $w_{1\to 2}$ denote the signed flow ("signed" means positive or negative) from point 1 to point 2.

(i) Write an equation for $w_{1\to 2}$ as function of $p_1 - p_2$, when $p_1 - p_2$ can be both positive and negative.

Hint: In other words, given two arbitrary numbers p_1 and p_2 (without assuming necessarily that $p_1 > p_2$), how would you compute the mass flow rate?

(ii) Verify that your proposed equation reduces to equation (3.11), when $p_1 > p_2$.

(iii) Write an equation for $w_{2\to 1}$ and explain how it relates to $w_{1\to 2}$.

Answer:

(i) We propose

$$w_{1\to 2} = \operatorname{sign}(p_1 - p_2)\frac{1}{r}(|p_1 - p_2|)^{1/\alpha} = \begin{cases} \frac{1}{r}(p_1 - p_2)^{1/\alpha} & \text{if } p_1 \geq p_2 \\ -\frac{1}{r}(p_2 - p_1)^{1/\alpha} & \text{if } p_2 > p_1 \end{cases} \tag{3.65}$$

(ii) When $p_1 > p_2$, we compute

$$w_{1\to 2} = \operatorname{sign}(p_1 - p_2)\frac{1}{r}(|p_1 - p_2|)^{1/\alpha} = \frac{1}{r}(p_1 - p_2)^{1/\alpha} \tag{3.66}$$

(iii) $w_{2\to 1} = -w_{1\to 2} = \operatorname{sign}(p_2 - p_1)\frac{1}{r}(|p_1 - p_2|)^{1/\alpha}$

▼

E3.4 **Heat and fluid flow in a water heater.** Consider a water tank above a coil heater as in Figure 3.21.

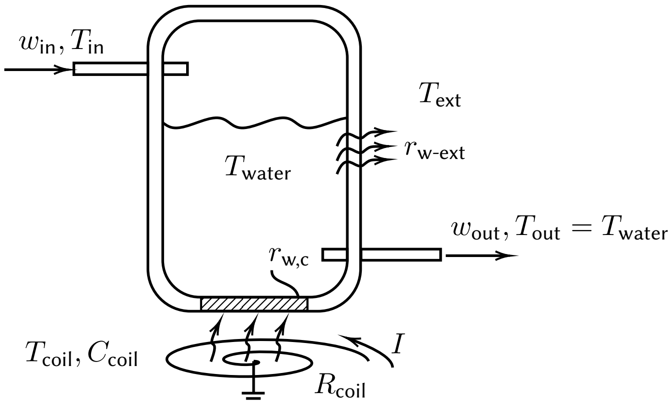


Figure 3.21: A water heater model.
Electrical current I enters the coil, which has resistance R_{coil}, resulting in heat generation. Internal coil temperature is denoted T_{coil}, with heat capacity C_{coil}. Water enters with temperature T_{in} and exits with temperature $T_{\text{out}} = T_{\text{water}}$.
The water in the tank is thermally connected to (i) a coil heater through a thermal resistance $r_{\text{w,c}}$ and (ii) the external environment through a thermal resistance $r_{\text{w-ext}}$.
Let T_{water}, T_{coil} and T_{ext} denote the temperatures of water, coil heater, and external environment, respectively.

(i) Using Fourier's law, write an equation for the heat flow $q_{\text{coil}\to\text{water}}$ from coil to tank, and $q_{\text{water}\to\text{ext}}$ from tank to external environment.

Now, assume that the coil heater warms up due to Joule resistive heating with its electrical resistance R_{coil} when a current I is applied. This means that the energy produced per second, that is, power generated is:

$$P = R_{\text{coil}} I^2 \qquad \text{(measured in watts = Joule per sec)}$$

Let C_{coil} be the total heat capacity of the coil heater. Let m denote the water mass in the tank, let c_{water} denote the *specific heat capacity*, that is, the amount of heat required per unit mass to raise the temperature of water by one-degree Celsius. Therefore, mc_{water} is the total heat capacity of the water in the heater. For now, assume no water flows into and out of the tank (both valves are closed).

(ii) Write an equation for the evolution of the temperature of the tank water and of the coil.

