UC Santa Barbara, Department of Mechanical Engineering
ME103 Dynamical Systems. Slides by Chapter.

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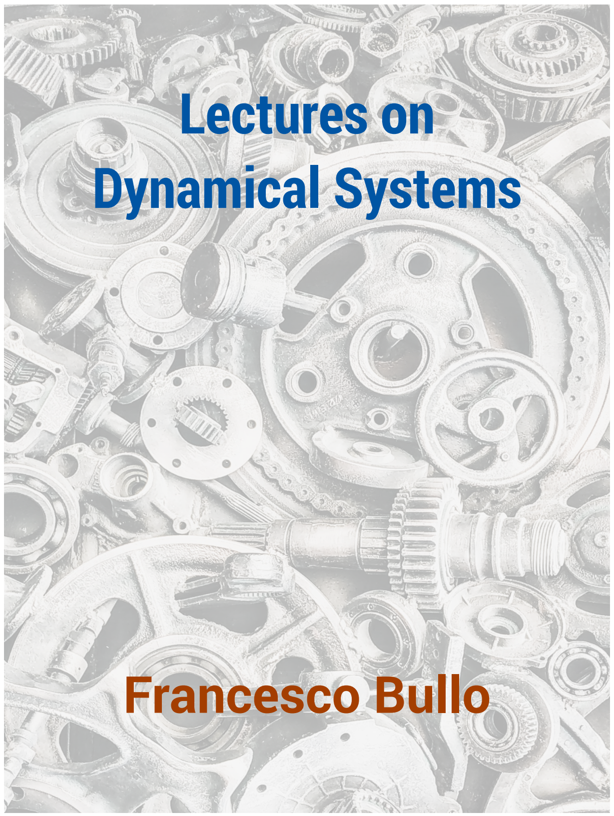
http://motion.me.ucsb.edu/ME103-Fall2024/syllabus.html



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Chapter 8

Control Systems Design

8.1 Introduction

In this chapter we use transfer functions and block diagram in the Laplace domain to reason about interconnections of dynamical systems and design of control systems. With the combined tools of block diagrams and transfer functions, we study control system design and analysis. We present strategies to design PID controllers, which are the most widely used control strategy in engineering.

8.2 Feedback diagrams in the Laplace domain

We report here Figure 7.11 and the system of equations (7.19) from Chapter 7:

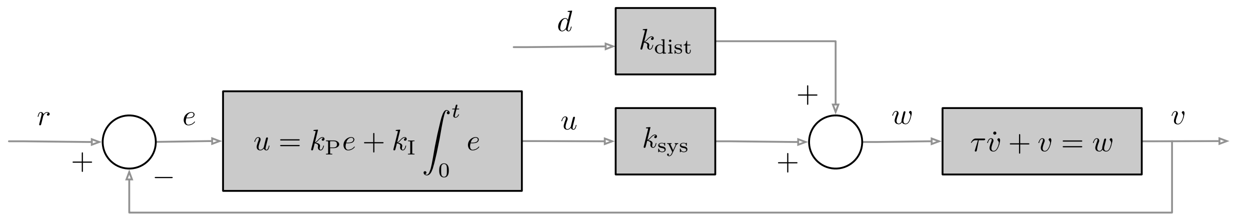

Figure 8.1: Closed-loop PI control of the dynamic car velocity model:

$$\tau \dot{v}(t) + v(t) = k_{\text{sys}} u(t) + k_{\text{dist}} d(t),$$
$$e(t) = r - v(t),$$
$$u(t) = k_{\text{P}} e(t) + k_{\text{I}} \int_0^t e(\sigma) d\sigma.$$

In this chapter we argue that it is convenient to represent feedback control systems via block diagrams in the Laplace domain. For example, in the Laplace domain all blocks are multiplications.

