UC Santa Barbara, Department of Mechanical Engineering
ME103 Dynamical Systems. Chapter Slides.

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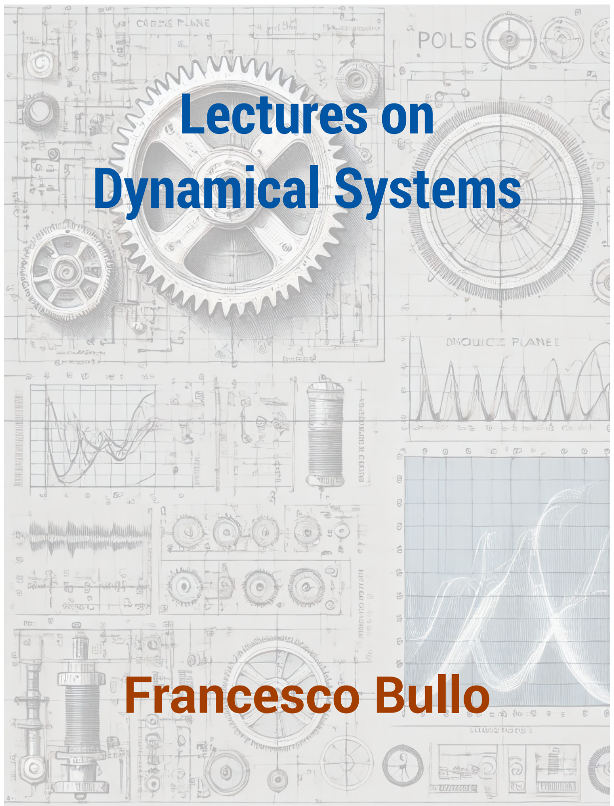
http://motion.me.ucsb.edu/ME103-Fall2025/syllabus.html



Contents

Chapter 2

Mechanical and Electromechanical Systems

This chapter delves into the fundamental principles and applications of dynamical systems, exploring *mechanical*, *electrical*, and *electromechanical systems* through the lens of Newton's laws and subsequent advancements.

In mechanical systems, the analysis begins with the study of first-order models, where the *time constant* τ determines the rate at which the system responds to changes. Second-order models build on *Newton's second law* $F = ma$ and include damped harmonic oscillators, classified as *underdamped*, *overdamped*, or *undamped*. For these systems, we introduce the *natural frequency* ω_n as a critical parameter shaping the system response.

The discussion extends to two-degree-of-freedom systems, exemplified by vehicle suspensions, which require complex modeling to account for multiple interacting components like springs and shock absorbers. Numerical simulations are used to analyze these systems' responses to varying conditions, highlighting the importance of dynamics in structural design.

The chapter further explores *rotational mechanical systems*, where torque and moment of inertia govern motion, as demonstrated by pendulums and gear systems. These systems illustrate the transition between *stable* and *unstable* equilibrium points and the role of *gear* ratios in transmitting motion.

Finally, in electrical systems, the focus shifts to components such as resistors, capacitors, and inductors, with differential equations describing their behavior in circuits. The analogy between electrical circuits and mechanical oscillators is drawn, particularly in RLC circuits. Electromechanical systems, like DC motors, are then examined, showcasing the interaction between electrical inputs and mechanical outputs through the Lorentz force.

2.1 Mechanical systems: One degree of freedom

Newton's law is the starting point for any analysis of mechanical systems. For a *body* composed of a single particle or rigidly interconnected particles moving in a single direction, the law is

$$F = ma \tag{2.1}$$

where:

- F is the resultant force (the algebraic sum of all forces) applied to the body, measured in Newtons (N),
- m is the mass of the body, measured in kg, and
- $a = a(t) = \ddot{x}(t)$ is the acceleration of the body, which is the second time derivative of the body's position $x(t)$, measured in $\mathrm{m/s^2}$.

As in Chapter 1, the independent variable t is time and the position $x(t)$ is the dependent variable. The mass m is most often treated as a constant parameter.[1] The force F is independent of Newton's law; the force may be an external force generated by some unspecified means. Equation (2.1) is referred to as the *equation of motion* because the equation describes the evolution of the position $x(t)$. When no force is applied to the particle, the solution is a translation at constant velocity.

[1] There are problems where the mass is not constant, e.g., consider the mass of a rocket burning fuel.

2.1.1 First-order systems

A *damper* is a mechanical element that dissipates energy. A classic example of a translational damper is a piston connected to a rod and an oil-filled cylinder. The oil resists any relative motion between the piston and the cylinder. Typically, one approximates the force generated by the damper linearly:

$$F_{\text{damper}} = -b\dot{x}(t) \tag{2.2}$$

where $b > 0$ is the *damping coefficient*, also known as the viscous friction coefficient or the mechanical resistance.

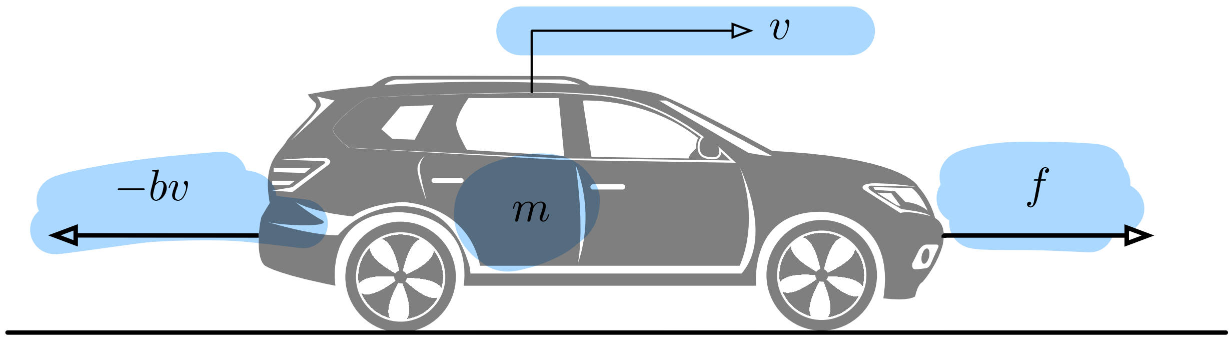


Figure 2.1: Moving car with engine propulsion force f and linear drag force $-bv$.

In a moving car, energy is dissipated by the interaction between the air and the moving car. Assuming the dissipation is linearly proportional to the car speed (with damping coefficient b) and assuming the motor produces a constant force f, the equations of motion are

$$m\ddot{x}(t) = -b\dot{x}(t) + f. \tag{2.3}$$

If we concern ourselves only with velocity $v(t) = \frac{d}{dt}x(t)$, we can write the *car velocity system* as

$$m\dot{v}(t) = -bv(t) + f. \tag{2.4}$$

This system is a *first-order system*. The key feature of a first-order system is that *one variable* is required and sufficient to describe the storage of position, velocity, energy, or mass.

input

In class assignment

Is there any difference between this model and the linear growth/decay model in Chapter 1?

(1) $\dot{x} = rx$ $r > 0$, $r < 0$

(2) $m\dot{v} = -bv + f$

differences:

- (2) has an input
- (2) has 2 parameters

Numerical simulation of car velocity system with switching force

```
# Python libraries
import numpy as np; import matplotlib.pyplot as plt; from scipy.integrate import odeint
plt.rcParams.update({"text.usetex": True, "font.family": "serif", "font.serif": ...
    ["Computer Modern Roman"] })

# Differential equation model of the dynamical system
def model(v, t, b, m, f):
    dvdt = - (b/m) * v + f(t)/m
    return dvdt

# Parameters and time array
b = 4; m = 3; t = np.linspace(0, 10, 500)
# Force with a step change at time 5
def f(t):
    if t < 5:
        return 20
    else:
        return 30

# Initial conditions
v0_values = [2, 3, 4, 5, 6, 7, 8];

# Numerically integrate and plot solutions for each initial condition.
plt.figure(figsize=(7, 4.2));  # Colors are (from darker to lighter)
oranges = ['#471b00', '#752d00', '#a43e00', '#d35000', '#ff6100', '#ff7f1a', '#ff9b56']
for idx, v0 in enumerate(v0_values):
    v = odeint(model, v0, t, args=(b, m, f))
    plt.plot(t, v, label=f'$v_0={v0}$', color=oranges[idx])

# Annotate and save the plot
plt.title('Solutions to the car velocity system',fontsize=14)
plt.xlabel('time $t$', fontsize=16); plt.ylabel('state $v(t)$', fontsize=16); ...
    plt.legend(); plt.grid(True);
plt.xlim(0, 10); plt.savefig("first-order-ode.pdf", bbox_inches='tight')

# Second figure: Illustrate time constant
tau = m / b;  v0 = 1
def fzero(t):
    return 0

# Numerically integrate and plot solution for the given initial condition
plt.figure(figsize=(10, 3));
v = odeint(model, v0, t, args=(b, m, fzero))
plt.plot(t, v, label=f'v0={v0}', color='#0085ff', zorder=2)

# Add tangent line at t=0. For x(t) = e^(-t/tau), the derivative at t=0 is dx/dt|_{t=0} ...
    = -1/tau
# So the tangent line is: y = x(0) + (dx/dt|_{t=0}) * t = 1 - t/tau
t_tangent = np.linspace(0, 2*tau, 100); tangent_line = v0 - t_tangent/tau
plt.plot(t_tangent, tangent_line, color=oranges[2], linestyle='--', linewidth=1.5,
         label='Tangent at $t=0$', zorder=1)

# Highlighting the time constants
time_constants = [tau, 2*tau, 3*tau, 4*tau, 5*tau]
for tc in time_constants:
    plt.axvline(x=tc, color=oranges[4], linestyle='--', linewidth=0.75)
plt.xticks(time_constants, [r'${} \tau$'.format(int(tc/tau)) for tc in time_constants])

# Drawing the 1% horizontal dashed line
one_percent_value = 0.01; exp_value = np.exp(-1);
plt.axhline(y=one_percent_value, color=oranges[2], linestyle='--', linewidth=1.5)
plt.annotate(r'$1\% > \mathrm{e}^{-5} \approx 0.67\%$', (6*tau, one_percent_value), ...
    textcoords="offset points", xytext=(0,10), ha='left', color=oranges[2])
plt.axhline(y=exp_value, color=oranges[2], linestyle='--', linewidth=1.5)
plt.annotate(r'$\mathrm{e}^{-1}\approx 36.8\%$', (6*tau, exp_value), textcoords="offset ...
    points", xytext=(0,10), ha='left', color=oranges[2])

# Annotate and save the plot
plt.title(r'Solution to unforced first-order system: $\quad\tau \dot{x} = -x$',fontsize=11)
plt.xlabel('time $t$',fontsize=11); plt.ylabel('state $x(t)$',fontsize=11);
plt.grid(True); plt.xlim(0, 6); plt.ylim(-0, 1)
plt.savefig("first-order-ode-timeconstant.pdf", bbox_inches='tight')
```

Listing 2.1: Python script generating Figure 2.2. Available at `first-order-ode.py`

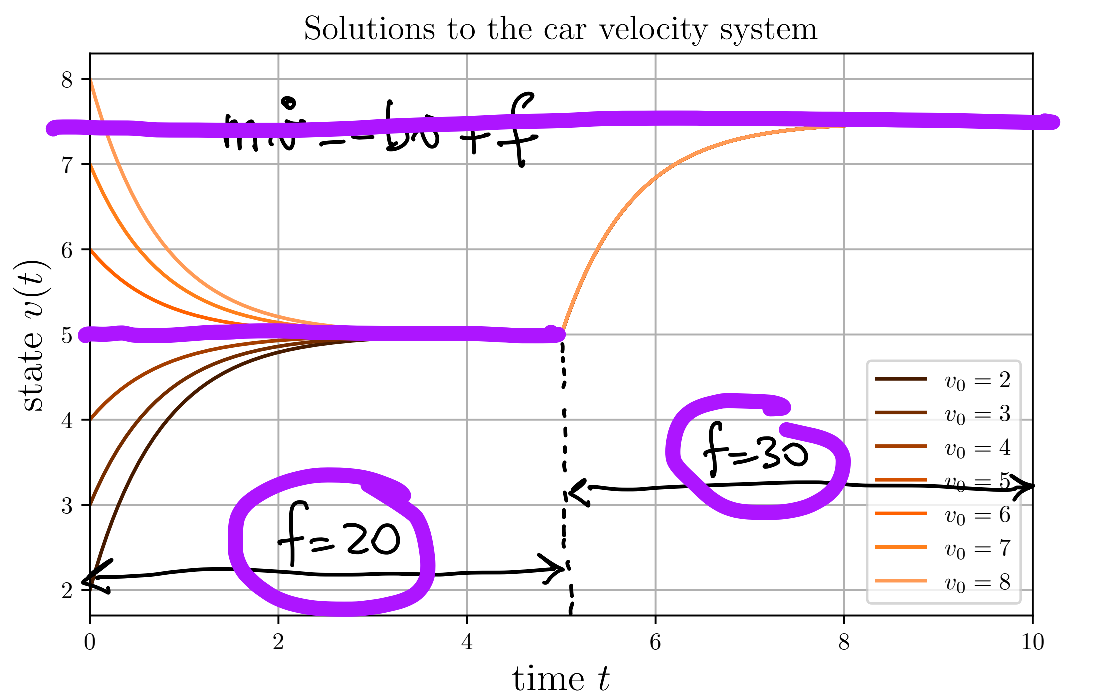


Figure 2.2: Solutions to the first-order equation (2.4): $m\dot{v}(t) = -bv(t) + f$ for $m = 3$ and $b = 4$. When the force is $f = 20$, the final value is $v_{\text{final}} = f/b = 20/4 = 5$. When the force changes to $f = 30$, the final value is $v_{\text{final}} = f/b = 30/4 = 7.5$. Loosely speaking, the speed at which the solution starting at $x(0) = 8$ drops to the value 5 is the same as the speed with which the solution starting at $x(0) = 2$ rises to the value 5.

$$v_{final} = \frac{f}{b} = \begin{cases} \frac{20}{4} = 5 & \text{if } f = 20 \\ \frac{30}{4} = 7.5 & \text{if } f = 30 \end{cases}$$

Mathematical analysis: Change of coordinates into an error system

Consider the affine[2] first-order system

$$m\dot{v}(t) = -bv(t) + f \quad \iff \quad \ddot{x}(t) = -\frac{b}{m}\dot{x}(t) + \frac{f}{m}, \tag{2.5}$$

with constant coefficients m, b, and f. We saw in the numerical simulation that, for each constant force f, the solution converges to a constant *final value*. In other words, the system has a stable equilibrium point, which is easily computed. By setting $\dot{v} = 0$ and solving for v, we define the *final value* (or *steady-state value*) of the model (2.5) to be

$$v_{\text{final}} = f/b. \tag{2.6}$$

Next, we consider a *change of coordinates* into a *relative velocity* (velocity relative to the final value)

$$v_{\text{relative}}(t) = v(t) - v_{\text{final}} = v(t) - f/b. \tag{2.7}$$

The relative velocity plays the role of an *error variable*, measuring the error from the current position to the final position. Assuming f is constant, we can compute:

$$\frac{d}{dt}v_{\text{relative}}(t) = \frac{dv}{dt}(t) - 0 = \frac{1}{m}(-bv(t) + f) = -\frac{b}{m}(v_{\text{relative}}(t) + v_{\text{final}}) + \frac{f}{m} = -\frac{b}{m}v_{\text{relative}}.$$

In summary, the *error system* is an *unforced first-order system*

$$\dot{v}_{\text{relative}}(t) = -\frac{b}{m}v_{\text{relative}}(t). \tag{2.8}$$

[2] A function is affine if it is the sum of a linear function and a constant.

Mathematical analysis: Time constant of unforced first-order systems

We rewrite the first-order system with a useful new parameter:

$$\dot{x} = -rx \qquad \iff \qquad \tau\dot{x} = -x \qquad \implies \qquad x(t) = \mathrm{e}^{-t/\tau}\, x(0), \tag{2.9}$$

t/τ

where we define the *time constant* $\tau = 1/r$ and we call the following equation the *canonical form of a first-order system*:

$$\tau\dot{x} = -x \tag{2.10}$$

(i) For an unforced system from a nonzero initial condition, τ is the time required for the system's response $x(t)$ to decay to $\mathrm{e}^{-1} \approx 36.8\%$ of the initial value $x(0)$.

(ii) At time $t = 5\tau$, the distance to the final value reaches a value $\mathrm{e}^{-5} \approx 0.67\% < 1\%$ of the initial value $x(0)$. The rule of thumb is that, *after five time constants, the error has practically vanished*.

Assume $x(0) = 1$.

at $t = \tau$: $x(\tau) = x(0)\, e^{-t/\tau}\Big|_{x(0)=1,\ t=\tau} = e^{-1}$

$\cong 0.368 = 36.8\%$

at $t = 5\tau$: $x(5\tau) = e^{-5} \cong 0.0067 = 0.67\% < 1\%$

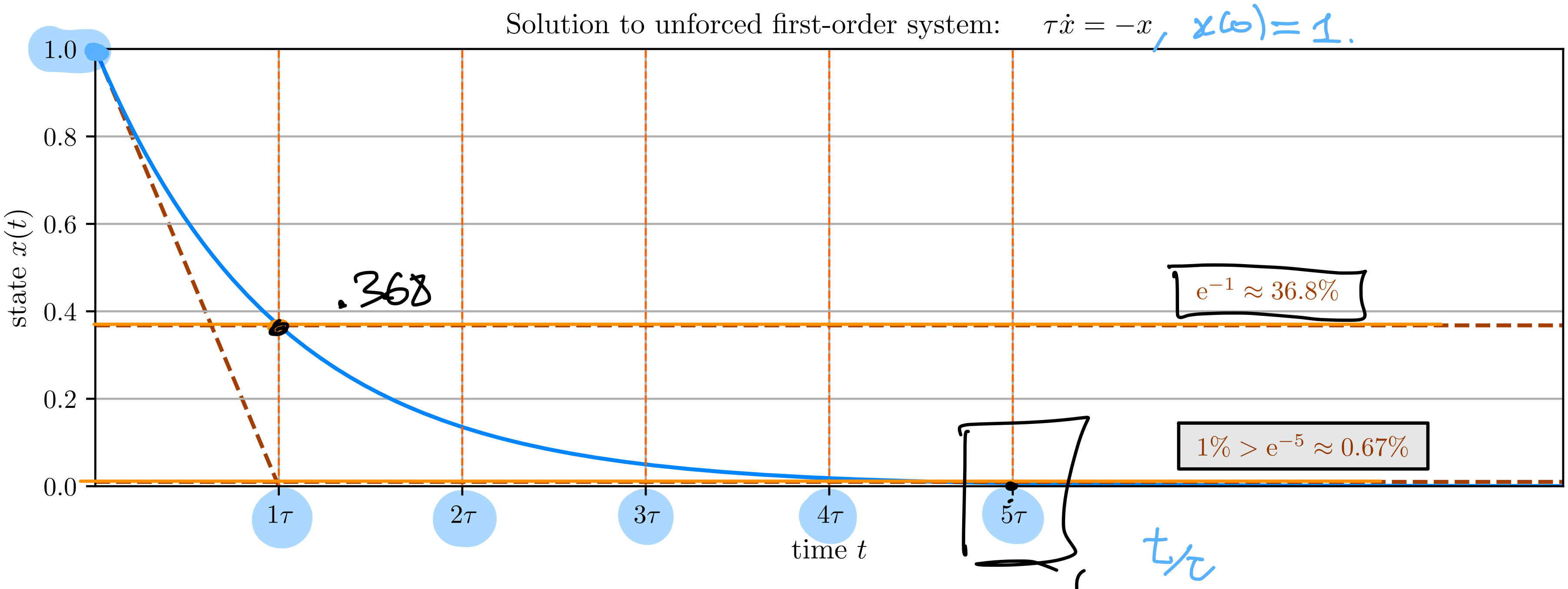


Figure 2.3: Illustrating the time constant of an unforced first-order system $\tau\dot{x} = -x$, $x(0) = 1$.
Note $x(\tau) = e^{-1}\,x(0)$ and $x(5\tau) = e^{-5}\,x(0)$. Hence, the state is equal to $e^{-1} \approx 36.8\%$ at $t = \tau$ and is below 1% at and after $t = 5\tau$.
For the forced system $\tau\dot{x} = -x + 1$ and $x(0) = 0$, the constant τ is the time required for $x(t)$ to reach approximately $1 - e^{-1} \approx 63.2\%$ of its final value.
Note that the tangent to $x(t)$ at time $t = 0$ intersects the horizontal axis at time equal to τ. This equality provides an empirical way to determine τ: (i) draw the tangent to $x(t)$ at $t = 0$ and extend it until it intersects the time axis, (ii) the time coordinate of this intersection provides an empirical estimate of τ.

$$\tau \dot{x} = -x + k u$$

2.1.2 Second-order systems: harmonic oscillators

A *spring* is a mechanical element that stores energy. For now, we focus on translational springs. Typically, a spring has a natural *rest length* with the property that, at rest length, the spring produces no force. When the spring is stretched or compressed, the spring produces a restoring force which is proportional to the displacement. Assume that the first end of the spring is fixed and the second end is at position 0. When the second end is attached to a body at position x, then the spring force on the body is

$$F_{\text{spring}} = -kx \tag{2.11}$$

where $k > 0$ is a *stiffness* or *spring constant*.