Finally, open the both valves and assume $w_{\text{in}} > 0$ and $w_{\text{out}} > 0$ are incoming and outgoing mass flow rates, as in the mass balance equation (3.9). Let T_{in} denote the temperature of the incoming water. Note that

- the *total thermal energy* of the tank water is $c_{\text{water}} m T_{\text{water}}$,
- due to the incoming water, energy is added to the water tank at a rate $q_{\text{in}} = w_{\text{in}} c_{\text{water}} T_{\text{in}}$ and subtracted at a rate $q_{\text{out}} = w_{\text{out}} c_{\text{water}} T_{\text{water}}$.

(iii) Write the balance equation for the time-derivative of the total thermal energy.

(iv) Using the formula from (iii) and the mass balance equation $\dot{m} = w_{\text{in}} - w_{\text{out}}$, obtain a single equation for $\dot{T}_{\text{water}}$.

(v) Collect the various results and write the resulting control system with three variables (m, T_{coil}, and T_{water}) with two inputs (w_{in}, I^2) and a disturbance (w_{out}).

Section 3.3: Linearization of nonlinear systems for small signals

E3.5 **Linearization of a mass-spring-damper system with a nonlinear spring .** Consider a nonlinear spring with zero rest length and restoring force equal, in magnitude, to

$$f_{\text{nonlinear-spring}}(x) = k_1 x + k_2 x^3 \tag{3.67}$$

where x is the displacement. Given a constant force f, consider a mass-spring-damper system with this nonlinear spring:

$$m\ddot{x} + b\dot{x} + k_1 x + k_2 x^3 = f. \tag{3.68}$$

(i) Assume $k_1 = 0$. For each value of f (positive and negative) and $k_2 > 0$, compute each possible equilibrium point of the mass-spring-damper system.
Hint: As a minor point, recall that an equilibrium point for this dynamical system is not just a position x^*.

(ii) If $k_1 > 0$, $k_2 > 0$, and $f = k_1 + k_2$, what is the equilibrium? Let $(x^*, 0)$ denote this equilibrium point.

(iii) Compute the small signal linearization of the mass-spring-damper system at the equilibrium point $(x^*, 0)$ when $f = k_1 + k_2$.

E3.6 **Equilibrium points and linearization of the water tank dynamics with input.** Recall that, given two positive coefficients a, b, the water tank dynamics is

$$\dot{h}(t) = -a\sqrt{h(t)} + bw(t) \tag{3.69}$$

where $h(t)$ is the water height and $w(t) \geq 0$ is the input signal.

(i) Compute each possible equilibrium point of the system.
Hint: Recall that an equilibrium point is a pair (h^*, w^*).

(ii) Compute the small signal linearization of the system at an equilibrium point such that $w^* > 0$.

(iii) Is it possible to compute the small-signal linearization at the equilibrium point such that $w^* = 0$? It not, explain why not. Otherwise compute it.
Note: *As discussed after the flow resistance model (3.11), when h is small and the flow is slow, then the assumption $\alpha = 2$ is inappropriate.*

E3.7 **Equilibrium points and linearization of a two-dimensional nonlinear system.** Consider the nonlinear system:

$$\begin{aligned}\dot{x}_1 &= x_1 + 0.5x_1x_2\\ \dot{x}_2 &= x_2 - 0.5x_1x_2\end{aligned}$$

(i) Compute all equilibrium points of this system.

(ii) Compute the linearization of this system around each equilibrium point.

Answer:

(i) To find the equilibrium points, the derivatives of the state variables are set to zero:

$$\dot{x}_1 = x_1(1 + 0.5x_2) = 0$$
$$\dot{x}_2 = x_2(1 - 0.5x_1) = 0$$

The first equation, $x_1(1 + 0.5x_2) = 0$, implies that either $x_1 = 0$ or $1 + 0.5x_2 = 0$.

Case 1: $x_1 = 0$. Substituting $x_1 = 0$ into the second equation, $x_2(1 - 0.5x_1) = 0$, yields $x_2(1 - 0) = 0$, which simplifies to $x_2 = 0$. The resulting equilibrium point is $(0, 0)$.