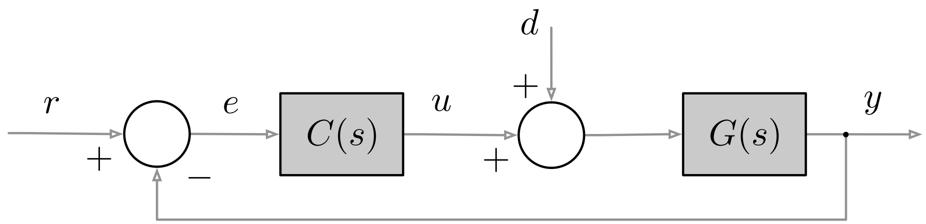

Figure 8.2: A simple feedback diagram. This block diagram is equivalent to the following equations:

$$Y(s) = G(s)(U(s) + D(s)),$$
$$E(s) = R(s) - Y(s),$$
$$U(s) = C(s)E(s),$$

where, as usual, we let $R(s)$ be a reference signal and $Y(s)$ be the system response.

The three equations in the caption of Figure 8.2 are essentially the same as the three equations in the caption of Figure 8.1. But it is substantially easier to manipulate multiplication and division by s, rather than differentiation and integration with respect to time. For example, simple manipulations show:

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)} \tag{8.1}$$

8.2.1 Block diagram algebra for interconnected transfer functions

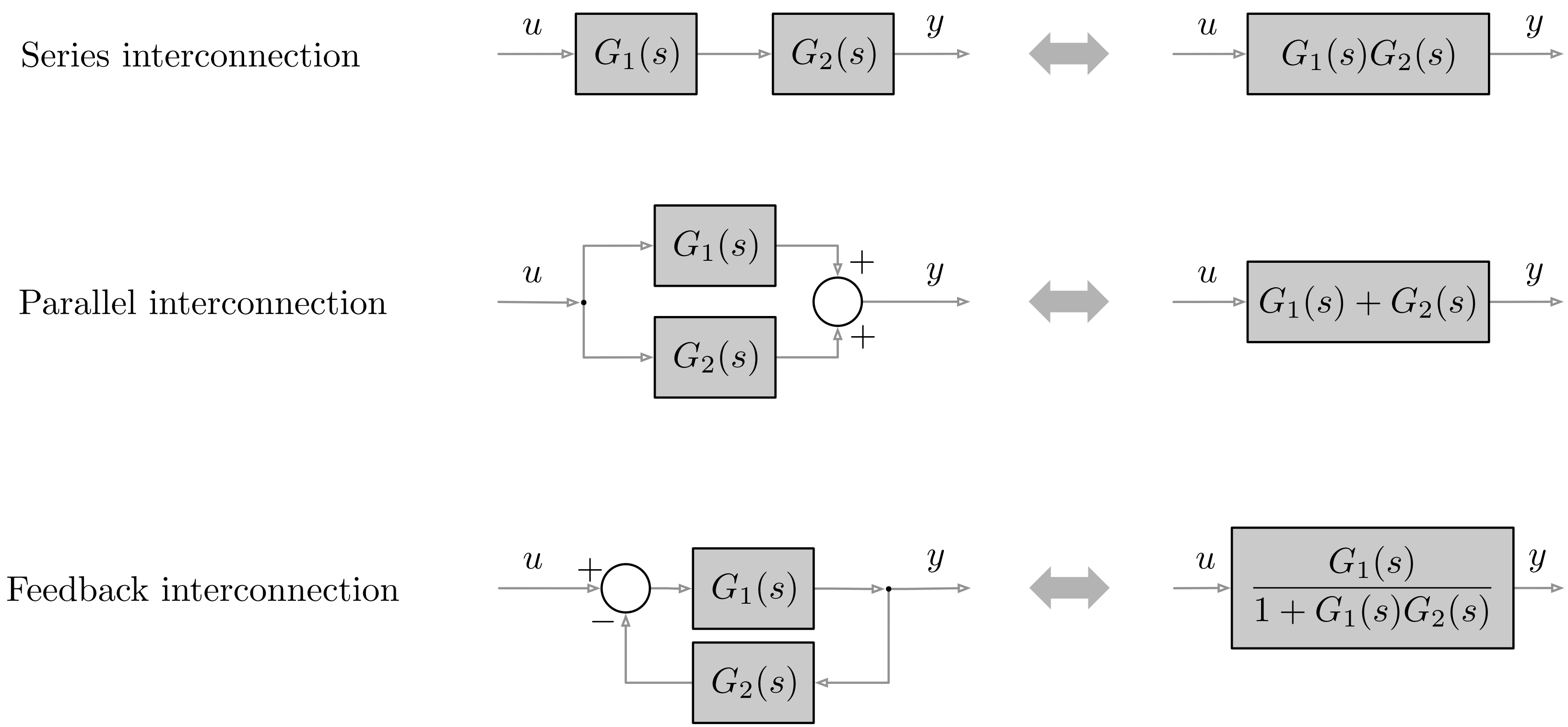

Figure 8.3: The three basic interconnections: series, parallel and feedback.

The first two results in Figure 8.3 are easy to see. To verify the formula for the feedback interconnection, we compute

$$Y = G_1(s)\big(U(s) - G_2(s)Y\big) \quad \Longrightarrow \quad \big(1 + G_1(s)G_2(s)\big)Y = G_1(s)U(s) \quad \Longrightarrow \quad Y = \frac{G_1(s)}{1 + G_1(s)G_2(s)}U(s).$$

This calculation also confirms the correctness of equation (8.1).

8.2.2 Transfer functions of PID controllers