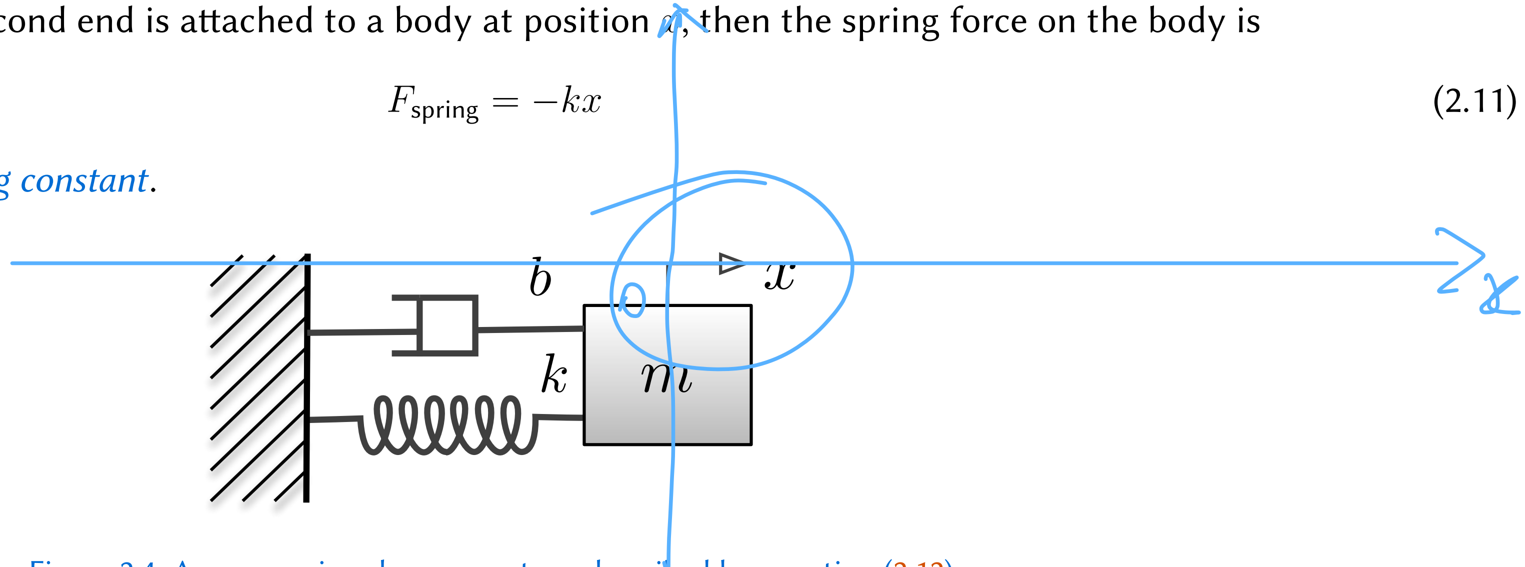


Figure 2.4: A mass-spring-damper system, described by equation (2.12).

When a body (translating along a single axis) is connected to both a spring and a damper, the resulting dynamics are called the *mass-spring-damper system* or, equivalently, the *damped harmonic oscillator*:

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0. \tag{2.12}$$

where b is the *damping coefficient* describing the damper.

It is possible and useful to rewrite this second-order differential equation as a first-order equation in two variables. As before, we define the velocity variable $v(t) = \dot{x}(t)$ and write

$$\dot{x}(t) = v(t) \tag{2.13a}$$

$$\dot{v}(t) = -(b/m)v(t) - (k/m)x(t) \tag{2.13b}$$

Since the system requires two variables, the system is said to have *dimension* 2.

The key feature of a *second-order system* is that *two variables* are required and sufficient to describe the storage of position, velocity, energy, or mass.

Numerical analysis of the damped harmonic oscillator: *Underdamped oscillator*

```python
# Python libraries
import numpy as np; import matplotlib.pyplot as plt; from ...
    scipy.integrate import odeint
plt.rcParams.update({"text.usetex": True, "font.family": "serif", ...
    "font.serif": ["Computer Modern Roman"] })

# Constants
m = 1.0    # Mass
b = 0.5    # Damping coefficient
k = 2.0    # Stiffness

# Differential equations for damped harmonic oscillator
def damped_oscillator(y, t, b, k, m):
    x, v = y
    dxdt = v
    dvdt = -(b/m) * v - (k/m) * x
    return [dxdt, dvdt]

# Time vector
t = np.linspace(0, 14, 1000)

# Six different initial conditions [x0, v0]
initial_conditions = [ [2, 0],  [1, 0],  [.5, 0],  [.1, 0], [-1, 0]]
colors = ['#471b00','#752d00', '#a43e00', '#d35000', '#ff6100', '#ff7f1a']

# Plotting solutions as a function of time
plt.figure(figsize=(6,3)); plt.xlim(0, 14);
for idx, init_cond in enumerate(initial_conditions):
    sol = odeint(damped_oscillator, init_cond, t, args=(b, k, m))
    plt.plot(t, sol[:, 0], label=f'$x_0={init_cond[0]}$, ...
        $v_0={init_cond[1]}$', color=colors[idx])

plt.ylabel('displacement $x(t)$',fontsize=12); plt.legend(); ...
    plt.xlabel('time $t$',fontsize=12); plt.grid(True);
plt.savefig('harmonic-damped.pdf', bbox_inches='tight')

# Phase portrait
X, Y = np.meshgrid(np.linspace(-2, 2, 20), np.linspace(-3, 3, 20))
U = Y; V = -(b/m) * Y - (k/m) * X; magnitude = np.sqrt(U**2 + V**2)

plt.figure(figsize=(12,6)); plt.grid(True)
plt.xlim(-2, 2); plt.ylim(-3, 3); plt.scatter(0, 0, color='black', ...
    s=50, zorder=5)
plt.streamplot(X, Y, U, V, density=0.75, color='#0085ff', ...
    arrowsize=1.5, linewidth=magnitude)
plt.xlabel('displacement $x$',fontsize=24); plt.ylabel('velocity ...
    $v$',fontsize=24)
plt.tick_params(axis='both', which='major', labelsize=24)
plt.savefig('harmonic-damped-phase.pdf', bbox_inches='tight')
```

Listing 2.2: Python script generating Figure 2.5. Available at `harmonic-damped.py`

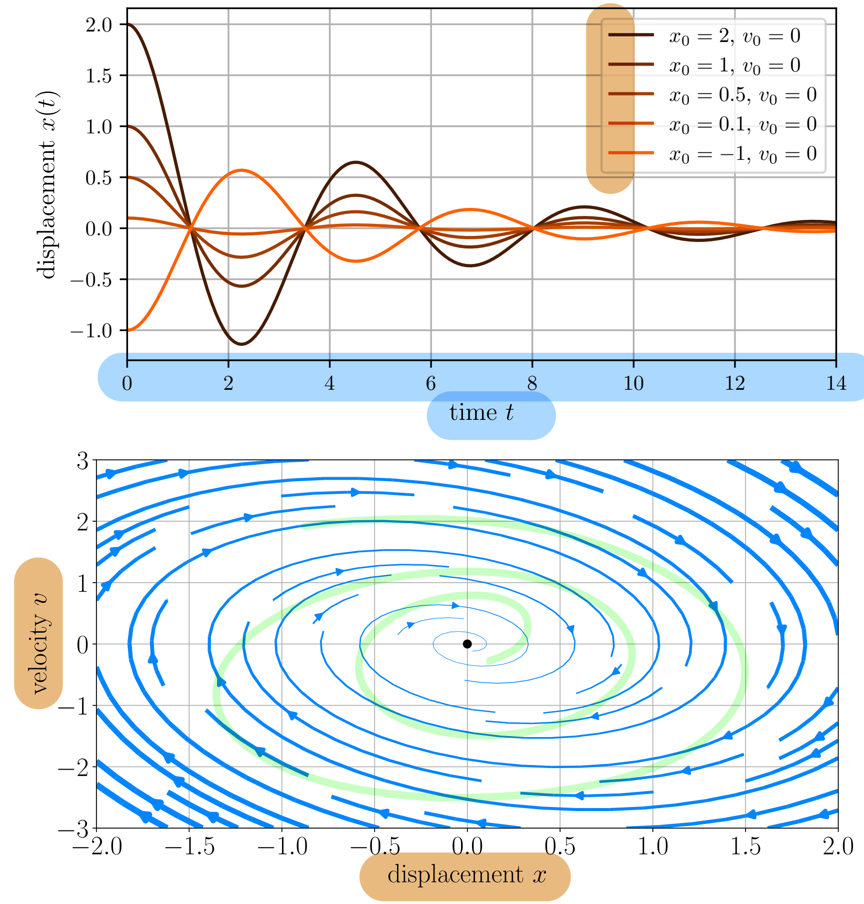


Figure 2.5: Solutions (from zero initial velocity) and phase portrait for the damped harmonic oscillator (2.13), with a low value of the damping coefficient: $m = 1$, $b = 0.5$, and $k = 2.0$. When there are oscillations, the system is said to be *underdamped*.

Numerical analysis of the damped harmonic oscillator: *Overdamped oscillator*

```python
# Python libraries
import numpy as np; import matplotlib.pyplot as plt
from scipy.integrate import odeint

# Constants
m = 1.0    # Mass
b = 3.0    # Damping coefficient
k = 2.0    # Stiffness

# Differential equations for overdamped harmonic oscillator
def overdamped_oscillator(y, t, b, k, m):
    x, v = y
    dxdt = v
    dvdt = -(b/m) * v - (k/m) * x
    return [dxdt, dvdt]

# Time vector
t = np.linspace(0, 14, 1000)

# Six different initial conditions [x0, v0]
initial_conditions = [ [2, 0],  [1, 0],  [.5, 0],  [.1, 0], [-1, 0] ]
colors = ['#471b00','#752d00', '#a43e00', '#d35000', '#ff6100', '#ff7f1a']

# Plotting solutions as a function of time
plt.figure(figsize=(6,3)); plt.xlim(0, 14);
for idx, init_cond in enumerate(initial_conditions):
    sol = odeint(overdamped_oscillator, init_cond, t, args=(b, k, m))
    plt.plot(t, sol[:, 0], label=f'$x_0={init_cond[0]}$, ...
        $v_0={init_cond[1]}$', color=colors[idx])

plt.ylabel('displacement $x(t)$',fontsize=12); plt.legend(); ...
    plt.grid(True); plt.xlabel('time $t$',fontsize=12)
plt.savefig('harmonic-overdamped.pdf', bbox_inches='tight')

# Phase portrait
X, Y = np.meshgrid(np.linspace(-2, 2, 20), np.linspace(-3, 3, 20))
U = Y; V = -(b/m) * Y - (k/m) * X; magnitude = np.sqrt(U**2 + V**2)

plt.figure(figsize=(12,6)); plt.grid(True)
plt.xlim(-2, 2); plt.ylim(-3, 3); plt.scatter(0, 0, color='black', ...
    s=50, zorder=5)
plt.streamplot(X, Y, U, V, density=0.75, color='#0085ff', ...
    arrowsize=1.5, linewidth=magnitude)
plt.xlabel('displacement $x$',fontsize=24); plt.ylabel('velocity ...
    $v$',fontsize=24)
plt.tick_params(axis='both', which='major', labelsize=24)
plt.savefig('harmonic-overdamped-phase.pdf', bbox_inches='tight')
```

Listing 2.3: Python script generating Figure 2.6. Available at `harmonic-overdamped.py`

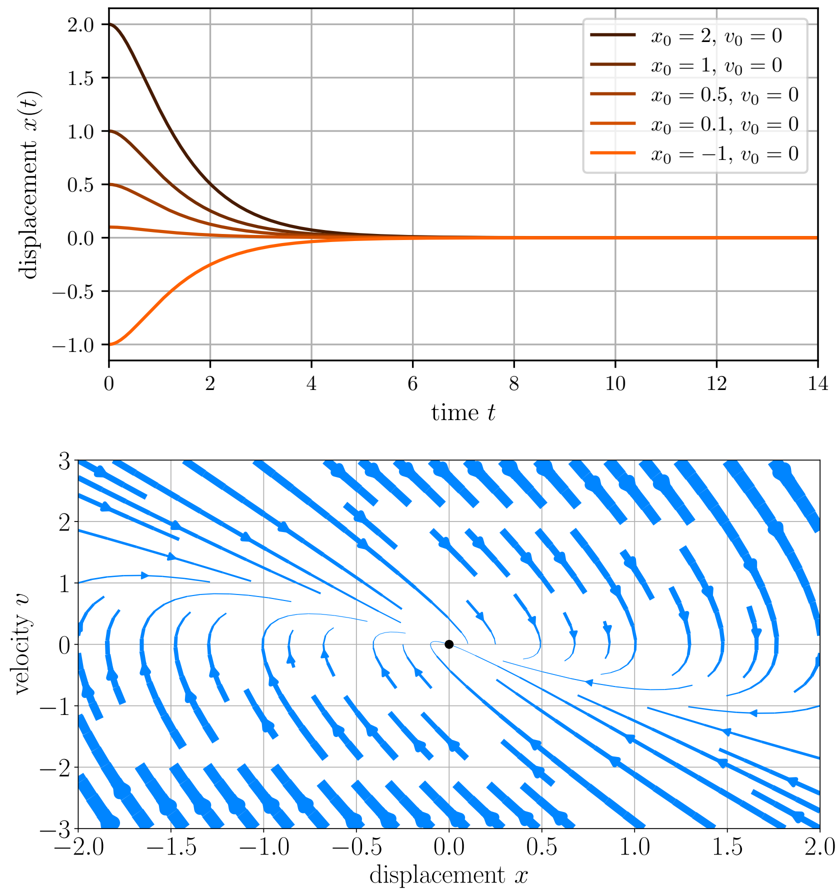


Figure 2.6: Solutions (from zero initial velocity) and phase portrait for the damped harmonic oscillator (2.13) with a high value of the damping coefficient: $m = 1$, $b = 3.0$, and $k = 2.0$. When there are no oscillations, the system is said to be *overdamped*.

It is also possible to set the damping coefficient to zero ($b = 0$) and consider the *undamped harmonic oscillator*:

$$m\ddot{x}(t) + kx(t) = 0. \tag{2.14}$$

Writing this second-order differential equation as a first-order equation in two variables, we get

$$\dot{x}(t) = v(t) \tag{2.15a}$$

$$\dot{v}(t) = -(k/m)x(t) \tag{2.15b}$$

Numerical analysis of the damped harmonic oscillator: *Undamped oscillator*

```
# Python libraries
import numpy as np; import matplotlib.pyplot as plt
from scipy.integrate import odeint
plt.rcParams.update({"text.usetex": True, "font.family": "serif", ...
    "font.serif": ["Computer Modern Roman"] })

# Constants
m = 1.0    # Mass
k = 2.0    # Stiffness

# Differential equations for undamped harmonic oscillator
def undamped_oscillator(y, t, k, m):
    x, v = y
    dxdt = v
    dvdt = - (k/m) * x
    return [dxdt, dvdt]

# Time vector
t = np.linspace(0, 14, 1000)

# Six different initial conditions [x0, v0]
initial_conditions = [ [2, 0],  [1, 0],  [.5, 0],  [.1, 0], [-1, 0]]
colors = ['#471b00','#752d00', '#a43e00', '#d35000', '#ff6100', '#ff7f1a']

# Plotting solutions as a function of time
plt.figure(figsize=(6,3)); plt.xlim(0, 14);
for idx, init_cond in enumerate(initial_conditions):
    sol = odeint(undamped_oscillator, init_cond, t, args=(k, m))
    plt.plot(t, sol[:, 0], label=f'$x_0={init_cond[0]}$, ...
        $v_0={init_cond[1]}$', color=colors[idx])

plt.ylabel('displacement $x(t)$',fontsize=12); plt.legend(); ...
    plt.grid(True); plt.xlabel('time $t$',fontsize=12)
plt.savefig('harmonic-undamped.pdf', bbox_inches='tight')

# Phase portrait
X, Y = np.meshgrid(np.linspace(-2, 2, 20), np.linspace(-3, 3, 20))
U = Y; V = - (k/m) * X; magnitude = np.sqrt(U**2 + V**2)

plt.figure(figsize=(12,6)); plt.grid(True)
plt.xlim(-2, 2); plt.ylim(-3, 3); plt.scatter(0, 0, color='black', ...
    s=50, zorder=5)
plt.streamplot(X, Y, U, V, density=0.75, color='#0085ff', ...
    arrowsize=1.5, linewidth=magnitude)
plt.xlabel('displacement $x$',fontsize=24); plt.ylabel('velocity ...
    $v$',fontsize=24)
plt.tick_params(axis='both', which='major', labelsize=24)
plt.savefig('harmonic-undamped-phase.pdf', bbox_inches='tight')
```

Listing 2.4: Python script generating Figure 2.7. Available at `harmonic-undamped.py`

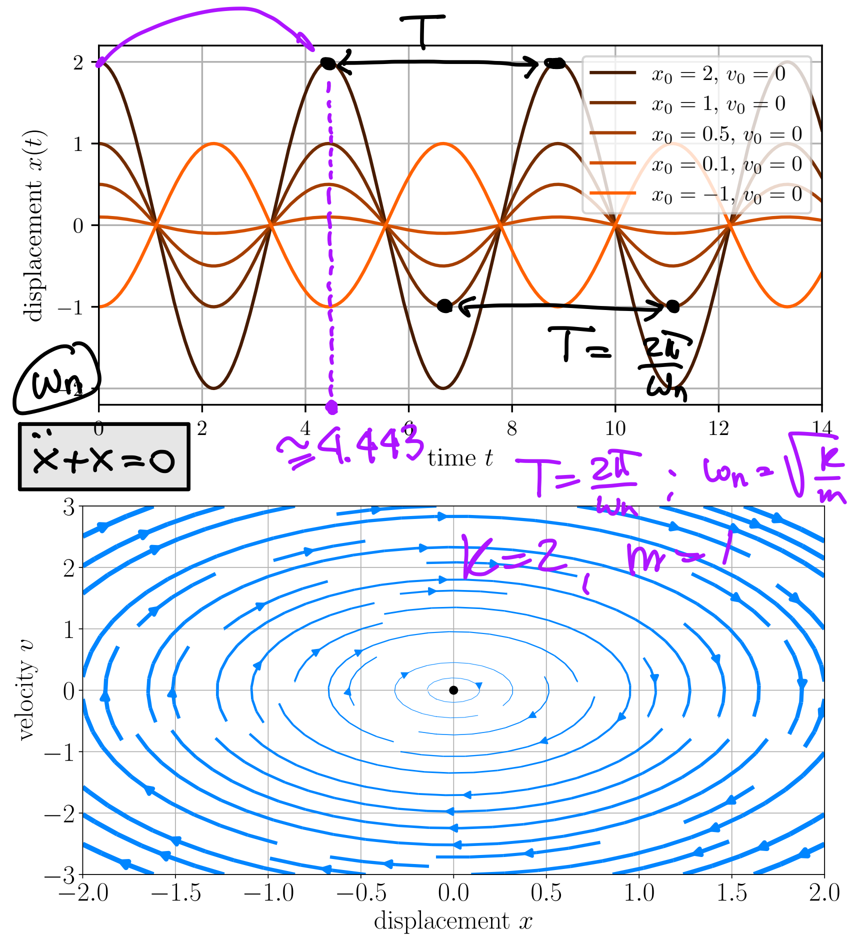
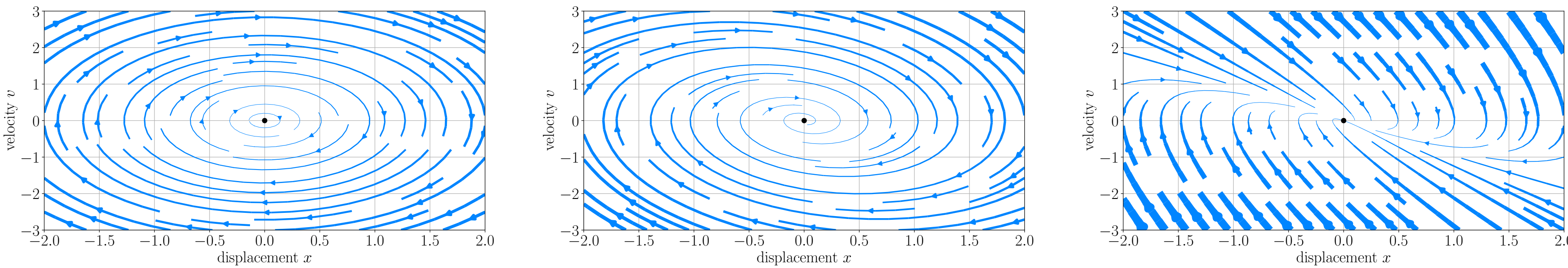


Figure 2.7: Solutions (from zero initial velocity) and phase portrait for the undamped harmonic oscillator (2.15), with $m = 1$ and $k = 2.0$.