Case 2: $1+0.5x_2 = 0$. The equation $1+0.5x_2 = 0$ solves to $x_2 = -2$. Substituting $x_2 = -2$ into the second equation, $x_2(1-0.5x_1) = 0$, yields $-2(1-0.5x_1) = 0$. The term $(1 - 0.5x_1)$ must be zero, which gives $x_1 = 2$. The resulting equilibrium point is $(2, -2)$.

In summary, the system has two equilibrium points: $(0, 0)$ and $(2, -2)$.

(ii) To linearize the nonlinear system, we first compute the Jacobian matrix J. The system functions are $f_1(x_1, x_2) = x_1 + 0.5x_1x_2$ and $f_2(x_1, x_2) = x_2 - 0.5x_1x_2$. The Jacobian matrix is:

$$J(x_1, x_2,) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 + 0.5x_2 & 0.5x_1 \\ -0.5x_2 & 1 - 0.5x_1 \end{bmatrix}.$$

Next, we evaluate the Jacobian matrix at each equilibrium point.

Regarding the linearization around the equilibrium point $(0, 0)$, we evaluate the Jacobian matrix at $(0, 0)$:

$$A_1 = J(0, 0) = \begin{bmatrix} 1 + 0.5(0) & 0.5(0) \\ -0.5(0) & 1 - 0.5(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The linearized system around $(0, 0)$ is $\frac{d}{dt}\begin{bmatrix}\delta x_1 \\ \delta x_2\end{bmatrix} = A_1 \begin{bmatrix}\delta x_1 \\ \delta x_2\end{bmatrix}$, or

$$\frac{d}{dt}\delta x_1 = \delta x_1$$
$$\frac{d}{dt}\delta x_2 = \delta x_2$$

Regarding the linearization around the equilibrium point $(2,-2)$, we evaluate the Jacobian matrix evaluated at $(2,-2)$:

$$A_2 = J(2,-2) = \begin{bmatrix} 1+0.5(-2) & 0.5(2) \\ -0.5(-2) & 1-0.5(2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

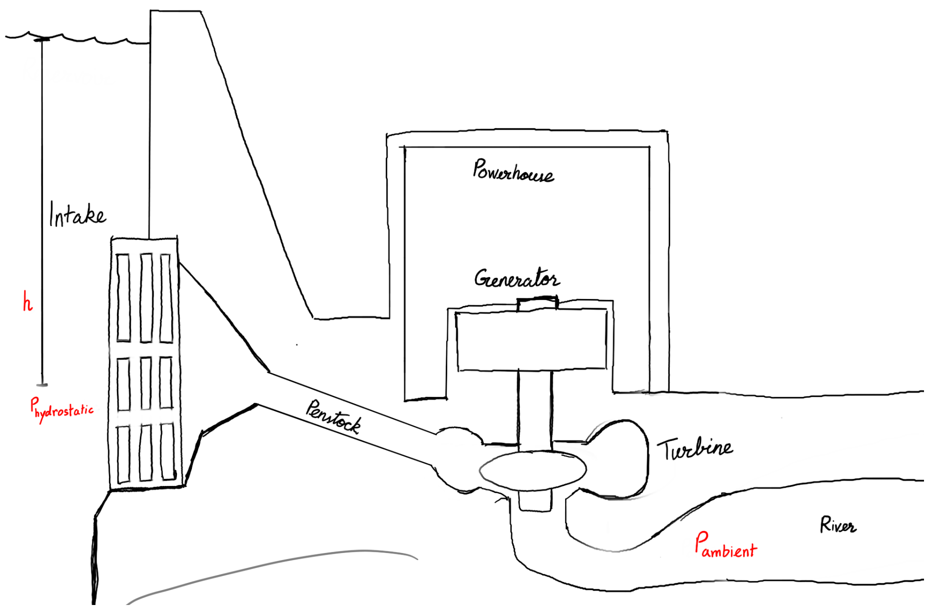
The linearized system around $(2,-2)$ is $\frac{d}{dt}\begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = A_2 \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$, or