From Chapter 7, we remember the definition of proportional control and of integral control. It is natural to consider also the notion of derivative control. Here are their definitions and their transfer functions. Given proportional gain k_P, integral gain k_I, and derivative gain k_D, we have:

proportional control (P action)	$u(t) = k_\mathsf{P} e(t)$	$\frac{U(s)}{E(s)} = k_\mathsf{P}$
integral control (I action)	$u(t) = k_\mathsf{I} \int_0^t e(\sigma) d\sigma$	$\frac{U(s)}{E(s)} = \frac{k_\mathsf{I}}{s}$
derivative control (D action)	$u(t) = k_\mathsf{D} \frac{de(t)}{dt}$	$\frac{U(s)}{E(s)} = k_\mathsf{D} s$

When PI control is applied (as we did in the previous chapter), we mean to set $u(t) = k_\mathsf{sys} e(t) + k_\mathsf{I} \int_0^t e(\sigma) d\sigma$ or, in other equivalent words, $U(s)/E(s) = k_\mathsf{sys} + k_\mathsf{I}/s$. Therefore, PI control is equivalent to implementing proportional control and integral control in parallel (not series). Therefore, when the controller is designed utilizing P, I, or D actions (or any combinations thereof), the simple feedback diagram in Figure 8.2 is updated to the one in Figure 8.4.

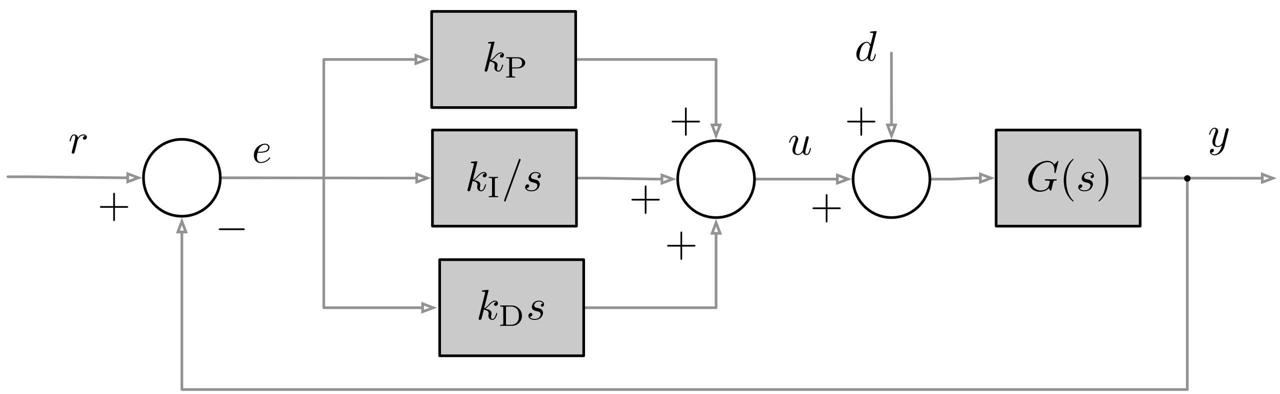

Figure 8.4: A simple feedback diagram with a PID controller. Note: A PI controller has $k_\mathsf{D} = 0$.