Transition from zero to high damping values



(a) Undamped system: $b = 0$ (b) Underdamped system: small $b > 0$ (c) Overdamped system: large $b > 0$

2.1.3 Mathematical analysis: Harmonic solutions

Each solution to the undamped harmonic oscillator

$$m\ddot{x} + kx = 0 \tag{2.16}$$

is of the form

$$x(t) = a\sin(\omega_n t) + b\cos(\omega_n t) \tag{2.17}$$

where

- $\omega_n = \sqrt{\frac{k}{m}}$ is called the *natural frequency*, measured in radians per second,
- the parameters a and b in equation (2.17) are uniquely determined by the initial condition $(x(0), \dot{x}(0))$,
- the *period of oscillation* is defined as

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}}. \tag{2.18}$$

Natural frequency and period of oscillation are intrinsic to the system and independent of initial conditions,

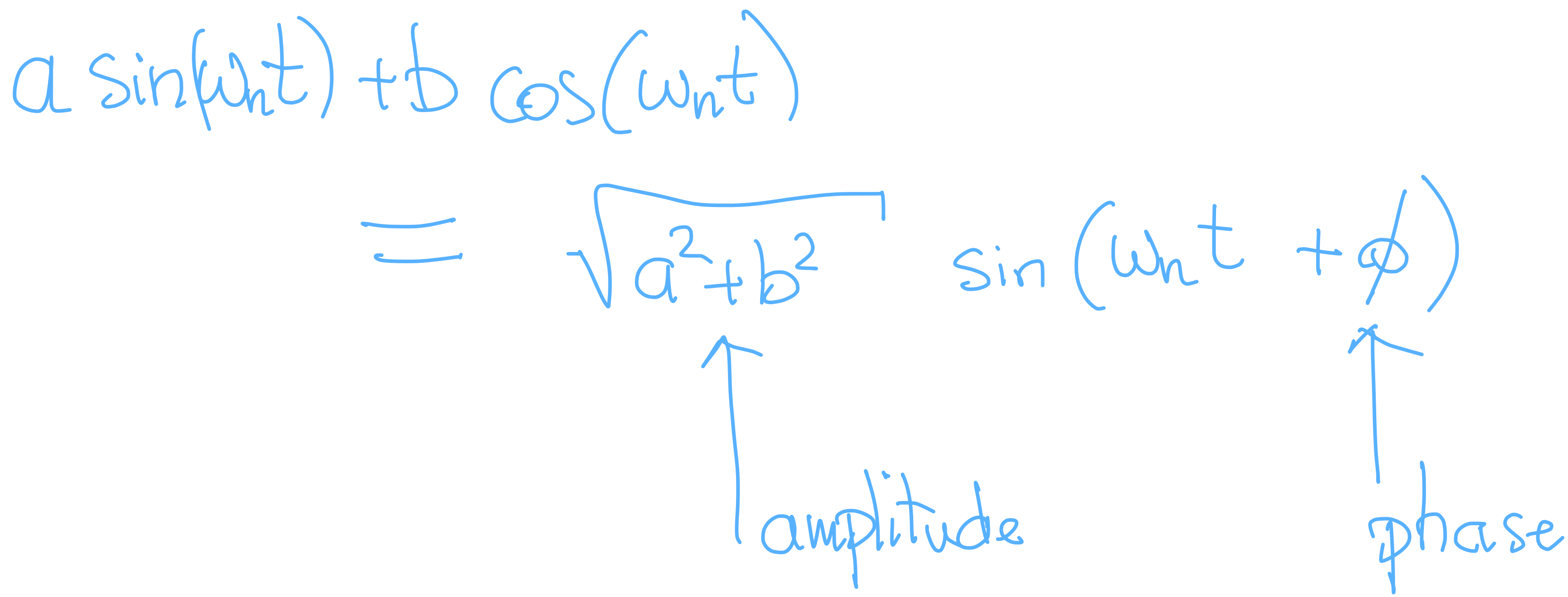
$$\omega_n = \sqrt{\frac{k}{m}}$$

- the sinusoidal function $a\sin(\omega t) + b\cos(\omega t)$ is called a *harmonic motion*. Each harmonic motion is determined by a frequency, magnitude, and phase. For[3] $\phi = \text{arctan}_2(b, a)$, one can show that

$$a\sin(\omega t) + b\cos(\omega t) = \sqrt{a^2 + b^2}\sin(\omega t + \phi) \tag{2.19}$$

For example, for the undamped harmonic oscillator in the numerical example in Figure 2.7, the parameters are $m = 1$ and $k = 2$. Therefore, the natural frequency is $\omega_\mathsf{n} = \sqrt{k/m} = \sqrt{2} \approx 1.414$ rad/s. The period of oscillation is $T = 2\pi/\omega_\mathsf{n} = 2\pi/\sqrt{2} = \sqrt{2}\pi \approx 4.443$ s. In Figure 2.7, one can observe that the solution completes one full cycle in approximately 4.4 seconds, which is consistent with the calculated period.

A later chapter will cover harmonic, overdamped, and underdamped oscillators in detail.



[3] The function $\text{arctan}_2(y, x)$ computes the angle of the point (x, y) in the Cartesian coordinate system, measured counterclockwise from the positive x-axis. The function returns the angle in the range $(-\pi, \pi]$, taking into account the signs of both x and y to correctly determine the quadrant. When both x and y are positive, $\text{arctan}_2(y, x) = \text{arctan}(y/x)$.

2.2 Mechanical systems: Two degrees of freedom and the suspension example

In the previous section we analyzed single–degree-of-freedom models, where the system's motion could be described by a single generalized coordinate. Many practical systems, however, require multiple coordinates to capture their dynamics. A two–degree-of-freedom (2-dof) model can represent the interaction between two moving bodies, including coupling effects such as shared forces or displacements.

As an example, we examine a vehicle suspension system. A suspension consists of springs, shock absorbers, and linkages connecting the vehicle body to its wheels. Its primary functions are to absorb and dissipate energy from road irregularities, and to maintain consistent contact between the tires and the road surface. A well-engineered suspension improves ride comfort, enhances handling, and contributes to overall vehicle safety.

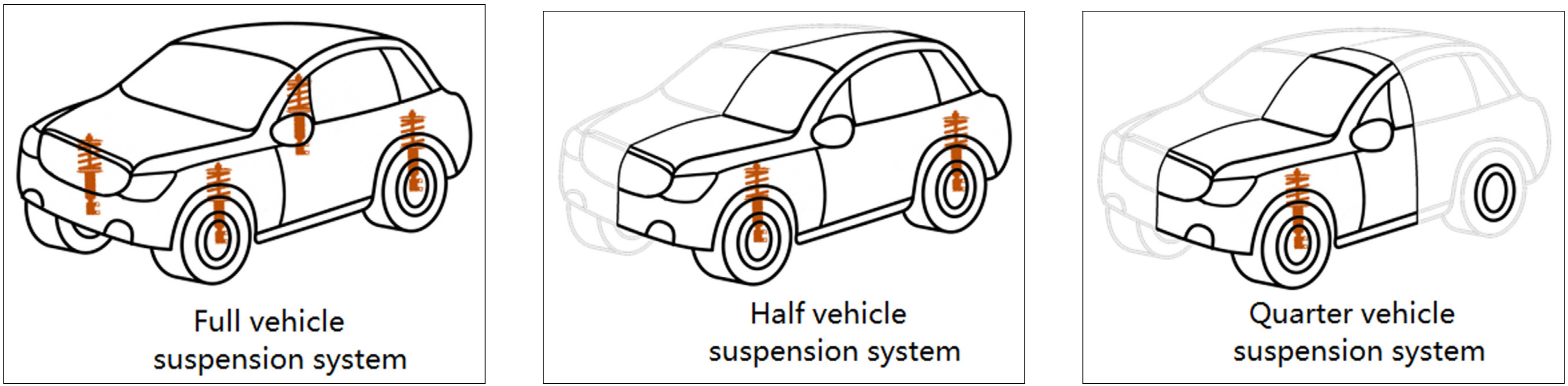


Figure 2.8: Illustration of full, half and quarter vehicle suspension system. We focus on the quarter suspension system. Image sourced from (Zhang et al., 2020) without permission.

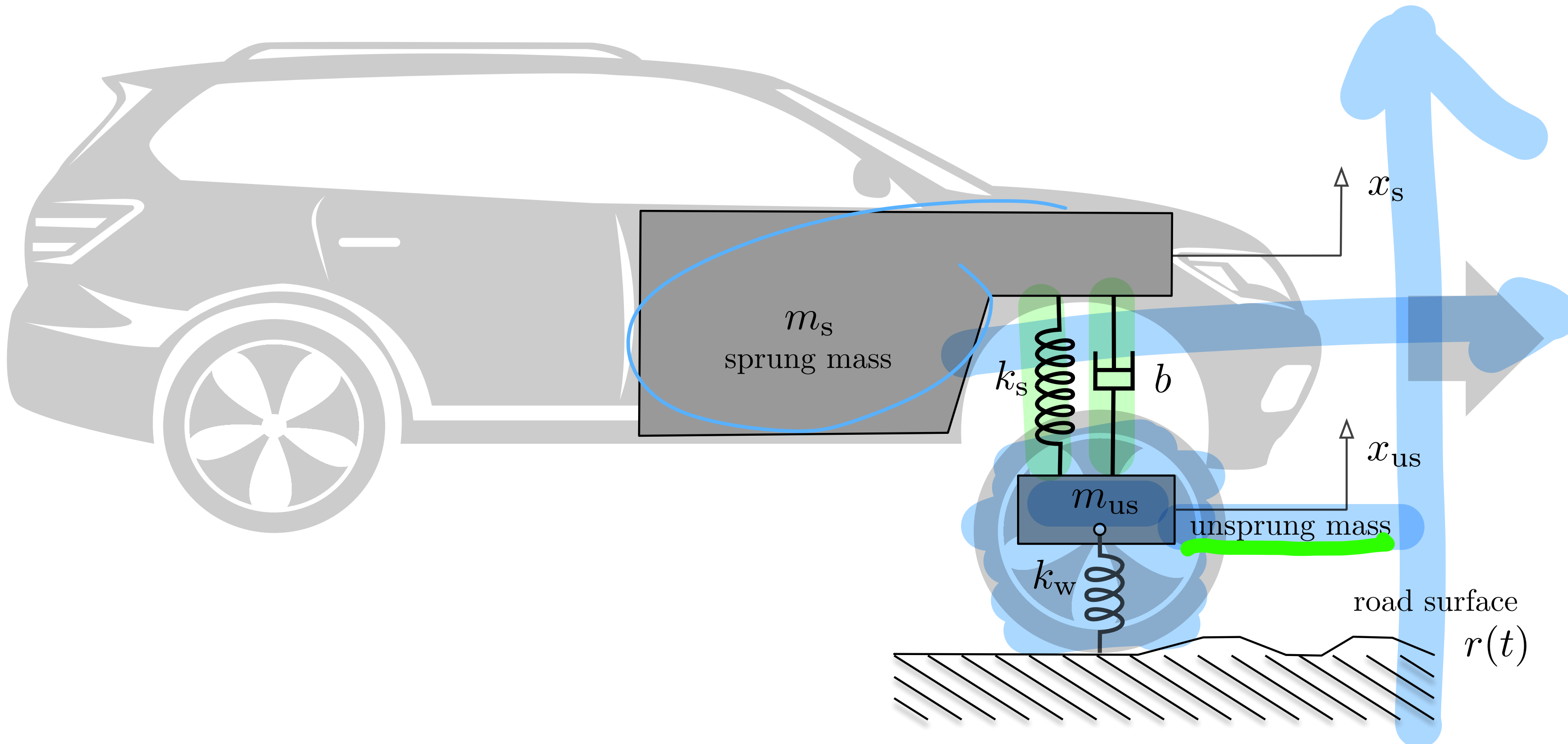


Figure 2.9: The suspension in an automobile.

The body of the vehicle, passengers, and cargo is called the *sprung mass* and is denoted by m_s; the vertical position of the sprung mass is denoted by x_s.

The wheels, axles, and other parts that are directly connected to the road surface are called the *unsprung mass*, denoted by m_{us}; the vertical position is x_{us}.

The interaction between the wheel and the road is usually described as a spring with stiffness k_w.

The suspension includes a shock absorber, which is a damper with coefficient b, and a spring with stiffness k_s.

We assume the automobile is moving forward at a constant speed over a possibly-uneven terrain described by the road height function $r(t)$.

As in Figure 2.9, let x_s and x_{us} denote the vertical displacements of the two masses from their equilibrium positions. (The equilibrium position accounts for gravity and a counterbalancing compression of the two springs; therefore we do not consider gravity). Let $r(t)$ denote the height of the road surface as a function of time. It is important to understand that the sprung mass moves relative to the unsprung mass. From Newton's law, we know that the equations will be of the form

$$m_s \ddot{x}_s = \text{resultant force on sprung mass, and}$$
$$m_{us} \ddot{x}_{us} = \text{resultant force on unsprung mass.}$$

There are three forces acting on the sprung and unsprung masses:

- $k_w(x_{us} - r(t))$ = force generated by the wheel/road interaction with stiffness k_w,
- $k_s(x_{us} - x_s)$ = force generated by the spring in the shock absorber with stiffness k_s, and
- $b(\dot{x}_{us} - \dot{x}_s)$ = force generated by the damper in the shock absorber with damping coefficient d.

b

To determine the sign of the forces as they act on the two bodies, one can draw a free body diagram, see Exercise E2.4. We describe a short-cut in Remark 2.1.

$x(t) \rightarrow \frac{d}{dt} x(t) = -x + x^2$

In summary, the *suspension dynamics* are:

$$m_s \ddot{x}_s + b(\dot{x}_s - \dot{x}_{us}) + k_s(x_s - x_{us}) = 0, \tag{2.20a}$$
$$m_{us} \ddot{x}_{us} + b(\dot{x}_{us} - \dot{x}_s) + k_s(x_{us} - x_s) + k_w x_{us} = k_w r(t), \tag{2.20b}$$

Note: Just like the force in the simulation of the car velocity system, the road surface $r(t)$ is now an *input* into the dynamical system. An input signal is an external signal that affects the system's behavior but is not influenced by the system's state.

$m\ddot{x} + b\dot{x} + kx = 0$

4th order, 4 variables: x_s, $\dot{x}_s$, x_{us}, $\dot{x}_{us}$

params: m_s, m_{us}, b, k_s, k_w = 5 params

input: $r(t)$

Remark 2.1 (How to get the correct signs). *Recall the damped harmonic oscillator in* (2.12)*:* $m\ddot{x} + b\dot{x} + kx = 0$*. Similarly, to ensure that the signs are correct in the first equation* (2.20a)*, note that the acceleration, velocity, and position terms in* x_s *and its derivatives need to be multiplied by positive coefficients. The same is true in equation* (2.20b) *for the coefficients of* x_{us} *and its derivatives.* •

Remark 2.2 (Absolute versus relative effects). *Consider a body with position* x_1 *and velocity* v_1*. Note the difference between an "absolute force" like*

$$-kx_1 \quad \text{or} \quad -bv_1 \tag{2.21}$$

and a "relative force" (due to the interconnection with a second body with position x_2 *and velocity* v_2*):*

$$-k(x_1 - x_2) \quad \text{or} \quad -b(v_1 - v_2) \tag{2.22}$$

For clarity, all springs and dampers generate only relative forces, which are forces proportional to relative position and relative velocity. The reason the forms in (2.21) *appear is because the second body is assumed to be at zero position and zero velocity* ($x_2 = v_2 = 0$)*.* •

Remark 2.3. *A choice of realistic automobile parameters taken from (Franklin et al., 2015, Section 2.2) is:*

sprung mass	m_s	*1500 kg*
unsprung mass	m_{us}	*80 kg*
wheel stiffness	k_w	*1,000,000 N/m*
suspension stiffness	k_s	*130,000 N/m*
suspension damping	b	*9800 N·s/m*

•

Remark 2.4. *It is usually preferable to have low unsprung weight (and a high sprung to unsprung weight ratio) in order to allow the suspension to respond more effectively to road imperfections, improving ride quality and handling.* •

Numerical analysis of the suspension system

```python
import numpy as np; from scipy.integrate import odeint;
import matplotlib.pyplot as plt
plt.rcParams.update({"text.usetex": True, "font.family": "serif", ...
    "font.serif": ["Computer Modern Roman"] })

# Define the system of ODEs with state = [xs, xs_dot, xu, xu_dot]
def system_of_eqns(state, t, mu, ms, ks, b, kw, road):
    xs, xs_dot, xu, xu_dot = state
    xs_ddot = (-ks*(xs-xu) - b*(xs_dot-xu_dot)) / ms
    xu_ddot = (ks*(xs-xu) + b*(xs_dot-xu_dot) - kw*xu + kw*road(t)) / mu
    return [xs_dot, xs_ddot, xu_dot, xu_ddot]

# Parameters for a "quarter automobile" and time array
ms = 375       # Sprung mass (for a quarter of a car)
mu = 20        # Unsprung mass
kw = 1000000   # Wheel stiffness
ks = 130000    # Suspension stiffness
b  = 9800      # Suspension damping coefficient

# Initial conditions: [xs, xs_dot, xu, xu_dot]. positions in meters.
t = np.linspace(0, 1.4, 300); initial_conditions = [-0.1, 0.0, 0.00, 0.0]
sol = odeint(system_of_eqns, initial_conditions, t, args=(mu, ms, ks, b, ...
    kw, lambda t: 0))

# Plotting the unforced solution
plt.figure(figsize=(10, 5)); plt.plot(t, sol[:, 0], label='$x_s$ sprung')
plt.plot(t, sol[:, 2], label='$x_{us}$ unsprung'); plt.grid(True)
plt.xlabel('time $t$',fontsize=16); plt.ylabel('position ...
    (meters)',fontsize=16); plt.xlim(0, 1.4);
plt.title('Unforced suspension system',fontsize=16); plt.legend();
plt.savefig("suspension-unforced.pdf", bbox_inches='tight')

# Road surface: zero for first .5 seconds, then a sinusoidal bump
bumpheight = .116   # typical bump height = 4 inches = .116 meters
duration = .46 / 4.4  # typical bump width = 18 inches = 0.46 meters. 10 ...
    miles/hour = 4.4 meter/sec
def bump_road(t):
    if 0.5 ≤ t < 0.5 + duration:
        return bumpheight * np.sin((t - .5) * np.pi / duration)
    else:
        return 0

# Solving for forced case from equilibrium initial condition
initial_conditions_forced = [-0.1, 0.0, 0.0, 0.0]
sol_forced = odeint(system_of_eqns, initial_conditions_forced, t, ...
    args=(mu, ms, ks, b, kw, bump_road))
road_data = np.array([bump_road(time) for time in t])

# Plotting the forced solution
plt.figure(figsize=(10, 5)); plt.plot(t, sol_forced[:, 0], label='$x_s$ ...
    sprung mass')
plt.plot(t, sol_forced[:, 2], label='$x_{us}$ unsprung mass)'); plt.grid(True)
plt.plot(t, road_data, label='$r$ road surface', linestyle='--'); ...
    plt.xlim(0, 1.4);
plt.xlabel('time $t$',fontsize=16); plt.ylabel('position ...
    (meters)',fontsize=16); plt.title('Forced suspension system',fontsize=16)
plt.legend(); plt.savefig("suspension-forced.pdf", bbox_inches='tight')
```

Listing 2.5: Python script generating Figure 2.10. Available at suspension.py

free response (there is no input, r(t)=0)

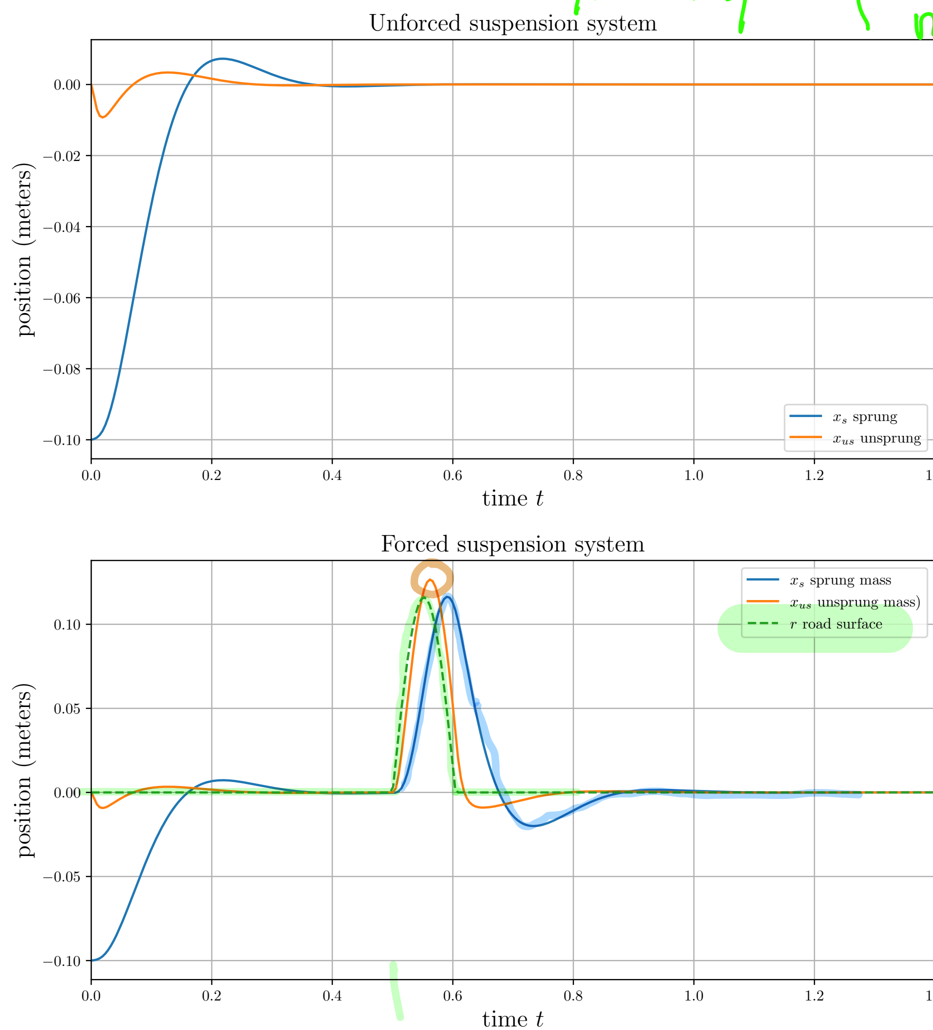


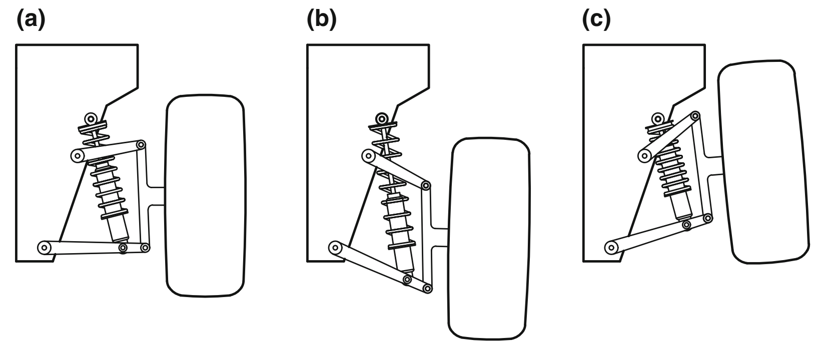
Figure 2.10: Solutions of the suspension system (2.20): unforced solution (road height = 0 for all time) and forced solution due to a speed bump at time 0.5.