$$\frac{d}{dt}\delta x_1 = \delta x_2$$
$$\frac{d}{dt}\delta x_2 = \delta x_1$$

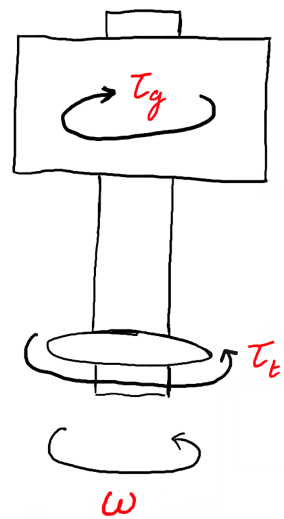
▼

E3.8 **Modeling (fluid flow and rotational dynamics) and linearization of a hydroelectric dam (Leonov et al., 2015).** Consider the hydroelectric dam shown in Figure 3.22a. When the intake is open, water flows from the reservoir through the penstock and out the outlet. The turbine and generator are coupled with a shaft so that energy is extracted from the flowing water and converted into electricity.

- The state variable for the hydroelectric dam control system is the rotation speed of the turbine-generator shaft ω.
- The hydraulic head h is the input signal to the hydroelectric dam control system.
- The pressure at the penstock intake is $p_{\text{hydrostatic}}$, and the pressure at the outlet is p_{ambient}. The combined flow resistance of the penstock and turbine is r, and the flow behavior parameter is $\alpha = 2$ (recall the flow resistance law in equation (3.11)). The density of the water is ρ, and the acceleration due to gravity is g. As shown in Figure 3.22b, the shaft has moment of inertia I and experiences a torque τ_{t} due to the water flowing through the turbine which is counteracted by a constant torque τ_{g} due to the generator.



(a) Schematic of a hydroelectric dam



(b) Free body diagram of turbine-generator shaft

Figure 3.22: Schematic and free body diagram of a hydroelectric dam.

(i) Derive the equation of motion for the rotation speed ω of the turbine-generator shaft in terms of the moment of inertia I and the torques τ_{t}, τ_{g}.

(ii) The turbine torque $\tau_{\text{t}} = (c_t P_t)/\omega$ where c_t is an efficiency constant (unitless) and P_t is the mechanical power of the turbine (in watts). We have the following equation for P_t:

$$P_t = q(p_{\text{hydrostatic}} - p_{\text{ambient}})$$

where q is the volumetric flowrate[4] through the penstock (in cubic meters per second). Use these relations along with others from the chapter to write an expression for the turbine torque τ_{t} in terms of the state ω, the input h, and other defined quantities.

(iii) Substitute the expression found in part (ii) into the equation of motion found in part (i). Identify the equilibrium of the resulting system. Sketch a 1-dimensional phase portrait and classify the stability of the equilibrium.

(iv) Linearize the system about the identified equilibrium. Use the notation $\delta\omega, \delta h$ for the transformed coordinates. Use the linearized model to verify the stability or instability of the equilibrium.

Answer:

(i) Referring to Figure 3.22b, we can immediately write down the equation of motion

$$I\dot{\omega} = \tau_{\text{t}} - \tau_{\text{g}}$$

(ii) We can use the given relations to write

$$\tau_{\text{t}} = \frac{c_t q}{\omega}(p_{\text{hydrostatic}} - p_{\text{ambient}}).$$

Then we can utilize the law of nonlinear resistance given in equation (3.11) to find the volumetric flow rate through the penstock and turbine:

$$q = \frac{1}{r\rho}(p_{\text{hydrostatic}} - p_{\text{ambient}})^{1/2}$$

where we have substituted in the given flow behavior parameter $\alpha = 2$. Furthermore, equation (3.14) gives the hydrostatic pressure:

$$p_{\text{hydrostatic}} = p_{\text{ambient}} + \rho g h.$$

Putting everything together we arrive at

$$\tau_{\text{t}} = \frac{c_t\sqrt{\rho}(gh)^{3/2}}{\omega r}$$

(iii) Substitution yields the equation of motion for the hydrostatic dam:

$$\dot{\omega} = \frac{c_t\sqrt{\rho}(gh)^{3/2}}{I\omega r} - \frac{\tau_{\text{g}}}{I}$$

[4] For the purpose of this exercise, assume the volumetric flow rate $q = w/\rho$, where w is the flow rate and ρ the water density.