Interpretation of the three possible control actions in PID control

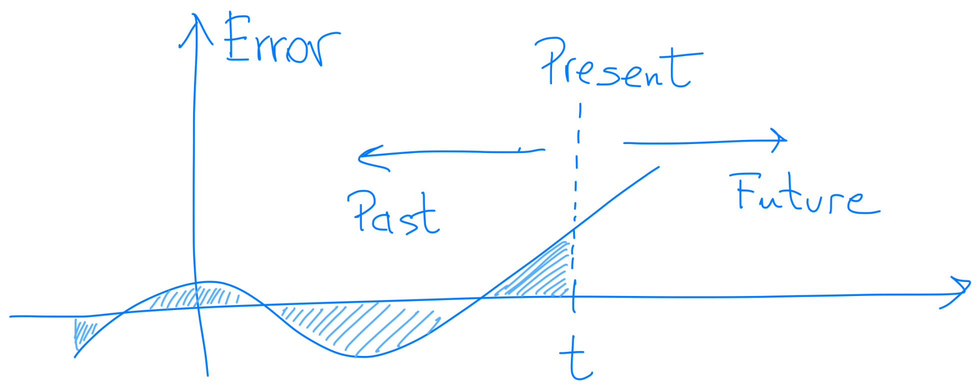

Figure 8.5: Information about present, past and future times are used in proportional, integral, and derivative control actions, respectively.

8.3 Control design strategies for first and second-order systems

In this section we consider various control designs for first and second order systems. We do not necessarily assume that the open-loop systems are stable. As in the canonical architecture in Figure 8.6, we consider PID controllers and their control gains k_P, k_I, and k_D. We consider also simplified versions of PID controllers, including P control ($k_I = k_D = 0$), PI ($k_D = 0$), PD ($k_I = 0$).

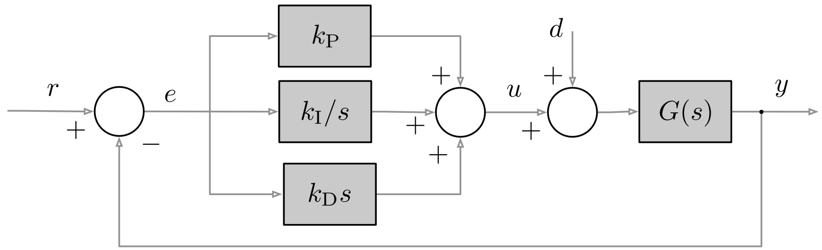

Figure 8.6: A simple feedback diagram with a PID controller.

In each case, we design controller gains by making the denominator of the closed-loop transfer function match a desirable given polynomial. This approach is called *pole placement*. The general form of a *stable polynomial of 2nd order* is

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \qquad \text{for appropriate positive parameters } \omega_n \text{and } \zeta. \tag{8.2}$$

The general form of a *stable polynomial of 3nd order* is

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)(s + \alpha\omega_n) \qquad \text{for appropriate positive parameters } \omega_n, \zeta, \text{ and } \alpha. \tag{8.3}$$

A special simplified case is when $\zeta = \alpha = 1$ and the 3rd order polynomial is

$$(s^2 + 2\omega_n s + \omega_n^2)(s + \omega_n) = s^3 + 3\omega_n s^2 + 3\omega_n^2 s + \omega_n^3 = (s + \omega_n)^3. \tag{8.4}$$

(It is also possible to use the simplified polynomial $s^3 + 2\omega_n s^2 + 2\omega_n^2 s + \omega_n^3$ with poles $-\omega_n$ and $\omega_n\big(-\frac{1}{2} \pm \frac{\sqrt{3}i}{2}\big)$.)