Comments on vehicle suspension systems

Vehicle suspension systems are designed to provide ride comfort, road handling, and stability by absorbing shocks from uneven surfaces and maintaining tire contact with the road. Among the various types, the MacPherson suspension and the double wishbone suspension are widely used in passenger vehicles. These systems rely on combinations of struts, control arms, and linkages to manage vertical wheel movement while supporting steering and load-bearing functions. We illustrate these concepts in Figure 2.11.

(a) *MacPherson suspension system* for a front wheel. It consists of: (i) a *strut* (coil spring and damper), with its upper part connected to the chassis through a pivoting bearing mount that allows steering rotation, and its lower part attached to the wheel hub via a ball joint; and (ii) a *lower control arm* (or *wishbone arm*), typically A- or V-shaped, connected to the chassis with revolute joints and to the wheel with a ball joint, providing lateral support and controlling vertical motion.

(b) An automotive *strut* combining a shock absorber and coil spring, sourced from `https://amazon.com` (image used without permission).



(c) A *double wishbone suspension system*, based on a four-bar linkage. From (Fernandes et al., 2019) (image used without permission). Shown in (a) equilibrium, (b) extended, and (c) compressed positions.

Figure 2.11: Illustrations of common vehicle suspension systems: MacPherson suspension, individual strut assembly, and double wishbone suspension.

2.3 Rotational mechanical systems

Newton's law also applies to rotational mechanical systems such as pendula, pulleys, and any mechanical system with an axis of rotation. The law is simply

$$\tau = I\ddot{\theta} \tag{2.23}$$

where

- τ is the resultant torque (the algebraic sum of all torques) applied to the body, measured in N·m,
- I is the moment of inertia of the body, measured in kg·m^2, and
- $\ddot{\theta}$ is the angular acceleration of the body, which is the second time derivative of the angular position $\theta(t)$, measured in rad/s^2.

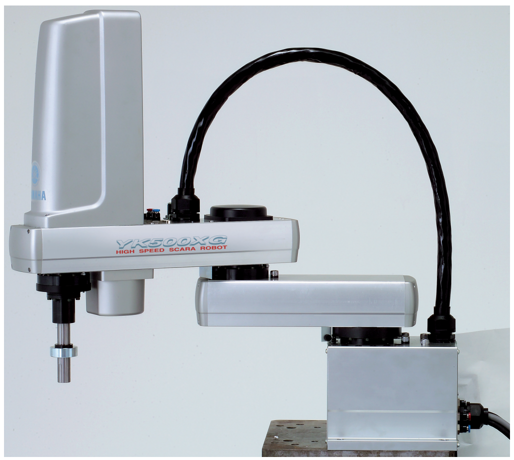
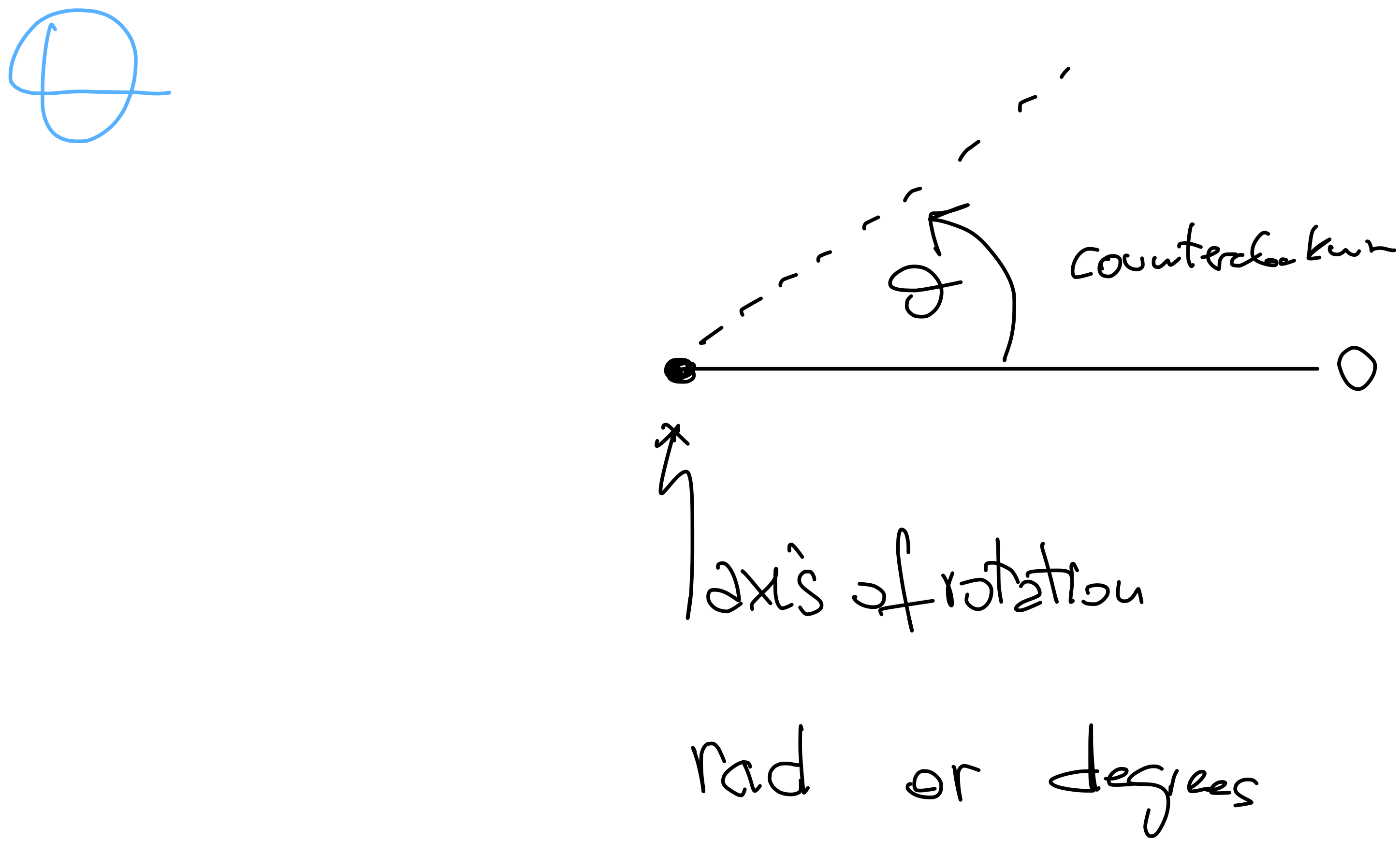


Figure 2.12: The Yamaha© YK500XG is a high-speed SCARA robot with two revolute joints and a vertical prismatic joint. Image courtesy of Yamaha Motor Co., Ltd, http://global.yamaha-motor.com/business/robot.

Figure 2.13: Illustration of NASA's Transiting Exoplanet Survey Satellite: TESS. Credit: NASA's Goddard Space Flight Center. Public domain image.

In class assignment

How many distinct pieces of information are required to unambiguously specify the meaning of an angle θ?
(e.g., one distinct piece of information is the direction of angle measurement: clockwise vs counterclockwise)



How to fully describe an angle

In order to unambiguously specify the meaning of an angle θ, one needs to specify:

(i) the reference angle: where the zero angle is,

(ii) the direction: counterclockwise (in this text, all angles are measured counterclockwise),

(iii) the unit: radians (not degrees),

(iv) the range: $(-\pi, \pi]$, and

(v) the axis of rotation (in three-dimensional diagrams).

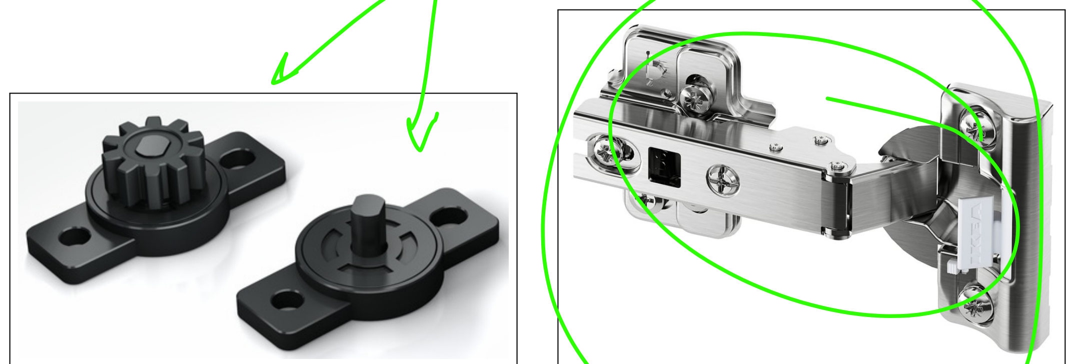
Rotary dampers and torsion springs

Just as we saw for translational motion, there exist dampers and springs for rotational motion, as shown in Figure 2.14. Therefore, even for rotational mechanics, it is possible and common to encounter damped harmonic oscillators:

$$I\ddot{\theta}(t) + b\dot{\theta}(t) + k\theta(t) = 0 \tag{2.24}$$

As for the translational system depicted in Figure 2.4, equation (2.24) is based on the assumption that the rotatory damper and torsional springs are connected at one end to a fixed body and at the other end to the rotating body at angle θ.

(a) Torsional springs store and release energy. Hair clips are a simple application of torsional springs. In automotive applications, torsional springs help balance the weight of the trunk or tailgate, making it easier to open and close.



(b) Exploiting fluid viscosity, rotary dampers slow down the motion of rotating parts. In furniture applications, cabinets, wardrobes, and other furniture with doors often utilize damped hinges for a smoother user experience. In automotive applications, rotary dampers are used for glove compartments, cup holders, and grad handles.

Figure 2.14: Torsional springs and Rotary dampers andplay the same role for rotational motion as springs and linear dampers.

2.3.1 The pendulum

As illustrated in Figure 2.15, consider a pendulum of length ℓ with a point mass m at its end. The pendulum is subject to gravity, with gravitational acceleration g, and linear friction from the air or the pivot point, described by a damping coefficient b. We note:

- the moment of inertia is $m\ell^2$, and
- the component of the gravitational force tangent to the circular motion of the pendulum is $mg\sin(\theta)$, which produces a torque on the pendulum of $m\ell g\sin(\theta)$.

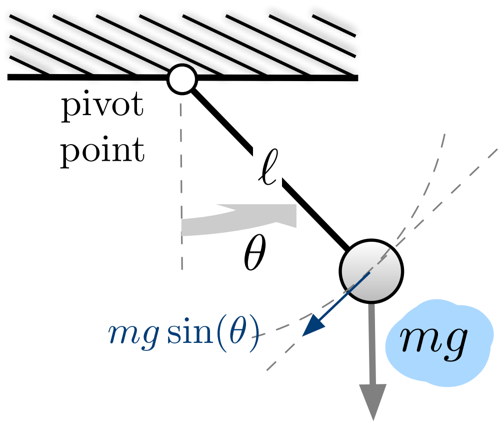


Figure 2.15: A pendulum subject to gravity, connected to a pivot point.
The system variable is the angle θ, measured counterclockwise from the vertical resting position.
The moment of inertia of the pendulum about the pivot point is $I = m\ell^2$.
The pendulum is subject to a gravitational force of magnitude mg, which translates into a restoring torque of magnitude $m\ell g\sin(\theta)$.

Equations of motion

The equations of motion for the pendulum are derived from Newton's law for rotational motion, as shown in equation (2.23).

> In summary, the *pendulum dynamics* are
>
> $$m\ell^2\ddot{\theta} + b\dot{\theta} + m\ell g \sin(\theta) = 0. \tag{2.25}$$

If there is no friction, the equation of motion simplifies to:

$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0. \tag{2.26}$$

We can write the equation in first-order form

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\frac{b}{m\ell^2}\omega - \frac{g}{\ell}\sin(\theta) \end{aligned} \tag{2.27}$$

or in vector form

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{b}{m\ell^2}\omega - \frac{g}{\ell}\sin(\theta) \end{bmatrix} \tag{2.28}$$

NONLINEAR
2nd order
2 variables
2 parameter
m, ℓ, b, g

Equilibrium points: pendulum down and up

To find the equilibrium points, we set the right-hand side of the vector form to zero, which yields

$$\omega = 0 \quad \text{and} \quad \sin\theta = 0 \iff \theta = n\pi \tag{2.29}$$

for any integer n. Restricting our attention to the range $-\pi < \theta \leq \pi$ yields two equilibria:

- the equilibrium point $\begin{bmatrix} \theta^*_{\text{down}} \\ \omega^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, corresponding to the pendulum in its *down position*,

- the equilibrium point $\begin{bmatrix} \theta^*_{\text{up}} \\ \omega^* \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$, corresponding to the pendulum in its *up position*.

Intuitively, the "pendulum down" equilibrium is stable, while the "pendulum up" equilibrium is unstable.

Numerical simulation of the pendulum without friction

```python
# Python libraries
import numpy as np; import matplotlib.pyplot as plt
from scipy.integrate import odeint
plt.rcParams.update({"text.usetex": True, "font.family": "serif", "font.serif": ...
    ["Computer Modern Roman"] })

# Pendulum dynamics
def pendulum(Y, t, g, ell):
    theta, omega = Y; dtheta = omega; domega = -g/ell * np.sin(theta)
    return [dtheta, domega]

# Parameters and time array
g   = 9.81     # gravity
ell = 1.0      # length of the pendulum
m   = 0.5      # mass (not directly used in the equations, but provided for ...
    completeness)
t = np.linspace(0, 10, 1000)

# Initial conditions: [theta0, omega0] and plot the solution
initial_conditions = [[.1*np.pi, 0], [.4*np.pi, 0], [.7*np.pi, 0], [.99*np.pi, 0]]
colors = ['#752d00', '#a43e00', '#d35000', '#ff6100']
plt.figure(figsize=(6, 3))

for idx, ic in enumerate(initial_conditions):
    Y = odeint(pendulum, ic, t, args=(g, ell))
    theta, omega = Y.T
    plt.plot(t, theta, label=f'theta0={ic[0]:.2f}, omega0={ic[1]:.2f}', ...
        color=colors[idx])

# Set y-ticks to be fractions of pi
plt.yticks( [-np.pi, -np.pi/2, 0, np.pi/2, np.pi],
            ['$-\pi$', '$-\pi/2$', '0', '$\pi/2$', '$\pi$'] )
plt.title('Undamped pendulum dynamics ($\\theta(t)$ vs $t$) and phase ...
    portrait',fontsize=12)
plt.xlabel('time $t$',fontsize=12); plt.ylabel('$\\theta(t)$',fontsize=12); ...
    plt.xlim(0, 10);
plt.grid(True); plt.savefig("pendulum.pdf", bbox_inches='tight')

# Phase portrait
theta_range, omega_range = np.meshgrid(np.linspace(-2*np.pi, 2*np.pi, 20), ...
    np.linspace(-7, 7, 20))
dtheta, domega = pendulum([theta_range, omega_range], 0, g, ell)
magnitude = np.sqrt(dtheta**2 + domega**2)/2; plt.figure(figsize=(12,6));
plt.streamplot(theta_range, omega_range, dtheta, domega, density=.5, ...
    linewidth=magnitude, color='#0085ff' , broken_streamlines=False, arrowsize=3)

# Plotting the trajectories in the phase portrait
for idx, ic in enumerate(initial_conditions):
    Y = odeint(pendulum, ic, t, args=(g, ell))
    theta, omega = Y.T
    plt.plot(theta, omega, color=colors[idx], label=f'theta0={ic[0]:.2f}, ...
        omega0={ic[1]:.2f}')

# Plotting the scatter points at theta = -2pi, -pi, 0, pi, 2pi
for scatter_theta in [-2*np.pi, -np.pi, 0, np.pi, 2*np.pi]:
    plt.scatter(scatter_theta, 0, color='black', s=50, zorder=5)
plt.xticks([-2*np.pi, -3*np.pi/2, -np.pi, -np.pi/2, 0, np.pi/2, np.pi, 3*np.pi/2, ...
    2*np.pi],
    ['$-2\pi$', '$-3\pi/2$', '$-\pi$', '$-\pi/2$', '0', '$\pi/2$', '$\pi$', ...
        '$3\pi/2$', '$2\pi$'])
plt.xlabel('$\\theta$',fontsize=24); plt.ylabel('$\omega$',fontsize=24); ...
    plt.xlim([-2*np.pi, 2*np.pi]); plt.ylim([-7, 7])
plt.tick_params(axis='both', which='major', labelsize=24)
plt.grid(True); plt.savefig("pendulum-phase.pdf", bbox_inches='tight')
```

Listing 2.6: Python script generating Figure 2.16. Available at pendulum.py

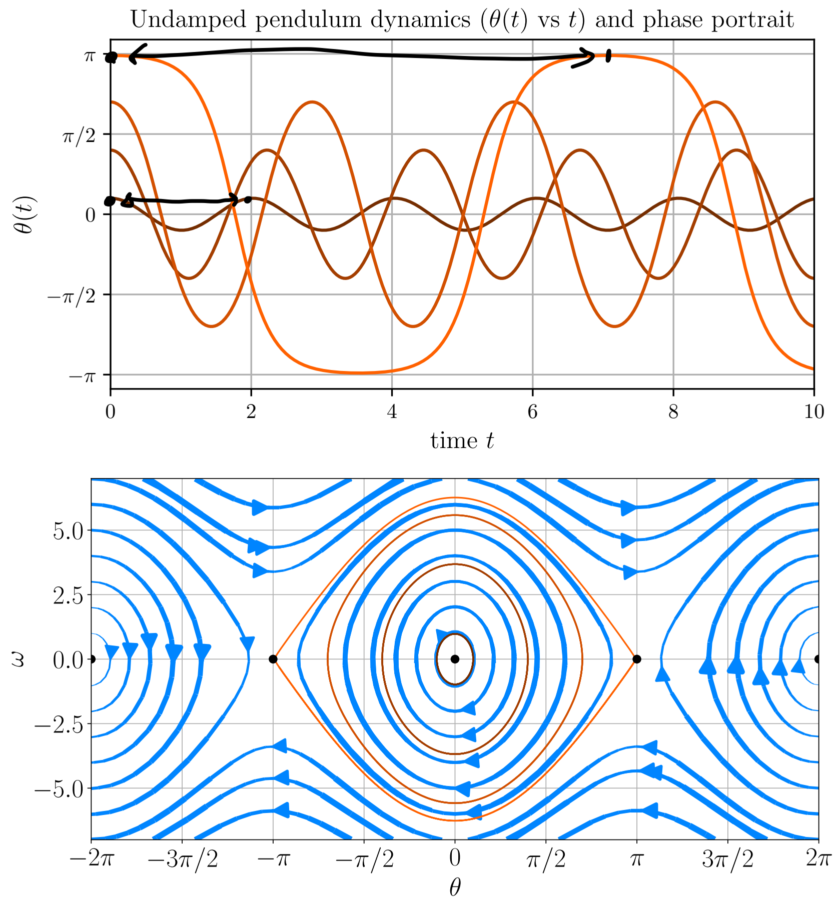


Figure 2.16: Solutions and phase portrait for the undamped pendulum dynamics (2.26). The top plot shows solutions for four initial conditions selected with angles in the range $[-\pi/2, \pi/2]$ and with low initial angular velocities.
The bottom plot shows the corresponding phase portrait with the four trajectories superimposed.