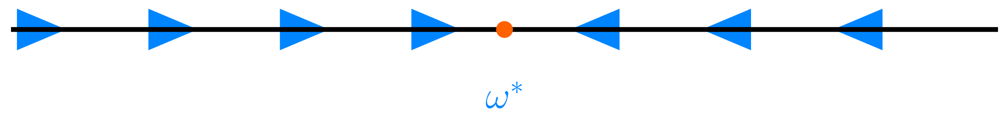


Figure 3.23: One-dimensional phase portrait for ω.

Setting $\dot{\omega} = 0$ yields the unique equilibrium

$$\omega^* = \frac{c_t\sqrt{\rho}g^{3/2}}{\tau_{\mathrm{g}} r}(h^*)^{3/2}$$

As shown in Figure 3.23, the phase portrait is identical to that of the water tank system from the chapter. Testing values of ω on either side of the equilibrium verifies its stability.

(iv) The system is described by the following equation of motion:

$$\dot{\omega} = \frac{c_t\sqrt{\rho}(gh)^{3/2}}{I\omega r} - \frac{\tau_{\mathrm{g}}}{I} = F(\omega, h).$$

Differentiation yields

$$\frac{\partial F}{\partial \omega} = -\frac{c_t\sqrt{\rho}(gh)^{3/2}}{Ir\omega^2}, \qquad \frac{\partial F}{\partial h} = \frac{3c_t\sqrt{\rho}g^{3/2}}{2Ir\omega}\sqrt{h}.$$

Evaluating these expressions at the equilibrium yields

$$\left.\frac{\partial F}{\partial \omega}\right|_* = -\frac{\tau_{\mathrm{g}}^2 r}{Ic_t\sqrt{\rho}g^{3/2}(h^*)^{3/2}}, \qquad \left.\frac{\partial F}{\partial h}\right|_* = \frac{3\tau_{\mathrm{g}}}{2Ih^*}.$$

Then we can write the linearized system:

$$\dot{\delta\omega} = -\frac{\tau_{\mathrm{g}}^2 r}{Ic_t\sqrt{\rho}g^{3/2}(h^*)^{3/2}}\delta\omega + \frac{3\tau_{\mathrm{g}}}{2Ih^*}\delta h$$

The negative sign in the first term indicates the stability of the equilibrium.

▼

E3.9 **Heat flow dynamics and linearization of a frying pan.** Consider a resistive coil heater and a frying pan, as shown in figure.

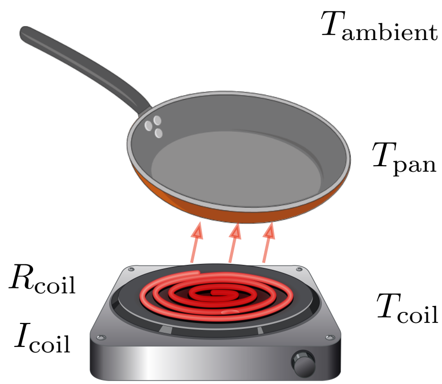


Figure 3.24: A frying pan over a coil.
let T_{pan} and c_{pan} be temperature and thermal capacity of the pan.
Let T_{coil} and c_{coil} be temperature and thermal capacity of the coil.
Let T_{ambient} denote the constant ambient temperature.

Resistive heating in coil: When the current I is applied, the coil with electrical resistance R_{coil} heats up due to Joule resistive heating. The heat generated is

$$q_{\text{electric}} = R_{\text{coil}} I^2$$

Conduction from coil to pan: Given the thermal resistance r between coil and pan, Fourier's law of heat conduction states

$$q_{\text{coil}\to\text{pan}} = \frac{1}{r}(T_{\text{coil}} - T_{\text{pan}})$$

Radiation from pan to ambient: Given a positive radiative coefficient $a > 0$, as discussed in Section 3.5.2, the Stefan-Boltzmann law of radiative heat transfer from the pan at temperature T_{pan} to the ambient at temperature T_{ambient} is:

$$q_{\text{pan}\to\text{ambient}} = a(T_{\text{pan}}^4 - T_{\text{ambient}}^4)$$

Assume no radiation from coil, only from the pan.