8.3.1 PI control of first-order systems

We consider

$$\text{first-order system} \qquad G(s) = \frac{Y(s)}{U(s)} = \frac{b}{s+a} \tag{8.5}$$

$$\text{PI controller} \qquad C(s) = k_{\mathsf{P}} + \frac{k_{\mathsf{I}}}{s} \tag{8.6}$$

The resulting closed-loop system[1] is

$$G_{\text{closed-loop}}(s) = \frac{Y(s)}{R(s)} = \frac{bk_{\mathsf{P}}s + bk_{\mathsf{I}}}{s^2 + (a + bk_{\mathsf{P}})s + bk_{\mathsf{I}}} \tag{8.7}$$

We now match the denominator to the 2nd-order polynomial (8.2). Given a desired natural frequency ω_{n} and damping ratio ζ, the control gains are:

$$\text{control gain selection} \qquad k_{\mathsf{P}} = \frac{2\zeta\omega_{\mathsf{n}} - a}{b} \qquad \text{and} \qquad k_{\mathsf{I}} = \frac{\omega_{\mathsf{n}}^2}{b} \tag{8.8}$$

Therefore, the PI controller achieves closed-loop stability, arbitrary pole placement, and exact reference tracking since $G_{\text{closed-loop}}(0) = \frac{bk_{\mathsf{I}}}{bk_{\mathsf{I}}} = 1$. Note: no assumptions are made on the sign of the system coefficient a. Hence, the PI controller with gains (8.8) stabilizes unstable first-order systems.

[1] This closed-loop transfer function gives the identical input/output relationship as we obtained in the time-domain analysis in equation (7.22) in Section 7.3.2 (where $d = 0$ and r is constant).

8.3.2 PD control of second-order systems

We consider

$$\text{second-order system} \qquad G(s) = \frac{Y(s)}{U(s)} = \frac{b}{s^2 + a_1 s + a_2} \tag{8.9}$$

$$\text{PD controller} \qquad C(s) = k_\mathrm{P} + k_\mathrm{D} s \tag{8.10}$$

The resulting closed-loop system is

$$G_{\text{closed-loop}}(s) = \frac{Y(s)}{R(s)} = \frac{bk_\mathrm{D}s + bk_\mathrm{P}}{s^2 + (a_1 + bk_\mathrm{D})s + (a_2 + bk_\mathrm{P})} \tag{8.11}$$

We now match the denominator to the 2nd-order polynomial (8.2). Given a desired natural frequency ω_n and damping ratio ζ, the control gains are:

$$\text{control gain selection} \qquad k_\mathrm{P} = \frac{\omega_\mathrm{n}^2 - a_2}{b} \qquad \text{and} \qquad k_\mathrm{D} = \frac{2\zeta\omega_\mathrm{n} - a_1}{b} \tag{8.12}$$

Therefore, the PD controller achieves closed-loop stability and arbitrary pole placement. Note: no assumptions are made on the sign of the system coefficients a_1 and a_2. Hence, the PD controller with gains (8.12) stabilizes unstable second-order systems.

However, the PD controller does not achieve exact reference tracking since:

$$G_{\text{closed-loop}}(0) = \frac{bk_\mathrm{P}}{a_2 + bk_\mathrm{P}} = 1 - \frac{a_2}{\omega_\mathrm{n}^2} \tag{8.13}$$

8.3.3 PID control of second-order systems

We consider

$$\text{second-order system} \qquad G(s) = \frac{Y(s)}{U(s)} = \frac{b}{s^2 + a_1 s + a_2} \tag{8.14}$$

$$\text{PID controller} \qquad C(s) = k_\mathsf{P} + \frac{k_\mathsf{I}}{s} + k_\mathsf{D} s \tag{8.15}$$