Numerical simulation of the pendulum with friction

```
# Python libraries
import numpy as np; import matplotlib.pyplot as plt
from scipy.integrate import odeint
plt.rcParams.update({"text.usetex": True, "font.family": "serif", "font.serif": ...
    ["Computer Modern Roman"] })

# Pendulum dynamics with damping
def pendulum(Y, t, g, ell, b):
    theta, omega = Y; dtheta = omega; domega = -g/ell * np.sin(theta) - ...
        b/(m*ell**2) * omega
    return [dtheta, domega]

# Parameters and time array
g   = 9.81    # gravity
ell = 1.0     # length of the pendulum
m   = 0.5     # mass
b   = 0.2     # damping coefficient (adjust this value as desired)
t = np.linspace(0, 10, 1000)

# Initial conditions: [theta0, omega0] and plot the solution
initial_conditions = [[.1*np.pi, 0], [.4*np.pi, 0], [.7*np.pi, 0], [.99*np.pi, 0]]
colors = ['#752d00', '#a43e00', '#d35000', '#ff6100']
plt.figure(figsize=(12, 6))
for idx, ic in enumerate(initial_conditions):
    Y = odeint(pendulum, ic, t, args=(g, ell, b)); theta, omega = Y.T
    plt.plot(t, theta, label=f'theta0={ic[0]:.2f}, omega0={ic[1]:.2f}', ...
        color=colors[idx])

plt.yticks([-np.pi, -np.pi/2, 0, np.pi/2, np.pi], ['$-\pi$', '$-\pi/2$', '0', ...
    '$\pi/2$', '$\pi$'])
plt.title('Pendulum dynamics ($\\theta(t)$ vs $t$) with damping',fontsize=24)
plt.xlabel('time $t$',fontsize=24); plt.ylabel('$\\theta(t)$',fontsize=24); ...
    plt.xlim(0, 10); plt.grid(True)
plt.tick_params(axis='both', which='major', labelsize=24)
plt.savefig("pendulum-damped.pdf", bbox_inches='tight')

# Phase portrait
theta_range, omega_range = np.meshgrid(np.linspace(-2*np.pi, 2*np.pi, 20), ...
    np.linspace(-7, 7, 20))
dtheta, domega = pendulum([theta_range, omega_range], 0, g, ell, b)
magnitude = np.sqrt(dtheta**2 + domega**2)/2; plt.figure(figsize=(12,6));
plt.streamplot(theta_range, omega_range, dtheta, domega, density=.5,
               linewidth=magnitude, color='#0085ff', broken_streamlines=False, ...
                   arrowsize=3)
for idx, ic in enumerate(initial_conditions):
    Y = odeint(pendulum, ic, t, args=(g, ell, b)); theta, omega = Y.T
    plt.plot(theta, omega, color=colors[idx], label=f'theta0={ic[0]:.2f}, ...
        omega0={ic[1]:.2f}')

# Plotting the scatter points at theta = -2pi, -pi, 0, pi, 2pi
for scatter_theta in [-2*np.pi, -np.pi, 0, np.pi, 2*np.pi]:
    plt.scatter(scatter_theta, 0, color='black', s=50, zorder=5)

plt.xticks([-2*np.pi, -3*np.pi/2, -np.pi, -np.pi/2, 0, np.pi/2, np.pi, 3*np.pi/2, ...
    2*np.pi],
           ['$-2\pi$', '$-3\pi/2$', '$-\pi$', '$-\pi/2$', '0', '$\pi/2$', ...
               '$\pi$', '$3\pi/2$', '$2\pi$'])
plt.xlabel('$\\theta$',fontsize=24); plt.ylabel('$\\omega$',fontsize=24); ...
    plt.xlim([-2*np.pi, 2*np.pi])
plt.tick_params(axis='both', which='major', labelsize=24)
plt.ylim([-7, 7]); plt.grid(True); plt.savefig("pendulum-damped-phase.pdf", ...
    bbox_inches='tight')
```

Listing 2.7: Python script generating Figure 2.17. Available at pendulum-damped.py

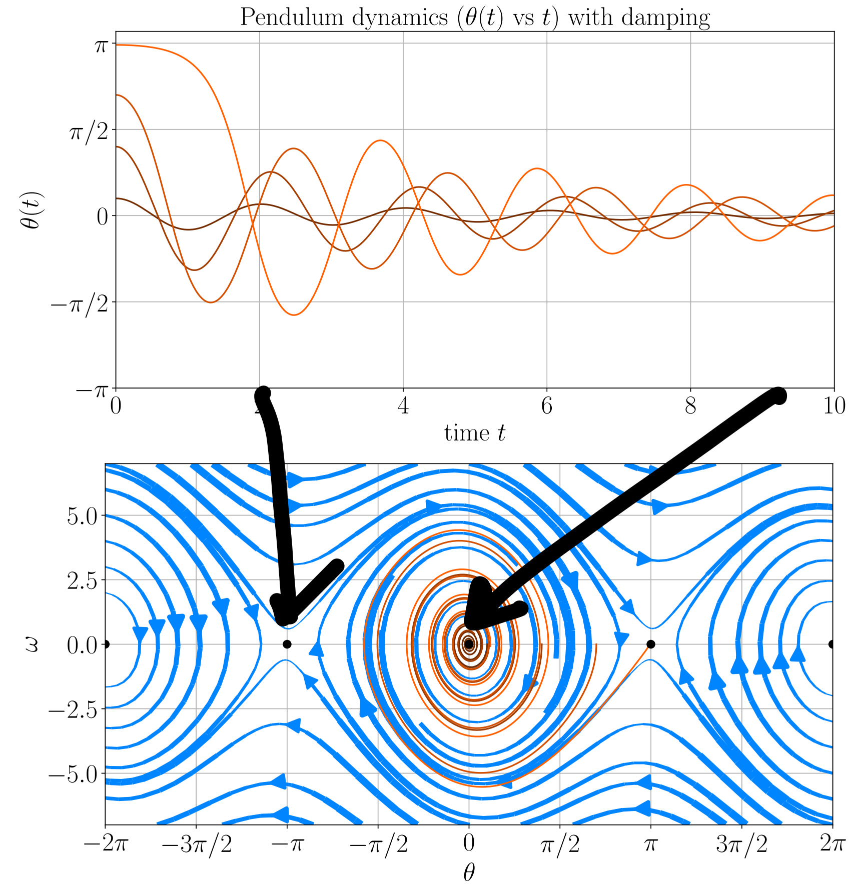
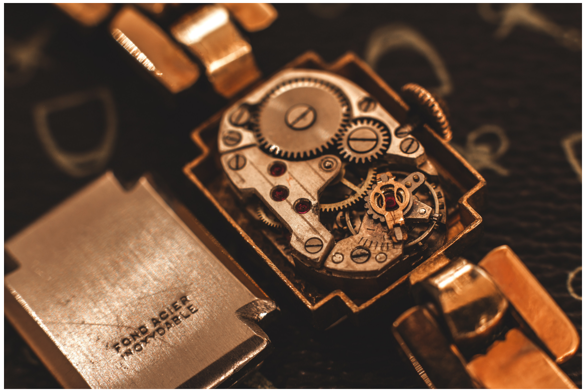


Figure 2.17: Solutions and phase portrait for the damped pendulum dynamics (2.25). The top plot shows solutions for four initial conditions selected with angles in the range $[-\pi/2, \pi/2]$ and with low initial angular velocities.
The bottom plot shows the corresponding phase portrait with the four trajectories superimposed.

2.3.2 Mechanical gears

- *Gears* are toothed mechanical elements used to transmit motion and power between rotating shafts. Gears work in pairs, with their teeth meshing to prevent slippage. Each gear is rigidly attached to a shaft.
- The *input gear*, also called the *driver*, transmits motion to the *output gear*, also called the *driven gear*. When gears mesh, they rotate in opposite directions.
- When interconnected gears have different sizes, they create a *mechanical advantage*, altering the output torque and rotational speed.
- Gears are widely used in devices such as mechanical clocks, windmills, bicycles, and automobile transmissions.



(a) Vintage internal clockwork (spring and toothed gearwheels inside a mechanical clock), sourced from https://unsplash.com.
Mechanical clocks rely on gear trains to transfer energy from a wound spring or suspended weight to the hands of the clock, ensuring precise movement. The gear design allows for accurate timekeeping by compensating for variations in power delivery, maintaining the clock's consistency over time.

(b) Bicycle drivetrain
The crankset and rear wheel of a bicycle are connected by a chain that engages with sprockets, commonly known as "chainrings" at the front and the "cassette" at the rear. The gear ratio determines how many times the rear wheel rotates for each full revolution of the crank.
On a single-speed bicycle, the gear ratio is fixed.
On a multi-speed bicycle, shifting the chain between sprockets alters the gear ratio, adjusting the bicycle's resistance. Depending on the terrain, the biker selects an optimal gear for slowly climbing hills or quickly riding on flat surfaces.

Figure 2.18: Two examples of gear systems. (a) Internal clockwork: precise timekeeping through mechanical gear trains. (b) Bicycle drivetrain: variable gear ratios enabling adaptation to variable terrain inclination.

Sprockets and chains function similarly to gears in transmitting rotational motion and torque. Gear configurations can also change the direction of rotation (e.g., via *bevel gears*) or convert between rotational and linear motion (e.g., via *rack and pinion systems*). The following videos provide more details:

- Basic gear types: This brief review illustrates the main types of gears and their characteristics: spur gears are simple, efficient, and widely used for parallel shaft applications; helical gears operate more quietly and smoothly than spur gears due to their angled teeth; double helical gears are even quieter and more stable, minimizing axial thrust; worm gears provide high gear reduction and can be self-locking; screw gears transmit motion between non-parallel, non-intersecting shafts; rack-and-pinion gears convert rotational motion to linear motion or vice versa; straight bevel gears transfer motion between intersecting shafts at various angles; helical bevel gears combine the advantages of bevel gearing with smoother, quieter operation; internal and external gears engage for compact power transmission and are often used in planetary gear systems.
- review of advanced gear types (short);
- explanations of gear trains and composite gear trains (8m 47s);
- the automobile differential is a gear system designed to allow two wheels to rotate at different speed: differential steering (3m 45s), and (short 49s); and
- a GOOGOL:1 reduction with Lego gears (recall a googol is 10^{100}) (9m 58s).

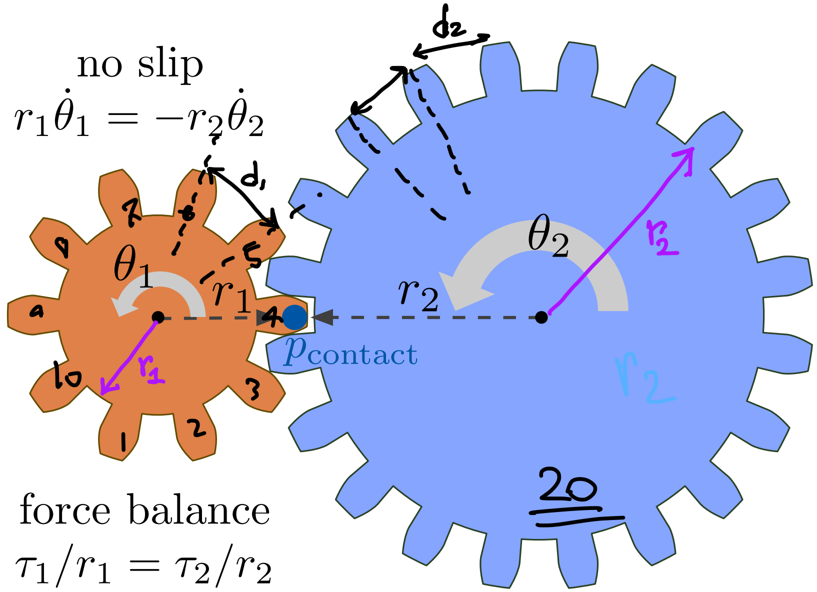


Figure 2.19: Two gears connecting two parallel shafts (not drawn).

The angular displacements θ_1 and θ_2 are measured counterclockwise. The two shafts rotate in opposite directions, so that $\dot\theta_1 > 0$ if and only if $\dot\theta_2 < 0$.

At the contact point p_{contact}:
(i) the no-slip condition is: $\text{velocity}_{\text{contact}} = r_1\dot\theta_1 = -r_2\dot\theta_2$.
(ii) the force-balance condition is: $\text{force}_{\text{contact}} = \frac{\tau_1}{r_1} = \frac{\tau_2}{r_2}$.

Note: As we discuss below, the number of teeth on each gear is proportional to its radius ($n_1 = 10$, $n_2 = 20$ implies $r_2 = 2r_1$).

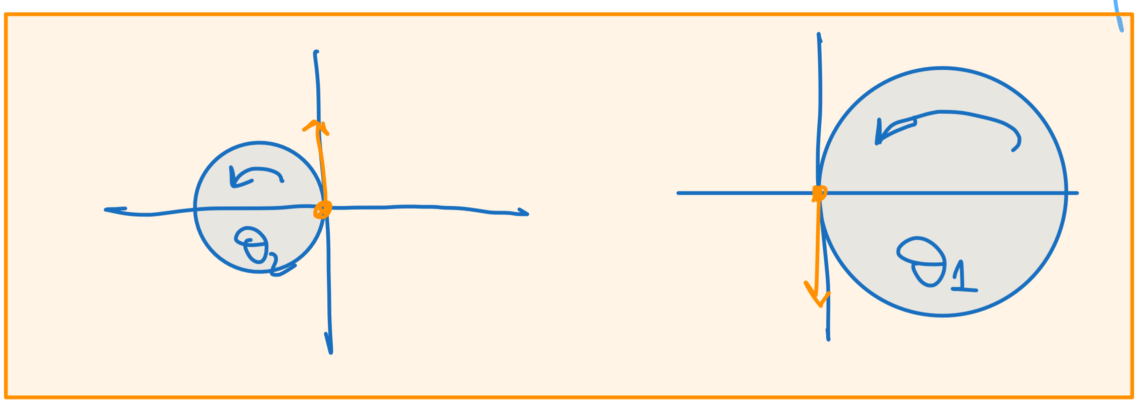
$n_1 = 10$
$n_2 = 20$

$\frac{r_2}{r_1} = \frac{n_2}{n_1} = 2$

$\frac{\text{input} = \text{orange} = 10}{\text{output} = \text{blue} = 20} = \frac{1}{2}$

$$\text{gear ratio} = \frac{n_{\text{input}}}{n_{\text{output}}}$$

and $\frac{\dot\theta_{\text{output}}}{\dot\theta_{\text{input}}} = -\text{gear ratio}$



$\Rightarrow r_1\dot\theta_1 = -r_2\dot\theta_2$

no slip

Mathematical analysis: transmission of velocities

We adopt the following notation:

- the input gear has angle θ_{input}, radius r_{input}, and n_{input} teeth.
- the output gear has angle θ_{output}, radius r_{output}, and n_{output} teeth.

(i) The teeth on each gear are evenly spaced. This property is referred to as *equal tooth pitch*. For two gears to mesh correctly, they must have the same tooth pitch. This requirement implies that the radius of a gear is proportional to its number of teeth:

$$\frac{r_{\text{input}}}{n_{\text{input}}} = \frac{r_{\text{output}}}{n_{\text{output}}} \iff \frac{r_{\text{output}}}{r_{\text{input}}} = \frac{n_{\text{output}}}{n_{\text{input}}}. \tag{2.30}$$

We define the *gear ratio* to be[4]

$$\boxed{\text{gear ratio} = \frac{n_{\text{input}}}{n_{\text{output}}}} \tag{2.31}$$

(ii) The gear interconnection is assumed to satisfy the *no-slip condition*, which states that the tangential velocities of the two gears are equal at the point of contact. Since linear velocity equals radius times angular velocity, we obtain:

$$r_{\text{input}}\,\dot{\theta}_{\text{input}} = -r_{\text{output}}\,\dot{\theta}_{\text{output}}. \tag{2.32}$$

Therefore, we can write

$$\frac{\dot{\theta}_{\text{output}}}{\dot{\theta}_{\text{input}}} = -\frac{r_{\text{input}}}{r_{\text{output}}} = -\frac{n_{\text{input}}}{n_{\text{output}}} = -\text{gear ratio} \tag{2.33}$$

A gear ratio greater than one indicates that the output gear rotates faster than the input gear, and in the opposite direction.

[4] It is important to note that some books and online documents use the opposite convention.

Mathematical analysis: transmission of torques

When two gears are in contact, they exert a contact force on each other. According to Newton's third law, this force is equal in magnitude and opposite in direction. Let τ_{input} be the torque applied to the input gear, and let τ_{output} be the torque transmitted to the output gear. Because both angles are measured in the same counterclockwise direction, as illustrated in Figure 2.19, we obtain

$$\frac{\tau_{\text{input}}}{r_{\text{input}}} = \frac{\tau_{\text{output}}}{r_{\text{output}}}. \tag{2.34}$$

In turn, the equal tooth pitch property implies

$$\frac{\tau_{\text{output}}}{\tau_{\text{input}}} = \frac{n_{\text{output}}}{n_{\text{input}}} = \frac{1}{\text{gear ratio}} \tag{2.35}$$

In the example in Figure 2.19, if we treat gear #1 (with $n_1 = 10$) as the input and gear #2 (with $n_2 = 20$) as the output, the gear ratio is $10/20 = 0.5$. Therefore,

(i) the angular velocity of gear #2 is half that of gear #1 (and in the opposite direction), and

(ii) a torque at gear #1 is perceived as twice as large at gear #2 (in the same direction).

This demonstrates the principle of *mechanical advantage:* meshing a smaller gear with a larger gear increases the output torque while reducing the output speed. Such a gear pair makes it easier to perform tasks like lifting heavy objects or climbing steep inclines in vehicles.

2.3.3 Dynamics of interconnected gears

This section examines the dynamical system formed by the interconnected gears shown in Figure 2.19. Specifically, a torque T is applied to the first gear, and we will derive the resulting dynamics for the second gear's angle, θ_2.

When the shafts are not interconnected, assuming moments of inertia I_1 and I_2, we have

$$I_1\ddot{\theta}_1 = T \tag{2.36}$$

$$I_2\ddot{\theta}_2 = 0 \tag{2.37}$$

When the gear interconnection is included, the contact torques τ_1 and τ_2 appear:

$$I_1\ddot{\theta}_1 = T + \tau_1 \tag{2.38}$$

$$I_2\ddot{\theta}_2 = \tau_2 \tag{2.39}$$

From the no-slip and force-balance conditions, we know $n_1\dot{\theta}_1 = -n_2\dot{\theta}_2$ and $n_2\tau_1 = n_1\tau_2$, which implies

$$\dot{\theta}_1 = -\frac{n_2}{n_1}\dot{\theta}_2, \quad \ddot{\theta}_1 = -\frac{n_2}{n_1}\ddot{\theta}_2 \quad \text{and} \quad \tau_1 = \frac{n_1}{n_2}\tau_2. \tag{2.40}$$

Plugging these expressions into the dynamics, we obtain:

$$I_1\left(-\frac{n_2}{n_1}\ddot{\theta}_2\right) = T + \frac{n_1}{n_2}\tau_2, \tag{2.41}$$

$$I_2\ddot{\theta}_2 = \tau_2. \tag{2.42}$$

To eliminate τ_2, we multiply the first equation by $-\frac{n_2}{n_1}$ and add the two resulting equations to get:

$$\boxed{\left(I_2 + \frac{n_2^2}{n_1^2}I_1\right)\ddot{\theta}_2 = (-\frac{n_2}{n_1})T} \tag{2.43}$$

The term $I_2 + \frac{n_2^2}{n_1^2}I_1$ in equation (2.43) is called the *equivalent moment of inertia* of the interconnected shafts.

2.4 Electrical systems

In this section we study electrical circuits and systems.

Figure 2.20: Passive elements (resistor, capacitor, and inductor) and active elements (voltage source and current source).

Components and their constitutive relations

Resistor: A resistor's behavior is described by *Ohm's law*, $v = ri$, where r is the *resistance* measured in Ohms (Ω), v is the voltage across the resistor, and i is the current flowing through it.

Capacitor: A capacitor's constitutive relation is $i = c\frac{dv}{dt}$, where the *capacitance* c is measured in Farads (F). This relation is equivalent to $v(t) = v(0) + \dfrac{1}{c}\displaystyle\int_0^t i(\tau)d\tau$. We assume ideal capacitors that store energy without loss.

Inductor: An inductor's constitutive relation is $v = \ell\frac{di}{dt}$, where the *inductance* ℓ is measured in Henrys (H). This relation is equivalent to $i(t) = i(0) + \dfrac{1}{\ell}\displaystyle\int_0^t v(\tau)d\tau$.

Voltage source: A *voltage source* provides a specified voltage, measured in Volts (V), irrespective of the current drawn from it. A battery is often modeled as a constant voltage source.

Current source: A *current source* supplies a specified current, measured in Amperes (A), irrespective of the voltage across it.

Kirchhoff's voltage law (KVL) KVL states that the algebraic sum of all voltages around a closed loop or closed path in a circuit is zero. Specifically, for voltages v_k across components around a given loop, measured with consistent reference direction (either all clockwise or all counterclockwise):

$$\sum_k v_k = 0.$$

Kirchhoff's current law (KCL) KCL states that the algebraic sum of all currents entering and leaving a node (or junction) in a circuit is zero. Specifically, for currents i_k associated with a given node k, currents entering the node are considered positive, and those leaving are considered negative (or vice versa, based on convention):

$$\sum_k i_k = 0.$$

In other words, the total current flowing into the node equals the total current flowing out.

RC circuit with a voltage generator

Consider a series circuit containing a voltage source, a resistor, and a capacitor, as illustrated in Figure 2.21. Let $v_{\text{input}}(t)$ be the input voltage, $r > 0$ be the resistance, $c > 0$ be the capacitance, and $v_{\text{output}}(t)$ be the output voltage across the capacitor.

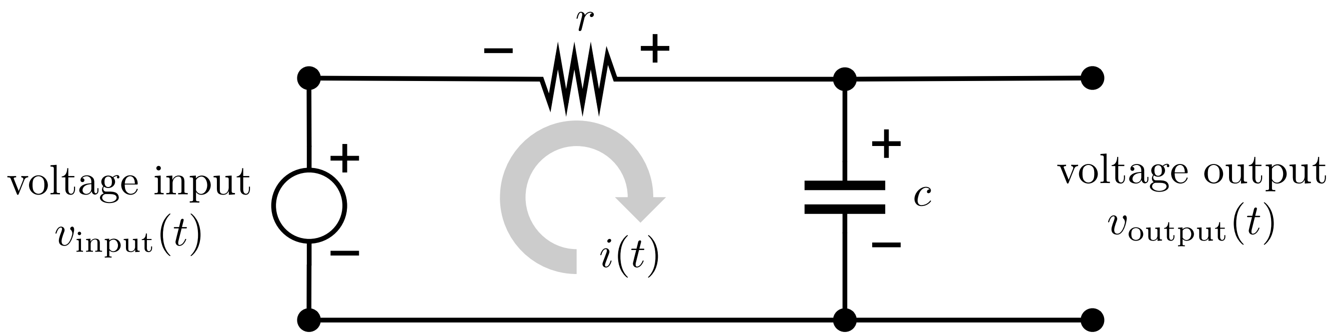


Figure 2.21: A series RC circuit. With the given voltage conventions, $v_{\text{resistor}} = ri$ and $i = -c\frac{d}{dt}v_{\text{output}}$.

From KVL, the sum of the voltage drops across the resistor and capacitor equals the source voltage v_{input}:

$$v_{\text{input}} + v_{\text{resistor}} - v_{\text{capacitor}} = 0.$$

Substituting the constitutive relation for the resistor and noting $v_{\text{capacitor}} = v_{\text{output}}$, we get:

$$v_{\text{input}} + ri(t) - v_{\text{output}} = 0.$$

Using the capacitor's constitutive relation, $i = -c\frac{d}{dt}v_{\text{output}}$, the governing differential equation for the RC circuit is

$$v_{\text{input}} - cr\dot{v}_{\text{output}} - v_{\text{output}} = 0 \quad \iff \quad \boxed{\dot{v}_{\text{output}}(t) + \frac{1}{rc}v_{\text{output}}(t) = \frac{1}{rc}v_{\text{input}}(t)} \tag{2.44}$$

This equation is a first-order model for the output voltage v_{output} with input v_{input}.