(i) Write the differential equations governing the evolution of T_{coil} and T_{pan} with input I.

(ii) Assume that $(T_{\text{pan}}^*, T_{\text{coil}}^*)$ and I^* is an equilibrium pair for the control system. Linearize the system around this equilibrium.
Hint: Remember to define and use the notation δT_{pan}, δT_{coil}, and δI for the coordinates of the linearized model.

Answer:

(i) The governing equations are:

$$c_{\text{coil}}\dot{T}_{\text{coil}} = \frac{1}{r}(T_{\text{pan}} - T_{\text{coil}}) + R_{\text{coil}}I^2 , \tag{3.70}$$

$$c_{\text{pan}}\dot{T}_{\text{pan}} = \frac{1}{r}(T_{\text{coil}} - T_{\text{pan}}) - a(T_{\text{pan}}^4 - T_{\text{ambient}}^4) . \tag{3.71}$$

(ii) Rearrange the differential equations and define functions F, G:

$$\begin{aligned}
\dot{T}_{\text{coil}} &= \frac{1}{c_{\text{coil}}r}(T_{\text{pan}} - T_{\text{coil}}) + \frac{R_{\text{coil}}}{c_{\text{coil}}}I^2 && \equiv F(T_{\text{coil}}, T_{\text{pan}}, I),\\
\dot{T}_{\text{pan}} &= \frac{1}{c_{\text{pan}}r}(T_{\text{coil}} - T_{\text{pan}}) - \frac{a}{c_{\text{pan}}}(T_{\text{pan}}^4 - T_{\text{ambient}}^4) && \equiv G(T_{\text{coil}}, T_{\text{pan}}).
\end{aligned}$$

Take partial derivatives:

$$\begin{aligned}
\frac{\partial F}{\partial T_{\text{coil}}} &= -\frac{1}{c_{\text{coil}}r}, & \frac{\partial F}{\partial T_{\text{pan}}} &= \frac{1}{c_{\text{coil}}r}, & \frac{\partial F}{\partial I} &= \frac{2R_{\text{coil}}}{c_{\text{coil}}}I,\\
\frac{\partial G}{\partial T_{\text{coil}}} &= \frac{1}{c_{\text{pan}}r}, & \frac{\partial G}{\partial T_{\text{pan}}} &= -\frac{1}{c_{\text{pan}}r} - \frac{4a}{c_{\text{pan}}}T_{\text{pan}}^3, & \frac{\partial G}{\partial I} &= 0.
\end{aligned}$$

Evaluate at the equilibrium point and assemble the linearized system:

$$\delta\dot{T}_{\text{coil}} = -\frac{1}{c_{\text{coil}}r}\delta T_{\text{coil}} + \frac{1}{c_{\text{coil}}r}\delta T_{\text{pan}} + \left(\frac{2R_{\text{coil}}}{c_{\text{coil}}}I^*\right)\delta I , \tag{3.72}$$

$$\delta\dot{T}_{\text{pan}} = \frac{1}{c_{\text{pan}}r}\delta T_{\text{coil}} - \left(\frac{1}{c_{\text{pan}}r} + \frac{4a}{c_{\text{pan}}}(T_{\text{pan}}^*)^3\right)\delta T_{\text{pan}} , \tag{3.73}$$

where the transformed coordinates are defined as $\delta T_{\text{coil}} = T_{\text{coil}} - T_{\text{coil}}^*$, $\delta T_{\text{pan}} = T_{\text{pan}} - T_{\text{pan}}^*$, $\delta I = I - I^*$.

▼

E3.10 **Modeling and linearization of an epidemic model.** In this exercise we explore the dynamics of epidemics and the occurrence of *epidemic outbreaks*. Given a population and a pathogen, we assume that each individual is in one of three possible states:

susceptible: the individual is not infected, but is vulnerable to being infected,
infected: the individual is infected and capable of transmitting the disease to susceptible individuals, and
recovered: the individual has been infected and has now recovered, gaining immunity to the pathogen.

Mathematically, we let s, x, r denote the fractions of *susceptible*, *infected*, and *recovered individuals*, respectively. Note that $s(t) + x(t) + r(t) = 1$ at all times t.