The resulting closed-loop system is

$$G_{\text{closed-loop}}(s) = \frac{Y(s)}{R(s)} = \frac{bk_\mathsf{D}s^2 + bk_\mathsf{P}s + bk_\mathsf{I}}{s^3 + (a_1 + bk_\mathsf{D})s^2 + (a_2 + bk)s + bk_\mathsf{I}} \tag{8.16}$$

We now match the denominator to the simplified 3nd-order polynomial (8.4). Given a desired natural frequency ω_n, the control gains are:

$$\text{control gain selection} \qquad k_\mathsf{P} = \frac{3\omega_\mathsf{n}^2 - a_2}{b} \qquad k_\mathsf{I} = \frac{\omega_\mathsf{n}^3}{b} \qquad \text{and} \qquad k_\mathsf{D} = \frac{2\omega_\mathsf{n} - a_1}{b}. \tag{8.17}$$

The PID controller achieves closed-loop stability, arbitrary pole placement, and exact reference tracking since $G_{\text{closed-loop}}(0) = 1$.

8.4 Instability lurks beneath

We discuss three cases where you think you designed a fully satisfactory controller and yet, instability lurks beneath an overly-simplified analysis.

8.4.1 Bounded control can stabilize an unstable system only locally

The bottom line is that, when the uncontrolled system is naturally unstable, then control strategies can only provide local stability because of an inevitable constraint: a maximum value on control actuation.

We consider the simplest example:

$$\dot{x} = rx + u \tag{8.18}$$

where the growth rate is positive $r > 0$ so that the system at $u = 0$ is unstable. We then assume that $u(t)$ cannot take values larger than $u_{\max}$ or smaller than $-u_{\max}$. For simplicity of notation, we set $r = u_{\max} = 1$ (the analysis of the general case is identical, but more cumbersome to follow).

As control design, we would like to design $u = -2x$, but we cannot apply a control signal larger than 1 and so the closed loop system is:

$$\dot{x} = x + \mathrm{sat}(-2x). \tag{8.19}$$

where we define the saturation nonlinearity as in Figure 8.7.

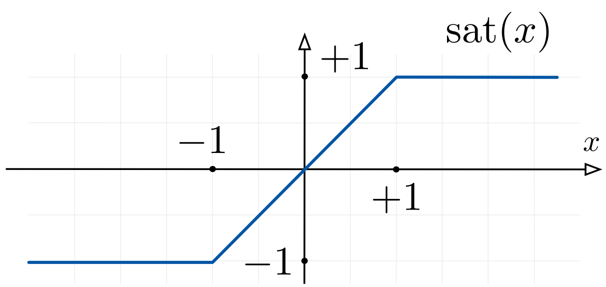

Figure 8.7: Consider the *saturation nonlinearity* $\mathrm{sat}(x) = \begin{cases} +1 & x > 1 \\ x & -1 \le x \le 1 \\ -1 & x < -1 \end{cases}$.

We now plot the right-hand side as a function of x.