RLC circuit with a voltage generator

We now consider a circuit with a voltage source, a resistor, an inductor, and a capacitor in series, as illustrated in Figure 2.22. Let $v_{\text{input}}(t)$ be the voltage at the input, $r > 0$ be a resistance, $c > 0$ be a capacitance, ℓ be an inductance, and $v_{\text{output}}(t)$ be the voltage at the output.

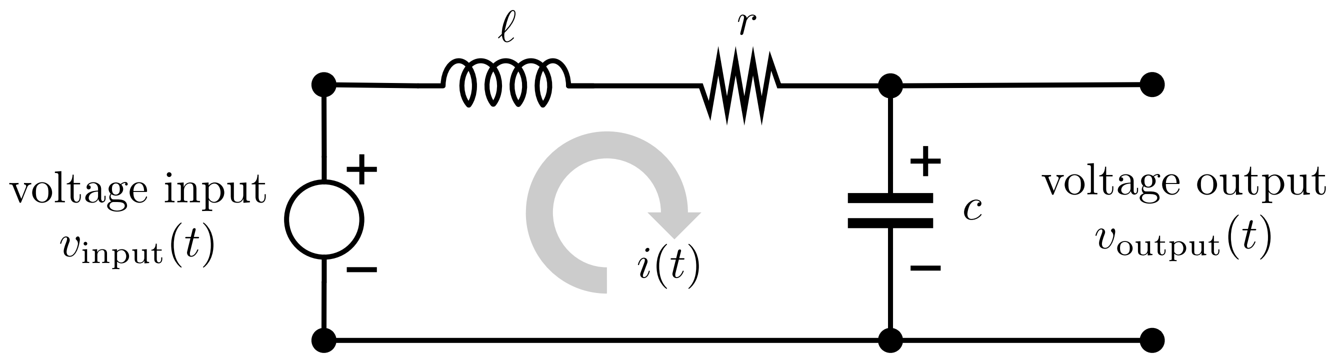


Figure 2.22: Series RLC circuit

As in Figure, we consider a *series RLC circuit with a voltage source* $v_{\text{input}}(t)$. The governing differential equation can be obtained from the KVL and from the constitutive relations for each element. In short:

$$v_{\text{input}}(t) = ri(t) + \ell\frac{di(t)}{dt} + \frac{1}{c}\int_0^t i(\tau)d\tau$$

Taking the derivative with respect to time[5] of both left and right had side, and rearranging terms, we obtain:

$$\ell\frac{d^2i(t)}{dt^2} + r\frac{di(t)}{dt} + \frac{1}{c}i(t) = \dot{v}_{\text{input}}(t) \tag{2.45}$$

This is a second-order model in the current i with input v_{input}. Following similar reasoning, we can obtain a second-order model for the output voltage:

$$\ddot{v}_{\text{output}} + \frac{r}{\ell}\dot{v}_{\text{output}} + \frac{1}{\ell c}\dot{v}_{\text{output}} = \frac{1}{\ell c}v_{\text{input}}(t). \tag{2.46}$$

[5] Recall that the fundamental theorem of calculus states $\frac{d}{dt}\int_0^t i(\tau)d\tau = i(t)$.

The equation (2.45) is analogous to the forced damped harmonic oscillator, described by the equation of motion

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

for a spring-mass-damper system subject to a force $f(t)$.

mass-spring-damper mechanical system		RLC electrical circuit	
m	mass	ℓ	inductance
b	damping coefficient	r	resistance
k	stiffness	$1/c$	inverse capacitance,
$f(t)$	external force	$\frac{d}{dt}v_{\text{input}}(t)$	forcing term

Table 2.1: Analogies between mechanical and electrical systems

2.5 Electromechanical systems and the DC motor

An *electromechanical system* is an engineering device composed of both electrical and mechanical components. Specifically, a *direct-current motor (DC motor)* converts electrical energy into mechanical energy or, more precisely, direct current into a torque. A DC motor is illustrated in Figure 2.23 and its functioning is illustrated in this wikipedia animation. Here are some highlights on how the DC motor functions:

Physical principle: A current-carrying conductor experiences a mechanical force when placed in a magnetic field. This force is called the Lorentz force.

From physical principle to engineering design: The Lorentz force cause the conductor to rotate, thus turning the motor's shaft and producing mechanical work.

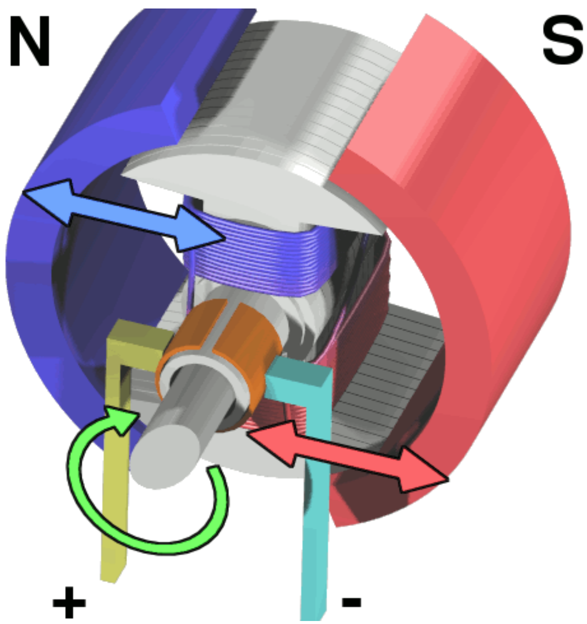


Figure 2.23: In a typical DC motor, the conductor is a coil, that is, a series of loops made from conductive wires wound around a core.

In a brushed DC motor, brushes are used to ensure that the current in the conductor is in the correct direction to produce maximum torque. In this image, a brushed DC electric motor generates torque from a supplied DC power, by using internal commutation (via brushes) and stationary permanent magnets.

Brushless DC motors (which do not use brushes) rely on electronic controllers to switch the current in the motor's windings.

Public domain image from Wikipedia. Also from Wikipedia: animation of a brushed DC electric motor generating torque from a DC power supply by using an internal mechanical commutation:

A complete derivation of the governing equations for a DC motor is outside the scope of these notes. Based upon the formulas for the Lorentz force and upon the geometry and design of the motor circuit, it suffices to say that

(i) *the current through the conductor i_{cond} generates a torque on the shaft*, with magnitude equal to $k_{\text{torque}} i_{\text{cond}}$ where $k_{\text{torque}} > 0$ is constant, and

(ii) *the shaft's angular velocity $\dot{\theta}_{\text{m}}$ generates a "back emf" voltage*[6] *on the conductor circuit*, with magnitude equal to $k_{\text{velocity}} \dot{\theta}_{\text{m}}$ with $k_{\text{velocity}} > 0$ and opposed to the voltage applied to the motor.

We assume some rotational damping with coefficient b and a moment of inertia I_m for the rotor. We also let ℓ denote the inductance and r denote the resistance of the conductor circuit.

In summary, the equations of motion for the DC motor are:

$$I_{\text{m}} \ddot{\theta}_{\text{m}}(t) + b \dot{\theta}_{\text{m}}(t) = k_{\text{torque}} i_{\text{cond}}(t) \tag{2.47a}$$

$$\ell \frac{d}{dt} i_{\text{cond}}(t) + r i_{\text{cond}}(t) = v_{\text{source}}(t) - k_{\text{velocity}} \dot{\theta}_{\text{m}}(t) \tag{2.47b}$$

where $v_{\text{source}}(t)$ is the externally applied voltage to the conductor circuit.

[6] "Back emf" stands for "back electromotive force."

These equations are *electromechanical* since they involve both mechanical and electrical quantities:

- equation (2.47a) with state θ_m is a rotational mechanical system with damping and with a forcing torque $k_\text{torque} i_\text{cond}$, and
- equation (2.47a) with state i_cond is an RL circuit with a forcing voltage source $v_\text{source}(t) - k_\text{velocity}\dot{\theta}_\text{m}$

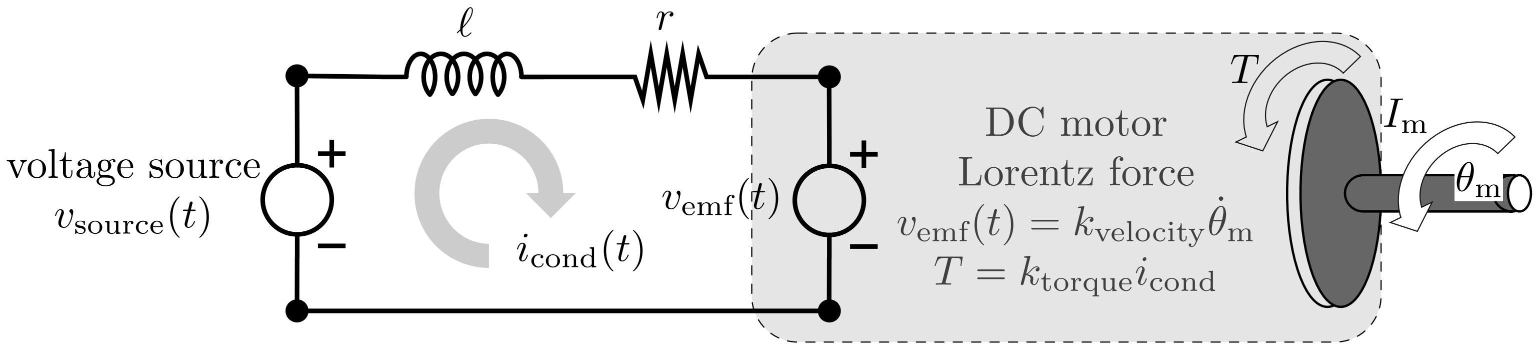


Figure 2.24: A DC motor relies upon the physical principle of the Lorentz force the transduce a voltage into a torque:
(i) the current through the conductor i_cond generates a torque on the shaft, with magnitude equal to $k_\text{torque} i_\text{cond}$, and
(ii) the shaft's angular velocity $\dot{\theta}_\text{m}$ generates a "back emf" voltage , with magnitude equal to $k_\text{velocity}\dot{\theta}_\text{m}$ and opposed to the voltage applied to the motor.

Numerical analysis of the DC motor

```python
import numpy as np; from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

def motor_dynamics(t, state, I_m, b, K_torque, L, R, K_velocity, V_input):
    theta_m, theta_m_dot, ic = state
    theta_m_ddot = (K_torque * ic - b * theta_m_dot) / I_m
    ic_dot = (V_input(t) - K_velocity * theta_m_dot - R * ic) / L
    return [theta_m_dot, theta_m_ddot, ic_dot]

# Parameters for the DC motor
I_m = 0.01          # Moment of inertia of the motor
b = 0.1             # Damping coefficient
K_torque = 0.01     # Torque constant
L = 0.5             # Motor inductance
R = 1               # Motor resistance
K_velocity = 0.01   # Back EMF constant

# Time array
t = np.linspace(0, 6, 1000)

# Voltage input: step function at 1V
V_input = lambda t: 1.0 if t > 1 else 0.0

# Initial conditions: [theta_m, theta_m_dot, ic]
initial_conditions = [0.0, 0.0, 0.0]
sol = solve_ivp(motor_dynamics, [t[0], t[-1]], initial_conditions, ...
    t_eval=t, args=(I_m, b, K_torque, L, R, K_velocity, V_input))

# Plotting
plt.figure(figsize=(8, 6)); plt.subplot(3, 1, 1); plt.xlim(0, 6)
plt.plot(sol.t, sol.y[0], label='Motor Position (rad)')
plt.grid(True); plt.ylabel('Position (rad)'); plt.legend()

plt.subplot(3, 1, 2)
plt.plot(sol.t, sol.y[1], label='Motor Speed (rad/s)'); plt.xlim(0, 6)
plt.grid(True); plt.ylabel('Speed (rad/s)'); plt.legend()

plt.subplot(3, 1, 3)
plt.plot(sol.t, sol.y[2], label='Current (A)', color='red'); ...
    plt.xlim(0, 6)
plt.grid(True); plt.xlabel('Time'); plt.ylabel('Current (A)'); plt.legend()

plt.tight_layout()
plt.savefig("dcmotor.pdf", bbox_inches='tight')
```

Listing 2.8: Python script generating Figure 2.25. Available at `dcmotor.py`

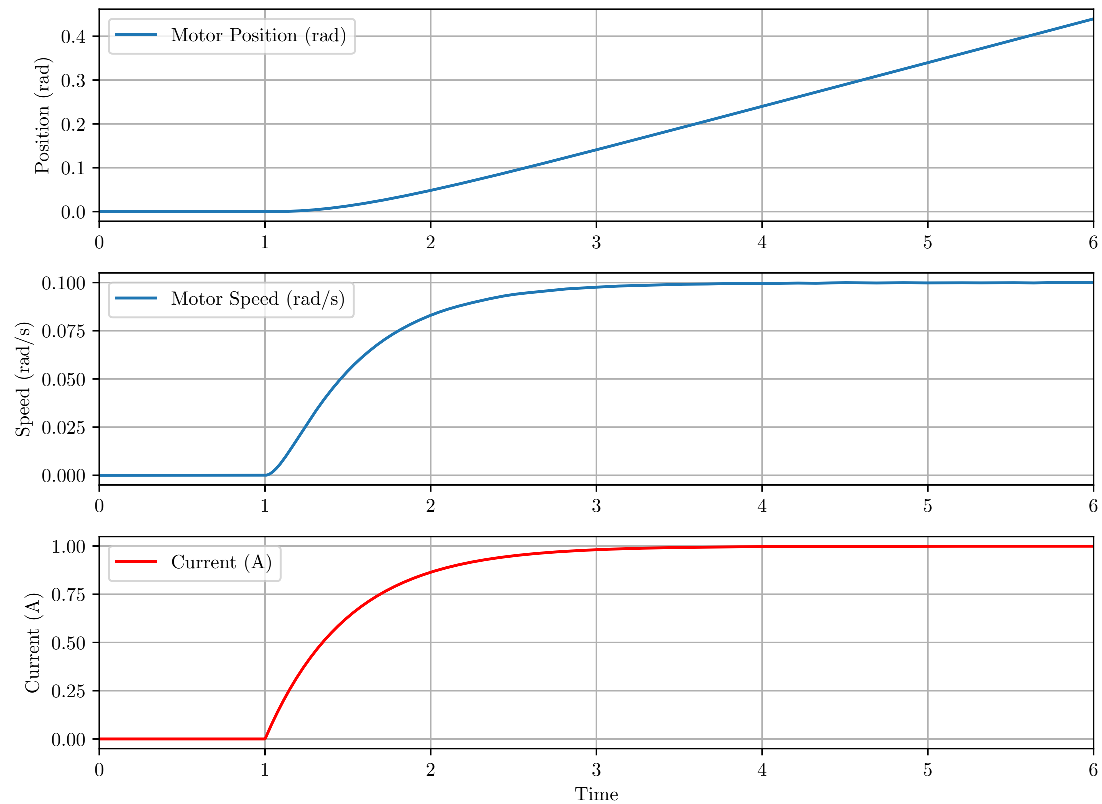


Figure 2.25: Solutions of the DC motor system (2.47): response to a 1V step input voltage at time $t = 1$s, that is, the input voltage satisfies $v_{\text{source}}(t) = 0$V from time $0 \leq t \leq 1$ and then $v_{\text{source}}(t) = 1$V for $t > 1$.

2.6 Historical notes and further resources

The historical context of the chapter is rooted in the foundational work of Isaac Newton, whose laws of motion, published in 1687, established the basis for classical mechanics. The study of stability in motion was significantly advanced by A. Lyapunov in 1892, whose work remains seminal in the field. The development of state-space representation by R. E. Kalman in 1960 and the control of nonlinear systems by R. W. Brockett in 1983 have further contributed to the understanding of dynamical systems. Seminal texts on control systems and electromechanical design include (Den Hartog, 1956; DiStefano et al., 1997; Ogata, 2003; Franklin et al., 2015; Åström and Murray, 2021).

The loss of the *Mars Climate Orbiter* in 1999 was a significant engineering failure due to a unit conversion error—specifically, a miscommunication between metric (SI) and imperial (U.S. customary) units. In this disaster, the problem was that NASA's Jet Propulsion Laboratory (JPL) used the metric system, while the spacecraft's contractor, Lockheed Martin, used imperial units. Lockheed Martin provided data for the spacecraft's thrusters in pounds of force, but NASA was expecting the data in Newtons (the SI unit for force). This discrepancy led to the orbiter's trajectory being incorrect, causing it to enter the Martian atmosphere at a much lower altitude than intended, leading to its destruction.

Instructive videos:

- how do car suspensions work (20m 23s) and (short) (2m 49s) with animations and explanation of different types of automobile suspensions;
- applications of rotary dampers and torsion springs: how to install rotary dampers (2m 26s);
- the only mechanical gears known to occur in nature are documented in this scientific article, review article, and video interview (3m 41s).

2.7 Exercises

Section 2.1: Mechanical systems: One degree of freedom

E2.1 **First-order system with piecewise-constant forcing.** Consider the first-order system

$$m\dot{v}(t) + bv(t) = f(t), \tag{2.48}$$

where $m > 0$ is the mass, $b > 0$ is the damping coefficient, $v(t)$ is the velocity, and $f(t)$ is an external force. Suppose $f(t)$ is piecewise constant:

$$f(t) = \begin{cases} F_1, & 0 \leq t < T, \\ F_2, & t \geq T, \end{cases} \tag{2.49}$$

with initial condition $v(0) = v_0$. Compute $v(t)$ for all $t \geq 0$ and express the result in piecewise form.

Answer: We solve the system in two intervals.

(i) Divide by m and define $\alpha := \frac{b}{m} > 0$. The equation becomes

$$\dot{v}(t) + \alpha v(t) = \frac{f(t)}{m}. \tag{2.50}$$

(ii) For $0 \leq t < T$, the force is $f(t) = F_1$, and the equation becomes

$$\dot{v}(t) + \alpha v(t) = \frac{F_1}{m}. \tag{2.51}$$

The general solution is

$$v(t) = C\,\mathrm{e}^{-\alpha t} + \frac{F_1}{b}. \tag{2.52}$$

Apply $v(0) = v_0$ to find

$$C = v_0 - \frac{F_1}{b}, \tag{2.53}$$

so

$$v(t) = \left(v_0 - \frac{F_1}{b}\right)\mathrm{e}^{-\alpha t} + \frac{F_1}{b}, \qquad 0 \leq t < T. \tag{2.54}$$

(iii) For $t \geq T$, define $v(T^-) = \left(v_0 - \frac{F_1}{b}\right)\mathrm{e}^{-\alpha T} + \frac{F_1}{b}$. Since $f(t) = F_2$, the equation becomes

$$\dot{v}(t) + \alpha v(t) = \frac{F_2}{m}, \tag{2.55}$$

with $v(T) = v(T^-)$. The general solution is

$$v(t) = A\,\mathrm{e}^{-\alpha(t-T)} + \frac{F_2}{b}. \tag{2.56}$$

Match $v(T) = v(T^-)$ to find

$$A = v(T^-) - \frac{F_2}{b}. \tag{2.57}$$

Substitute to obtain

$$v(t) = \left[v(T^-) - \frac{F_2}{b}\right]\mathrm{e}^{-\alpha(t-T)} + \frac{F_2}{b}, \qquad t \geq T, \tag{2.58}$$

and substitute $v(T^-)$:

$$v(t) = \left(\left(v_0 - \frac{F_1}{b}\right)\mathrm{e}^{-\alpha T} + \frac{F_1 - F_2}{b}\right)\mathrm{e}^{-\alpha(t-T)} + \frac{F_2}{b}. \tag{2.59}$$

(iv) The complete solution is

$$v(t) = \begin{cases} \left(v_0 - \frac{F_1}{b}\right)\mathrm{e}^{-\alpha t} + \frac{F_1}{b}, & 0 \leq t < T, \\ \left(\left(v_0 - \frac{F_1}{b}\right)\mathrm{e}^{-\alpha T} + \frac{F_1 - F_2}{b}\right)\mathrm{e}^{-\alpha(t-T)} + \frac{F_2}{b}, & t \geq T. \end{cases} \tag{2.60}$$

▼

E2.2 **Equivalent expressions for the solution to the harmonic oscillator.** Consider a mass-spring system described by the harmonic oscillator (i.e., an undamped harmonic oscillator) as in Figure 2.26.

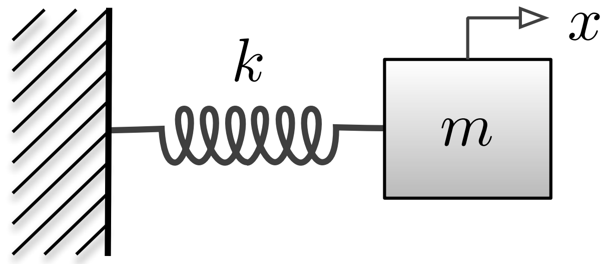


Figure 2.26: A mass-spring system described by the harmonic oscillator system $m\ddot{x} + kx = 0$.

Let m denote the mass of the oscillating object and k the stiffness of the spring. From equation (2.17), each solution to the undamped harmonic oscillator is of the form

$$x(t) = a\sin(\omega t) + b\cos(\omega t) \tag{2.61}$$

where the natural frequency is $\omega = \sqrt{k/m}$. Define the abbreviations: $x_0 := x(0)$ and $v_0 := \dot{x}(0)$.

(i) Write a formula for (a, b) as function of (x_0, v_0) and vice versa.