We call this simplified model the SIR model and we allow only two types of transitions:[5]

susceptible $\rightarrow$ infected: this transition is due to an interaction between one susceptible individual and one infected individual. Therefore, the transition rate depends upon the likelihood of contact between individuals in the two states. Mathematically, the transition rate equals βsx, where the *infection rate* $\beta > 0$ is describes how transmissible is the pathogen;

infected $\rightarrow$ recovered: this transition is spontaneous, independent of interactions, and proportional to the fraction of infected individuals. The transition rate is γx, where the *recovery rate* $\gamma > 0$ describes the decay rate of the infection.

We illustrate the transitions and their rates in Figure 3.25.

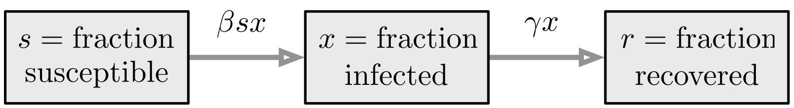


Figure 3.25: The three states, the two possible transitions, and the transition rates for the two possible transitions.

In summary, our proposed *SIR epidemic model* is

$$\dot{s} = -\beta sx \quad (3.74)$$
$$\dot{x} = \beta sx - \gamma x \quad (3.75)$$
$$\dot{r} = \gamma x. \quad (3.76)$$

(i) How many variables and parameters are in this epidemic system? (Here we mean variables dependent on time and we do not count time)

(ii) Identify all equilibria of the system.

(iii) Linearize the model around the equilibrium $(1, 0, 0)$.

(iv) Show that, for the linearized dynamics, the time derivative of the infected individuals δx is independent of the values of δs and δr.

(v) An *epidemic outbreak* is said to occur when: (i) all individuals are susceptible except for a small positive fraction of infected individuals, and (ii) the number of infected individuals starts to grow exponentially fast. Use the linearized model from questions (iii) and (iv) to find a condition on the ratio β/γ that leads to an epidemic outbreak.

***Note:** Some additional analysis explains that the ratio β/γ equals the R_0 value in the literature. The R_0 value equals the average number of individuals that an infected individual will infect. This value was widely publicized during the COVID-19 pandemic.*

[5] More realistic models include more states and more possible transitions.

Answer:

(i) 3 variables and 2 parameters

(ii) We can see that it is both necessary and sufficient that $x = 0$ for the system to be in equilibrium.

Thus, all points of the form $(a, 0, b)$ are equilibria.

(Since $s + x + r$ is a conserved quantity, we can equivalently write $(a, 0, 1 - a)$, assuming that the system starts from $(1, 0, 0)$.)

(iii) We compute

$$\text{Jacobian of the dynamics} = \begin{bmatrix} -\beta x & -\beta s & 0 \\ \beta x & \beta s - \gamma & 0 \\ 0 & \gamma & 0 \end{bmatrix} \tag{3.77}$$

Evaluating the Jacobian at the point $(1, 0, 0)$ and using equation (3.56) yields

$$\frac{d}{dt}\begin{bmatrix} \delta s \\ \delta x \\ \delta r \end{bmatrix} = \begin{bmatrix} 0 & -\beta & 0 \\ 0 & \beta - \gamma & 0 \\ 0 & \gamma & 0 \end{bmatrix} \begin{bmatrix} \delta s \\ \delta x \\ \delta r \end{bmatrix} \tag{3.78}$$

(iv) The equation $\dot{\delta x} = (\beta - \gamma)\delta x$ is decoupled from the evolution of the susceptible and recovered fractions.

(v) We can see that the linear system $\dot{\delta x} = (\beta - \gamma)\delta x$ is a linear growth model (studied in Chapter 1) and it is an unstable system if $\beta - \gamma > 0$. Manipulating this expression, we obtain the equivalent condition:

if $\beta/\gamma > 1$, then the linearized system is unstable

When this linearized system is unstable, $\delta x(t)$ grows exponentially fast (for small times, before we leave the neighborhood of the equilibrium point $(1, 0, 0)$) and so we have an epidemic outbreak.

This answer to question (v) shows that, when $R_0 > 1$, there is an epidemic outbreak.

▼

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