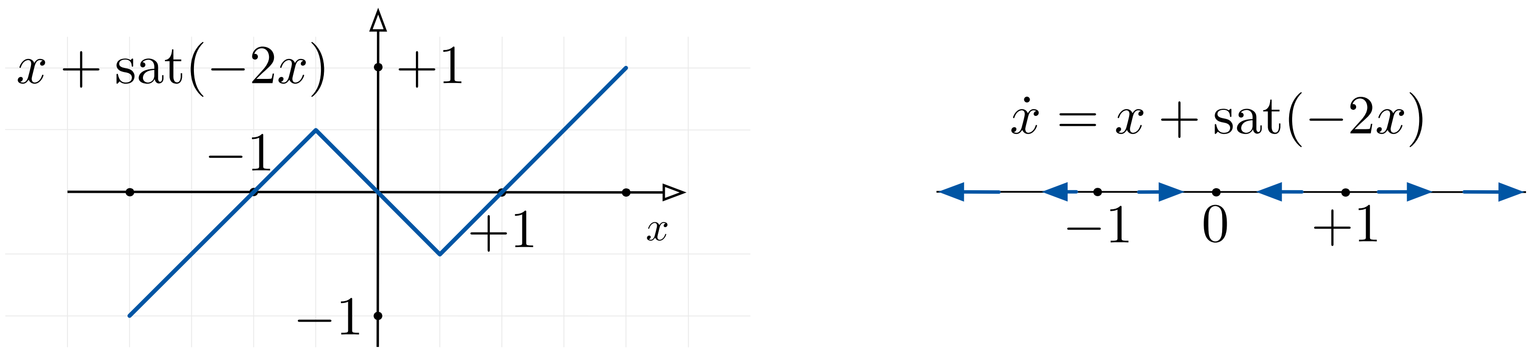

Figure 8.8: Saturation-induced instability. Left image: the right-hand side. Right image: the phase portrait.

This plot shows that the dynamical system $\dot{x} = x + \mathrm{sat}(-2x)$

(i) has three equilibrium points: -1, 0, and $+1$,

(ii) the equilibrium points 0 is stable,

(iii) the equilibrium points -1 and $+1$ are unstable,

(iv) each trajectory starting from $x(0) > 1$ goes to $+\infty$ and each trajectory starting from $x(0) < -1$ goes to $-\infty$.

Hence, the equilibrium point 0 is only *locally stable*. From this simple example we learn these lessons:

> *Unstable systems can never be operated without control systems regulating their behavior.*
>
> *However, unstable systems are fundamentally more difficult to control. Specifically, because of inevitable magnitude and rate constraints on the control signal, closed-loop systems with unstable processes are only locally stable.*

8.4.2 Control with bounded rate of change can stabilize an unstable system only locally

We now consider the same dynamical system as before

$$\dot{x} = x + u \tag{8.20}$$

but now assume that $\dot{u}(t)$ cannot take values larger than a constant $\dot{u}_{\text{max}}$, which again for simplicity take equal to 1.

It is hard to directly design a control signal u as a function of x with a limited derivative with respect to time. But there is a natural trick that simplifies the problem. Taking derivatives with respect to time, we obtain

$$\ddot{x} = \dot{x} + \dot{u}. \tag{8.21}$$

We now have an unstable second-order system with a bounded control. We would like to design $\dot{u} = -2\dot{x} - x$ because, without saturation, the resulting closed loop system would be: $\ddot{x} + \dot{x} + x = 0$, a stable underdamped second-order system. Because of the rate-saturation we obtain the closed-loop system:

$$\ddot{x} = \dot{x} + \text{sat}(-2\dot{x} - x). \tag{8.22}$$

(One detail is also important for the simulation analysis below. We also need to take into account the equality: $\dot{x}(0) = x(0) + u(0)$. In what follows we will assume that $u(0) = 0$ and focus on non-zero $x(0) = \dot{x}(0)$.)

We plot the phase portrait of the dynamics $\ddot{x} = \dot{x} + \text{sat}(-2\dot{x} - x)$ in Figure 8.9 and we realize that

(i) the equilibrium $(x, \dot{x}) = (0, 0)$ is locally stable,

(ii) each trajectory from initial condition $x(0) > 1$ diverges to $+\infty$, and

(iii) each trajectory from initial condition $x(0) < -1$ diverges to $-\infty$.

(iv) (The precise calculation of which initial conditions converge to $(0, 0)$ and which diverge is beyond this simplified discussion here.)

The lessons to be learned here are the same as in the previous example.