(ii) Consider the equality

$$a\sin(\omega t) + b\cos(\omega t) = A\sin(\omega t + \phi) \tag{2.62}$$

Write a formula for (a, b) as function of (A, ϕ) and vice versa.

Hint: Recall that in class we saw $A = \sqrt{a^2 + b^2}$. You will need to use trigonometric identities.

(iii) *Optional:* Show that the solution can also be written as

$$x(t) = C_1 \,\mathrm{e}^{-j\omega t} + C_2 \,\mathrm{e}^{j\omega t} \tag{2.63}$$

for appropriate complex numbers C_1 and C_2. Write C_1 and C_2 as a function of (x_0, v_0).

Hint: Recall Euler's formula for complex numbers.

Answer:

(i) From equation (2.61), the time derivative of $x(t)$ is given by

$$\dot{x}(t) = a\omega\cos(\omega t) - b\omega\sin(\omega t). \tag{2.64}$$

Evaluating equation (2.61) and equation (2.64) at time 0 we obtain:

$$\begin{aligned} x(0) &= a\sin(0) + b\cos(0) \\ \dot{x}(0) &= a\omega\cos(0) - b\omega\sin(0) \\ \implies & \boxed{b = x_0} \quad \text{and} \quad \boxed{a = \frac{v_0}{\omega}} \end{aligned}$$

Similarly, we can write (x_0, v_0) as a function of (a, b) by rearranging the terms above to find

$$\boxed{x_0 = b} \quad \text{and} \quad \boxed{v_0 = a\omega}$$

Finally, it is useful to substitute the expression for (a, b) into the formula for $x(t)$ to obtain:

$$x(t) = \frac{v_0}{\omega}\sin(\omega t) + x_0\cos(\omega t)$$

(ii) Recall that the angle sum trigonometric identity for the sinusoidal function is $\sin\big(\alpha+\beta\big) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. Using this identity, we expand the right hand side of the equation (2.62) to find

$$a\sin\big(\omega t\big) + b\cos\big(\omega t\big) = A\sin(\omega t)\cos(\phi) + A\cos(\omega t)\sin(\phi)$$

Therefore, by matching terms, we obtain

$$\boxed{a = A\cos(\phi)} \quad \text{and} \quad \boxed{b = A\sin(\phi)}$$

To write A in terms of (a, b) we make use of the fact that $\sin^2 + \cos^2 = 1$. Observe that

$$a^2 + b^2 = A^2\big(\cos^2(\phi) + \sin^2(\phi)\big) \quad \implies \quad \boxed{A = \sqrt{a^2 + b^2}}$$

Next, rearranging the expressions for (a, b) in terms of (A, ϕ) we see

$$\begin{aligned} a &= A\cos(\phi) \\ b &= A\sin(\phi) \quad \implies \quad \frac{b}{a} = \tan\phi \quad \implies \quad \boxed{\phi = \arctan(b/a)} \end{aligned}$$

(iii) We want to show that, for some complex numbers C_1 and C_2,

$$x(t) = a\sin(\omega t) + b\cos(\omega t) = C_1\, \mathrm{e}^{-j\omega t} + C_2\, \mathrm{e}^{j\omega t}$$

From Euler's formulas, we have the relationships

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) \qquad \text{and} \qquad e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t).$$

Substituting these equations into the exponential form (2.63) of the solution we have

$$\begin{aligned} x(t) &= C_1\big(\cos(\omega t) - j\sin(\omega t)\big) + C_2\big(\cos(\omega t) + j\sin(\omega t)\big) \\ &= (C_1 + C_2)\cos(\omega t) + j(C_2 - C_1)\sin(\omega t) \end{aligned}$$

Observe that the time derivative of this equation is given by

$$\dot{x}(t) = -\omega(C_1 + C_2)\sin(\omega t) + j\omega(C_2 - C_1)\cos(\omega t)$$

Similarly to before, by substituting our initial conditions for $(x(t), \dot{x}(t))$, we can write (C_1, C_2) as a function (x_0, v_0) as follows

$$\begin{aligned} x(0) &= \big(C_1 + C_2\big)\cos(0) + j\big(C_2 - C_1\big)\sin(0) \qquad \Longrightarrow \qquad C_1 + C_2 = x_0 \\ \dot{x}(0) &= -\omega\big(C_1 + C_2\big)\sin(0) + j\omega(C_2 - C_1)\cos(0) \qquad \Longrightarrow \qquad C_2 - C_1 = \frac{v_0}{j\omega} \end{aligned}$$

From here, we can find C_1 and C_2 independently to be

$$\big(C_1 + C_2\big) + \big(C_2 - C_1\big) = x_0 - j\frac{v_0}{\omega} \qquad \Longrightarrow \qquad \boxed{C_2 = \frac{1}{2}\big(x_0 - j\frac{v_0}{\omega}\big)}$$

$$C_1 = x_0 - C_2 \qquad \Longrightarrow \qquad \boxed{C_1 = \frac{1}{2}\big(x_0 + j\frac{v_0}{\omega}\big)}$$

▼

E2.3 **Mechanical modeling of a muscle.** A muscle connected to a fixed point and subject to a load force can be modeled by the equivalent mechanical system shown in the Figure 2.27. The key elements of the system are: (1) The muscle connects the fixed point to a mass m at position x. (2) The muscle is represented by the interconnection of two components, with the intermediate point at coordinate x_{mid}. (3) The muscle exerts a force F_{muscle} at the intermediate point. (4) A damper with damping coefficient b connects the intermediate point to the stationary point. (5) A spring with stiffness k and zero rest length connectsq the intermediate point to the mass. (6) The mass m is subject to a load force F_{load}.

(i) Write a differential equation for the mass acceleration $\ddot{x}$ as a function of the load force F_{load}, the force generated by the muscle F_{muscle}, and the velocity of the intermediate point $\dot{x}_{\text{mid}}$.

Hint: First, use the free body diagram of the mass m. Next, consider the free body diagram of the intermediate point with zero mass; the net force on this intermediate point must be 0.

(ii) Determine the equilibrium condition that relates F_{load} and F_{muscle} such that the system is at rest (i.e., at an equilibrium, no motion).

(iii) Assuming the equilibrium condition holds, find the final length of the spring.

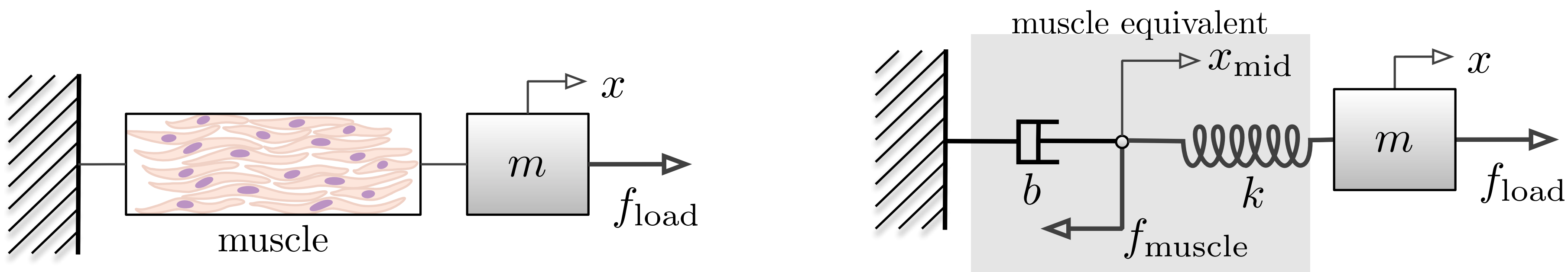


Figure 2.27: Left image: A muscle connected to a fixed point and subject to a load force. Right image: Equivalent mechanical system for the muscle excitation.

Section 2.2: Mechanical systems: Two degrees of freedom and the suspension example

E2.4 **Equations of motion for the suspension system.** Consider the suspension system studied in Section 2.2 and depicted in Figure 2.28: m_s and m_{us} are sprung mass and unsprung mass, respectively; k_s and k_w are the spring constants for m_s and m_{us} respectively, b is the damping coefficient, and $r(t)$ is the road surface.

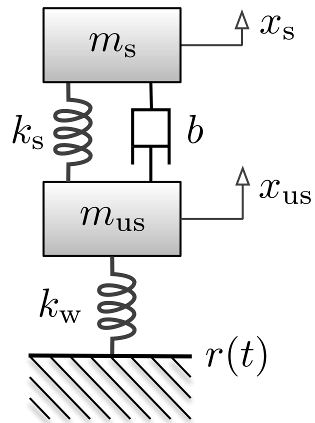


Figure 2.28: Suspension system

(i) Draw a free body diagram for each mass and accounting for each spring and damper (ignore gravity).

(ii) Write equations for each force acting on the masses m_s and m_{us}.

(iii) Write the equations of motion for the sprung and unsprung masses based on Newton's law.

Answer:

(i) Assume the positive axis is pointing upwards. In this system, we have spring and damping forces.

$$F_{\text{spring}} = (\text{stiffness coefficient}) \times \text{displacement} \qquad \text{and} \qquad F_{\text{damper}} = (\text{damping coefficient}) \times \text{velocity} \tag{2.65}$$

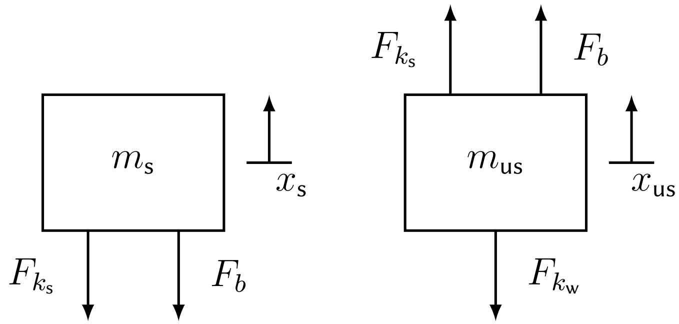


Figure 2.29: Free body diagram of suspension system

(ii) For mass m_s: There are two forces acting on the mass

$$F_{k_s} = k_s x_{us} - k_s x_s = k_s(x_{us} - x_s)$$
$$F_b = b\dot{x}_{us} - b\dot{x}_s = b(\dot{x}_{us} - \dot{x}_s)$$

so that $m_s \ddot{x}_s = F_{k_s} + F_b = k_s(x_{us} - x_s) + b(\dot{x}_{us} - \dot{x}_s)$.

For mass m_{us}: There are three forces acting on the mass. Specifically, the directions of the forces F_{k_s} and F_b are opposite

$$F_{k_s} = k_s x_s - k_s x_{us} = -k_s(x_{us} - x_s),$$
$$F_b = b\dot{x}_s - b\dot{x}_{us} = -b(\dot{x}_{us} - \dot{x}_s),$$
$$F_{k_w} = k_w r(t) - k_w x_{us} = -k_w(x_{us} - r(t)),$$

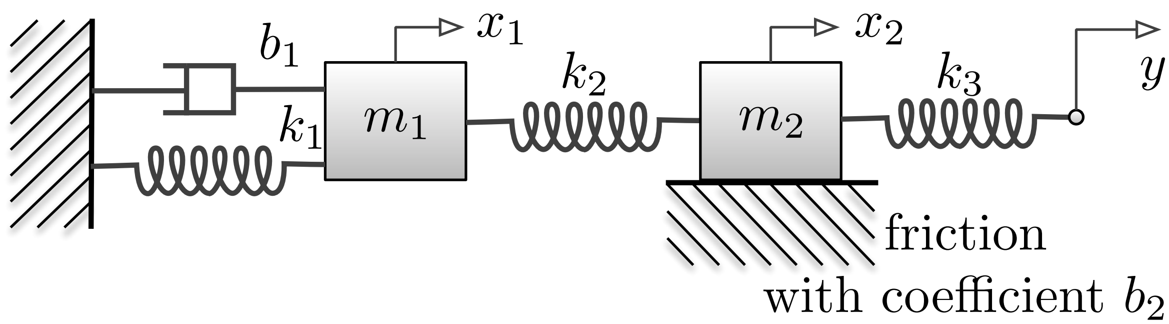
so that $m_{us}\ddot{x}_{us} = F_{k_s} + F_b + F_{k_w} = -k_s(x_{us} - x_s) - b(\dot{x}_{us} - \dot{x}_s) - k_w(x_{us} - r(t))$.

(iii) Rearrange the equations:

$$m_s \ddot{x}_s + b\left(\dot{x}_s - \dot{x}_{us}\right) + k_s\left(x_s - x_{us}\right) = 0$$
$$m_{us}\ddot{x}_{us} + b\left(\dot{x}_{us} - \dot{x}_s\right) + k_s\left(x_{us} - x_s\right) + k_w x_{us} = k_w r(t)$$

▼

E2.5 **Interconnected masses with friction.** Consider the mass-spring-damper system in figure, where the spring have zero rest length and the friction on the second mass acts like a damper with damping coefficient b_2.



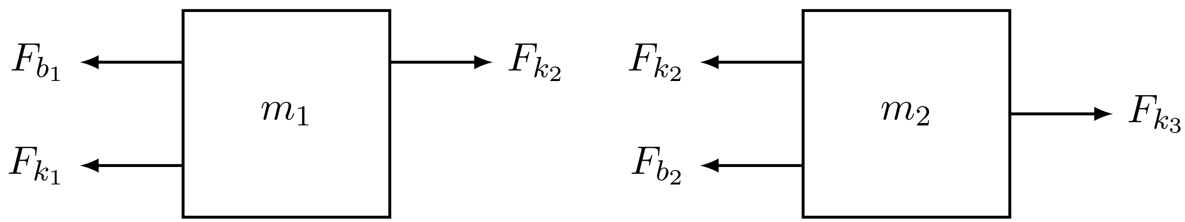
Perform the following steps:

(i) Draw the free body diagram for each mass.

(ii) Use Newton's 2nd Law to write the equations of motion for the system.

Answer:

(i) Here is the free body diagram:



Note that F_{k_2} is in opposite directions for first and second body.
Given the conventions in the drawing, we compute

$$F_{b_1} = b_1\dot{x}_1, \quad F_{k_1} = k_1\dot{x}_1, \quad F_{k_2} = k_2(x_2 - x_1) \tag{2.66}$$
$$F_{b_2} = b_2\dot{x}_2, \quad F_{k_3} = k_3(y - x_2). \tag{2.67}$$

(ii) Given the convention in the drawing, Newton's 2nd Law gives us

$$\begin{aligned} F_{k_2} - F_{b_1} - F_{k_1} &= m_1\ddot{x}_1 \\ F_{k_3} - F_{k_2} - F_{b_2} &= m_2\ddot{x}_2. \end{aligned}$$

Substituting expressions for each of the forces gives

$$\begin{aligned} k_2(x_2 - x_1) - b_1\dot{x}_1 - k_1 x_1 &= m_1\ddot{x}_1 \\ k_3(y - x_2) - k_2(x_2 - x_1) - b_2\dot{x}_2 &= m_2\ddot{x}_2. \end{aligned}$$

Putting these in standard form, we have

$$\ddot{x}_1 = -\frac{(k_1 + k_2)}{m_1}x_1 - \frac{b_1}{m_1}\dot{x}_1 + \frac{k_2}{m_1}x_2$$

$$\ddot{x}_2 = \frac{k_2}{m_2}x_1 - \frac{(k_2 + k_3)}{m_2}x_2 - \frac{b_2}{m_2}\dot{x}_2 + \frac{k_3}{m_2}y$$

▼

E2.6 **Inverted Pendulum Cart via Lagrangian Mechanics.** Consider the inverted pendulum on a cart, shown in Figure 2.30.

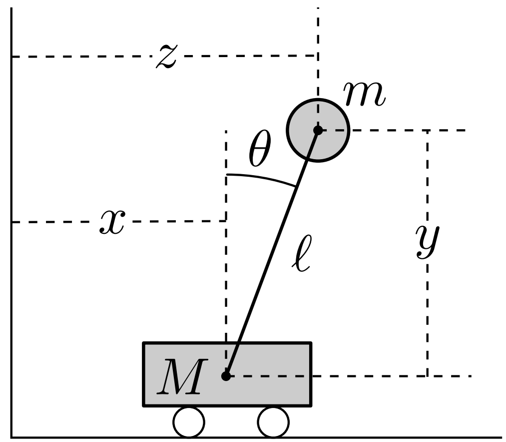


Figure 2.30: Inverted pendulum cart system. The pendulum is assumed to be a rigid rod of negligible mass, with all mass concentrated at its endpoint.

(i) Express the kinetic energy of the cart, the kinetic energy of the pendulum, the potential energy of the cart, and the potential energy of the pendulum in terms of the variables x, $\dot{x}$, y, $\dot{y}$, z, and $\dot{z}$. (Assume the rod has negligible mass and its rotational kinetic energy can be ignored.)

(ii) Rewrite the energies using only the variables x, $\dot{x}$, θ, and $\dot{\theta}$.

(iii) Compute the *Lagrangian* of the system, defined as

$$L := T - V,$$

where T is the total kinetic energy and V is the total potential energy.

(iv) Substitute the Lagrangian $L = L(x, \dot{x}, \theta, \dot{\theta})$ into the so-called *Euler–Lagrange equations*, carefully computing each derivative:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x},$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}.$$

Simplify the resulting two equations.

Note: *It is a standard result that the Euler–Lagrange equations are equivalent to Newton's laws.*

(v) Express the dynamics of the system in the form

$$\ddot{x} = f(x, \dot{x}, \theta, \dot{\theta}),$$
$$\ddot{\theta} = g(x, \dot{x}, \theta, \dot{\theta}),$$

where f and g are functions obtained from the simplified equations.

Hint: You should obtain two equations with two unknowns: $\ddot{x}$ and $\ddot{\theta}$.

(vi) Do the dynamics (i.e., the functions f and g) depend explicitly on x or $\dot{x}$? What does this imply about the structure of the system?

Note: *This alternative approach to deriving equations of motion for a mechanical system is called Lagrangian mechanics. Unlike Newtonian mechanics, which focuses on summing forces and torques, the Lagrangian formulation is based on energy functions (T and V). For many systems, the Lagrangian approach offers a more systematic and elegant way to obtain the dynamics. This is especially relevant for systems with multiple degrees of freedom or constraints. For further reading, see* `https://en.wikipedia.org/wiki/Lagrangian_mechanics`.

Answer:

(i) The energies of the system are:

$$T_{\text{cart}} = \tfrac{1}{2}M\dot{x}^2, T_{\text{pend}} \qquad = \tfrac{1}{2}m\left(\dot{y}^2 + \dot{z}^2\right),$$
$$V_{\text{cart}} = 0,$$
$$V_{\text{pend}} = mgy.$$

(ii) Using $y = \ell\cos\theta$ and $z = x + \ell\sin\theta$, and differentiating, we obtain

$$\dot{y} = -\ell\sin\theta\,\dot{\theta}, \qquad \dot{z} = \dot{x} + \ell\cos\theta\,\dot{\theta}.$$

Substituting these expressions into the energy formulas yields

$$T_{\text{cart}} = \tfrac{1}{2}M\dot{x}^2,$$
$$T_{\text{pend}} = \tfrac{1}{2}m\left(\ell^2\dot{\theta}^2 + \dot{x}^2 + 2\ell\dot{x}\cos\theta\,\dot{\theta}\right),$$
$$V_{\text{cart}} = 0,$$
$$V_{\text{pend}} = mg\ell\cos\theta.$$

(iii) The Lagrangian is

$$L = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\big(\ell^2\dot{\theta}^2 + \dot{x}^2 + 2\ell\dot{x}\cos\theta\,\dot{\theta}\big) - mg\ell\cos\theta.$$

(iv) Applying the Euler–Lagrange equations and simplifying yields:

$$\ell\ddot{\theta} + \ddot{x}\cos\theta - g\sin\theta = 0,$$
$$(M+m)\ddot{x} + m\ell\left(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right) = 0.$$

(v) Solving these two equations for $\ddot{x}$ and $\ddot{\theta}$ gives

$$\ddot{x} = \frac{m\ell\sin\theta\left(\dot{\theta}^2 - g\cos\theta\right)}{M+m-m\cos^2\theta},$$
$$\ddot{\theta} = \frac{\sin\theta\left((M+m)g - m\ell\cos\theta\,\dot{\theta}^2\right)}{\ell\left(M+m\sin^2\theta\right)}.$$

(vi) The dynamics do not depend on x or $\dot{x}$. This reflects the system's *translational invariance*: shifting the cart's position or velocity in the horizontal direction does not change the equations of motion. Physically, only the relative motion of the pendulum with respect to the cart matters.

▼

E2.7 **Rigid and flexible foundations in vibration isolation.** In many engineering applications, such as vehicles and machinery, controlling vibrations is critical to ensuring the stability and longevity of structures. Vibrating machinery is often mounted on structures, where it is necessary to reduce the transmission of vibrations. A common approach to isolating vibrations is by introducing a spring between the machine and the structure. In this exercise, you will design a system where an engine (i.e., a vibrating machine) is mounted on a structure. Our objective is to explore how the rigidity or flexibility of the foundation affects the system's dynamic properties.

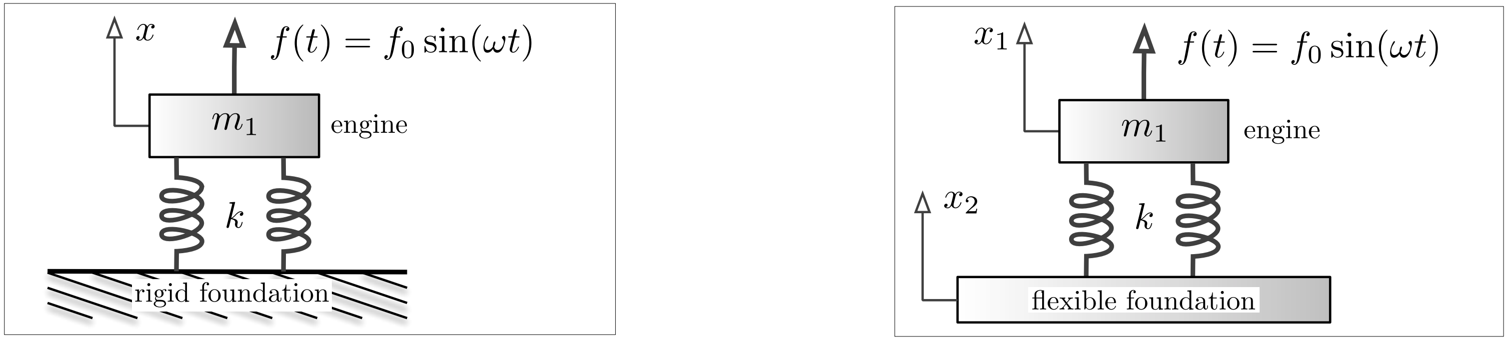


Figure 2.31: Engine mounted on a rigid (left) and flexible (right) foundation, e.g., in a car.
The parameter k is the total stiffness of the two springs; the rest length of the springs is zero. The engine mass is m_1, and, in the right image, m_2 is the mass of the foundation.

The system's motion is modeled using harmonic solutions of the form:

$$x_1 = x_{1m} \sin \omega t \tag{2.68}$$

$$x_2 = x_{2m} \sin \omega t \tag{2.69}$$

where x_{1m} and x_{2m} are the maximum oscillation amplitude of masses m_1 and m_2, respectively. (Clearly, $x_{2m} = 0$ when the foundation is rigid.)

#1) Rigid foundation system: Assume that the foundation m_2 is rigid.

(i) Write the differential equations for the rigid foundation system as shown in Figure 2.31(a).

(ii) Find the squared natural frequency ω_n^2 of the rigid system as shown in Figure 2.31(a).

#2) Flexible foundation system: Now, consider the flexible foundation system.

(iii) Write the differential equations for the flexible foundation system in Figure 2.31(b).

(iv) Solve for the natural frequency squared ω_n^2 of the flexible system in Figure 2.31(b). Your final answer should be in terms of only the variables k, m_1, and m_2.

#3) Comparative analysis: Finally, we perform a comparative analysis.

(v) If $m_1 = 10m_2$, what is the natural frequency of the system? How does it compare to the case where we assume the foundation is rigid?

(vi) Under what conditions is the natural frequency of the two systems equal?

Hint: Recall that the natural frequency is found by considering the solution to the un-forced system, i.e., the system with $f(t) = 0$.

Hint: To find the natural frequency, you will need to substitute our assumed solution into the dynamical system and solve for the frequency.

Section 2.3: Rotational mechanical systems

E2.8 **Equations of motion for the roll dynamics of a ship (Den Hartog, 1956, Section 3.4).** Consider an oceangoing ship. As shown in the left figure, when the ship is at rest, the balanced forces of weight W and buoyancy B act along the centerline of the ship, generating no torque. However, as shown in the right figure, when the ship is inclined at a roll angle θ (e.g., due to rough seas), the buoyancy force B shifts to the left, intersecting the centerline of the ship at a point called the *metacenter* and denoted as M. The hull of the ship is designed so that the metacenter M is above the ship's center of gravity C; this arrangement results in a restoring torque that stabilizes the ship. The vertical distance between the center of gravity C and the metacenter M is known as the *metacentric height* h.

(i) Derive the equation of motion for the inclined ship in terms of the inclination angle θ, the moment of inertia I_s, the weight W, and the metacentric height h. If no other forces are present, is there damping in this system?

(ii) Recall the small-angle approximations $\sin x \approx x$ and $\cos x \approx 1$ for x near zero. Assume a small inclination angle θ, and use the small-angle approximation to simplify the equation of motion from part (i). Identify the natural frequency ω_n of the system.

(iii) Write down the solution for the inclination angle θ as a function of time t, of the natural frequency ω_n and of the initial conditions $\theta(0), \dot{\theta}(0)$.

E2.9 **Equations of motion for interconnected shafts.** Consider two parallel shafts with meshed gears as in figure.

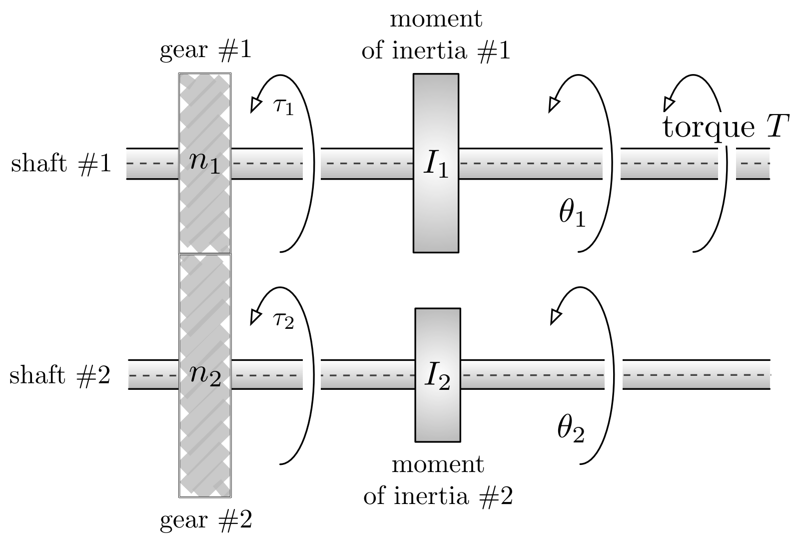


Figure 2.32: Two parallel shafts connected via gears: gear #1 on top transmits torque to gear #2 on bottom. (Both shafts are supported by bearings at either end, not drawn.) This setup models a typical mechanical transmission configuration for torque conversion and speed control.

Assume that

(i) the first shaft with angle θ_1 has moment of inertia I_1 and the second shaft with angle θ_2 has moment of inertia I_2;

(ii) the first shaft is subject to a external torque T, whereas no external torque is applied to the second shaft; and

(iii) the two shafts are interconnected via a pair of gears with n_1 teeth on the first gear and n_2 teeth on the second gear.

Show that the equations of motion are

$$\Big(I_1 + \frac{n_1^2}{n_2^2} I_2\Big)\ddot{\theta}_1 = T \tag{2.70}$$

Hint: Write the two equations of motion for the two shafts, including torques τ_1 and τ_2 generated by the meshed gears. Then, using the equalities $n_1\dot{\theta}_1 = -n_2\dot{\theta}_2$ and $n_2\tau_1 = n_1\tau_2$, eliminate the intermediate variables $\dot{\theta}_2$, $\ddot{\theta}_2$, τ_1 and τ_2.

Note: *The moment of inertia $I_1 + \frac{n_1^2}{n_2^2} I_2$ in equation (2.70) is the equivalent moment of inertia of the interconnected shafts.*

Answer: When the shafts are not interconnected, we have

$$I_1\ddot{\theta}_1 = T \tag{2.71}$$

$$I_2\ddot{\theta}_2 = 0 \tag{2.72}$$

Add the gear interconnection and therefore two torques, call them τ_1 and τ_2:

$$I_1\ddot{\theta}_1 = T + \tau_1 \tag{2.73}$$

$$I_2\ddot{\theta}_2 = \tau_2 \tag{2.74}$$

Since $n_1\dot{\theta}_1 = -n_2\dot{\theta}_2$ and $n_2\tau_1 = n_1\tau_2$, we know

$$\dot{\theta}_2 = -\frac{n_1}{n_2}\dot{\theta}_1, \quad \ddot{\theta}_2 = -\frac{n_1}{n_2}\ddot{\theta}_1 \quad \text{and} \quad \tau_2 = \frac{n_2}{n_1}\tau_1 \tag{2.75}$$

so that we can plug in and obtain

$$I_1\ddot{\theta}_1 = T + \tau_1 \tag{2.76}$$

$$I_2\Big(-\frac{n_1}{n_2}\ddot{\theta}_1\Big) = \frac{n_2}{n_1}\tau_1 \tag{2.77}$$

Equation (2.70) follows from multiplying the second equation by $-n_1/n_2$ and summing the two equations, thereby eliminating the intermediate variable τ_1. ▼

E2.10 **Sprockets and chains in bicycles.** In the image of a bicycle gear train below, the wheel angle θ_{wheel} and the crank angle θ_{crank} are measured counterclockwise (per convention in this text). In other words, while pedaling forward, both $\dot{\theta}_{\text{wheel}}$ and $\dot{\theta}_{\text{crank}}$ are negative, indicating counterclockwise motion.

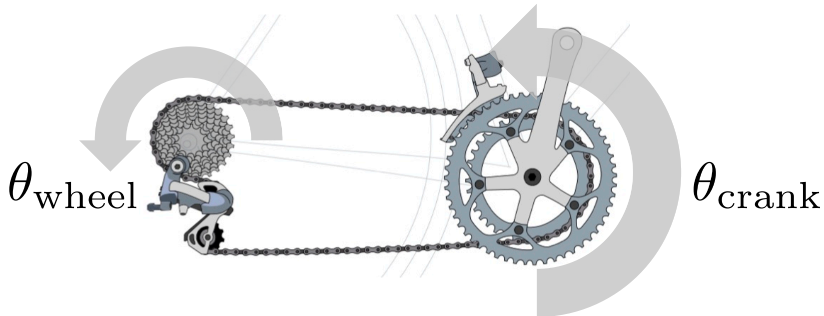


Figure 2.33: Bicycle gear train with crank and wheel gears described by θ_{crank} and θ_{wheel}, respectively.

(i) Do the wheel and crank sprockets satisfy the *equal tooth pitch* assumption? Justify your answer.

(ii) Write the no-slip condition for the interaction between the sprockets and the chain.

(iii) When biking uphill on a steep incline, explain which gear ratio is preferable. How does this compare to the preferable gear ratio when biking quickly on flat terrain? Explain your answer in terms of transmission of velocities and torques.

Hint: Recall that the gear ratio $= \dfrac{\text{number of teeth on the input sprocket (crank)}}{\text{number of teeth on the output sprocket (rear wheel)}}$.

Answer:

(i) *Equal Tooth Pitch Assumption:*

Yes, the wheel and crank sprockets have equal tooth pitch. This is because the teeth on both sprockets must engage the same chain links in a uniform manner.

(ii) *No-Slip Condition:* The no-slip condition requires that the linear velocity of the chain as it moves over the crank sprocket equals the linear velocity of the chain as it moves over the wheel sprocket. Mathematically, this can be expressed as:

$$r_{\text{crank}}\dot{\theta}_{\text{crank}} = r_{\text{wheel}}\dot{\theta}_{\text{wheel}} \tag{2.78}$$

where r_{crank} and r_{wheel} are the radii of the crank and wheel sprockets, respectively.

Note the difference in sign between the no-slip condition for sprockets (2.78) and the corresponding condition for gears (2.32). (In both cases, all angles are measured counterclockwise by convention.)

(iii) *Preferred Gear Ratios*

When biking on a steep uphill, a *low gear ratio* is preferable.

A low gear ratio is achieved by having few teeth on the crank sprocket and more teeth on the rear wheel sprocket. From

$$\text{Gear ratio} = \frac{\text{\# teeth on crank sprocket}}{\text{\# teeth on rear wheel sprocket}} = \frac{\dot{\theta}_{\text{wheel}}}{\dot{\theta}_{\text{crank}}}, \tag{2.79}$$

we see that a low gear ratio means $\dot{\theta}_{\text{crank}}$ is larger than $\dot{\theta}_{\text{wheel}}$, meaning the pedals rotate more compared to the wheel, with less force required per pedal stroke. Instead, when biking quickly on flat terrain, a higher gear ratio is preferable. A higher gear ratio is achieved by having more teeth on the crank sprocket and fewer teeth on the rear wheel sprocket. This setup allows the rider to cover more distance with each pedal stroke, making it easier to maintain high speeds. In this case, $\dot{\theta}_{\text{wheel}}$ becomes closer to or even larger than $\dot{\theta}_{\text{crank}}$, meaning the wheel rotates more for each pedal rotation, enabling faster motion.

▼

Notes: Many multi-speed bicycles feature a “50/34T crankset” coupled with an “11 speed 11-32 rear cassette.” These numbers means:

(i) the crankset has two “chainrings” with 50 and 34 teeth, respectively,

(ii) the rear cassette has 11 “cogs” ranging from 11 teeth up to 32 teeth (the exact number of teeth are 11/12/13/14/16/18/20/22/25/28/32).

The entries in the following table are all the possible gear ratios, calculated as the number of teeth on the front chainring divided by the number of teeth on the rear cog:

Chainring	11T	12T	13T	14T	16T	18T	20T	22T	25T	28T	32T
50T	$\frac{50}{11}=4.55$	$\frac{50}{12}=4.17$	$\frac{50}{13}=3.85$	$\frac{50}{14}=3.57$	$\frac{50}{16}=3.13$	$\frac{50}{18}=2.78$	$\frac{50}{20}=2.50$	$\frac{50}{22}=2.27$	$\frac{50}{25}=2.00$	$\frac{50}{28}=1.79$	$\frac{50}{32}=1.56$
34T	$\frac{34}{11}=3.09$	$\frac{34}{12}=2.83$	$\frac{34}{13}=2.62$	$\frac{34}{14}=2.43$	$\frac{34}{16}=2.13$	$\frac{34}{18}=1.89$	$\frac{34}{20}=1.70$	$\frac{34}{22}=1.55$	$\frac{34}{25}=1.36$	$\frac{34}{28}=1.21$	$\frac{34}{32}=1.06$

E2.11 **Gear train with three gears in contact.** Consider three gears on parallel shafts meshing each other, labeled Gear 1, Gear 2, and Gear 3. Let $\theta_1, \theta_2, \theta_3$ be their angular positions in radians, each measured counter-clockwise by convention. The numbers of teeth on the gears are n_1, n_2, and n_3, respectively.

(i) Write the no-slip constraint relating the angular velocities $\dot{\theta}_1$ and $\dot{\theta}_2$ for Gears 1 and 2 in mesh. Similarly, write the constraint for $\dot{\theta}_2$ and $\dot{\theta}_3$.

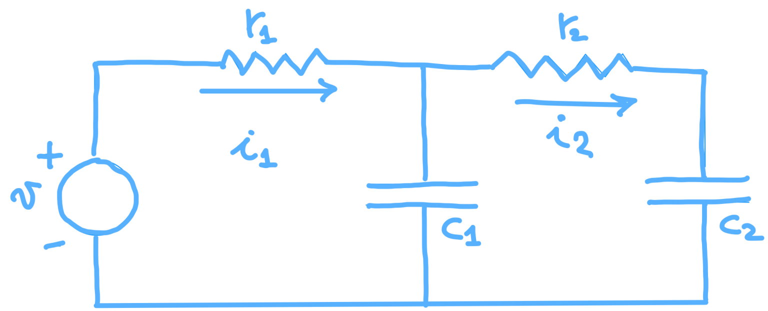
(ii) Assume Gear 1 is driven by an external torque τ_1, while Gears 2 and 3 have no external torques. Express the torques τ_2 and τ_3 acting on Gears 2 and 3 in terms of τ_1.

(iii) Note that Gears 1 and 3 rotate in the same direction. What can you conclude about the gear-ratio combination n_1, n_2, n_3? Explain briefly.

(iv) Denote the moments of inertia of the three gears by I_1, I_2, I_3. Derive the total equivalent moment of inertia I_{eq} "seen" at Gear 1 (neglect friction and other losses).

Section 2.4: Electrical systems

E2.12 **Governing equations for a cascade RC electric circuit .** Consider the following electrical circuit, composed of a left loop (with the voltage source, the resistor r_1, and the capacitor c_1) and a right loop (with capacitor c_1, resistor r_2, and capacitor c_2).



(i) Obtain two differential equations without integrals in the variables $i_1(t)$ and $i_2(t)$ and with input $v(t)$ (more precisely, the input will be $\dot{v}$).
Hint: Apply KVL to both loops and, to eliminate integrals (if you have any), differentiate with respect to time.

(ii) Write a single differential equation for the evolution of the current i_2 with $\dot{v}(t)$ as the input.
Hint: Use the two equations to eliminate i_1. Two useful preliminary stes are (1) differentiate the KVL for the right loop with respect to time, and (2) sum the two KVL equations from the previous question.

(iii) Do you obtain a first-order equation or a second-order equation?

Answer:

(i) Applying KVL and the constitutive relationships of resistors and capacitors on the left and right loop give:

$$\frac{1}{c_1}\int_0^t \big(i_1(\tau) - i_2(\tau)\big)d\tau + r_1 i_1 = v, \tag{2.80}$$

$$\frac{1}{c_1}\int_0^t \big(i_2(\tau) - i_1(\tau)\big)d\tau + r_2 i_2 + \frac{1}{c_2}\int_0^t i_2(\tau)d\tau = 0. \tag{2.81}$$

Next, to eliminate the integrals with respect to time, we differentiate:

$$\frac{1}{c_1}\big(i_1 - i_2\big) + r_1\frac{d}{dt}i_1 = \frac{d}{dt}v, \tag{2.82}$$

$$\frac{1}{c_1}\big(i_2 - i_1\big) + r_2\frac{d}{dt}i_2 + \frac{1}{c_2}i_2 = 0. \tag{2.83}$$

(ii) We differentiate the second KVL equation to obtain

$$\frac{1}{c_1}\frac{d}{dt}i_2 - \frac{1}{c_1}\frac{d}{dt}i_1 + r_2\frac{d^2}{dt^2}i_2 + \frac{1}{c_2}\frac{d}{dt}i_2 = 0 \tag{2.84}$$

and we reorganize the terms

$$r_2\frac{d^2}{dt^2}i_2 + \Big(\frac{1}{c_1} + \frac{1}{c_2}\Big)\frac{d}{dt}i_2 = \frac{1}{c_1}\frac{d}{dt}i_1. \tag{2.85}$$

We now sum the two equations (2.82) and (2.83) to obtain

$$r_1\frac{d}{dt}i_1 + r_2\frac{d}{dt}i_2 + \frac{1}{c_2}i_2 = \dot{v} \qquad \Longleftrightarrow \qquad r_1\frac{d}{dt}i_1 = -r_2\frac{d}{dt}i_2 - \frac{1}{c_2}i_2 + \dot{v}. \tag{2.86}$$

Finally, we take $\frac{d}{dt}i_1$ from equation (2.86) and plug it into equation (2.85) to obtain

$$r_2\frac{d^2}{dt^2}i_2 + \Big(\frac{1}{c_1} + \frac{1}{c_2}\Big)\frac{d}{dt}i_2 = \frac{1}{r_1c_1}\Big(-r_2\frac{d}{dt}i_2 - \frac{1}{c_2}i_2 + \dot{v}\Big). \tag{2.87}$$

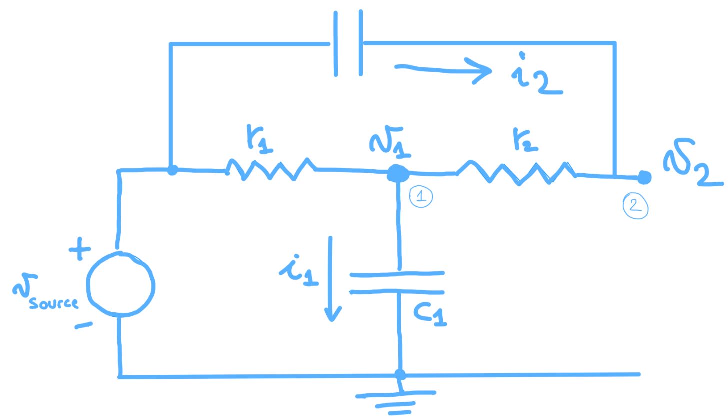
After rearranging the terms, and leaving the input v on the right hand side, we finally obtain:

$$r_2\frac{d^2}{dt^2}i_2 + \Big(\frac{1}{c_1} + \frac{1}{c_2} + \frac{r_2}{r_1c_1}\Big)\frac{d}{dt}i_2 + \frac{1}{r_1c_1c_2}i_2 = \frac{1}{r_1c_1}\dot{v} \tag{2.88}$$

(iii) Clearly, the model is second order, akin to a mass-spring-damper system.

▼

E2.13 **Governing equations for a bridge-tee electric circuit.**



(i) Write the governing equations for the voltages $v_1(t)$ and $v_2(t)$ with input $v_{\text{source}}(t)$.

(ii) Eliminate v_1 and obtain a single differential equation for $v_2(t)$ in terms of $v_{\text{source}}(t)$.

Answer:

(i) As first step, we write the KCL equation for the node :

$$\frac{v_1 - v_{\text{source}}}{r_1} + \frac{v_1 - v_2}{r_2} + c_1 \frac{dv_1}{dt} = 0 \tag{2.89}$$

Next, we write the KCL equation for the node :

$$\frac{v_2 - v_1}{r_2} + c_2 \frac{d(v_2 - v_{\text{source}})}{dt} = 0 \tag{2.90}$$

(ii) We now take v_1 from the second equation and plug it into the first. Solve the second equation for v_1:

$$v_1 = v_2 + r_2 c_2 \frac{d\big(v_2 - v_{\text{source}}\big)}{dt}. \tag{2.91}$$

Differentiate v_1 and substitute v_1 and $\dfrac{dv_1}{dt}$ into the first equation; after collecting like terms, the single governing equation for v_2 is

$$r_2 c_1 c_2 \frac{d^2 v_2}{dt^2} + \left(c_1 + c_2 + \frac{r_2 c_2}{r_1}\right) \frac{dv_2}{dt} + \frac{1}{r_1} v_2 = r_2 c_1 c_2 \frac{d^2 v_{\text{source}}}{dt^2} + \left(c_2 + \frac{r_2 c_2}{r_1}\right) \frac{dv_{\text{source}}}{dt} + \frac{1}{r_1} v_{\text{source}}. \tag{2.92}$$

▼

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