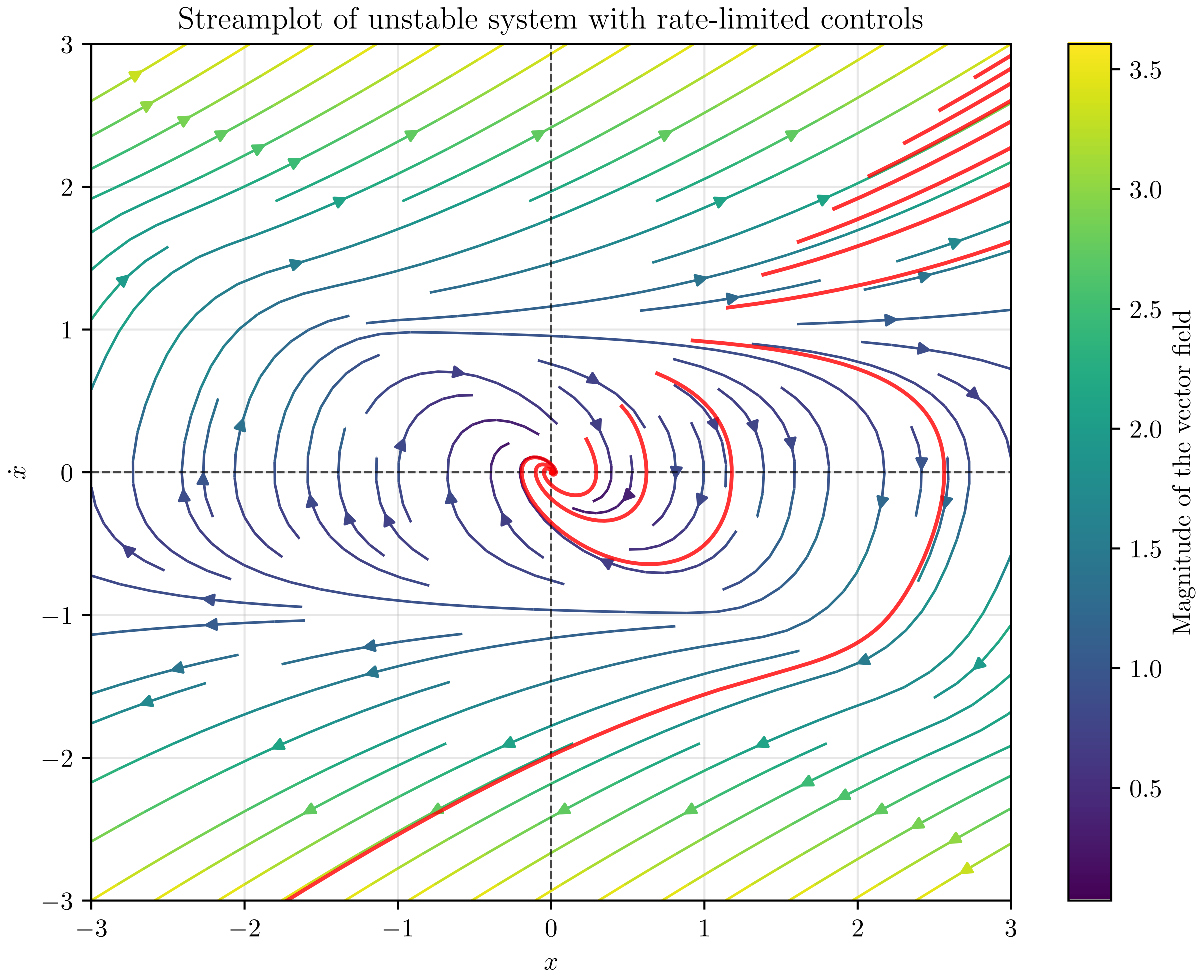

Figure 8.9: Saturation-induced instability: the rate limitation case. Note that initial conditions have $x(0) = \dot{x}(0)$ because we assume $u(0) = 0$.

8.4.3 Systems with unmodeled dynamics and destabilizing control design

This discussion is inspired by (Åström and Murray, 2021, Section 2.3). We now study (i) systems whose dynamics is not fully modeled or fully known, and (ii) the consequences of using large control gains on such systems.

(i) We assume that the real system is described by the transfer function

$$G_{\text{complete}}(s) = \frac{k_{\text{sys}}}{(s\tau + 1)(Ts + 1)} \tag{8.23}$$

with system gain k_{sys} and two time constants τ and T. This system has two poles at $-1/\tau$ and $-1/T$. We design a PI controller $C(s) = k_{\text{P}} + \frac{k_{\text{I}}}{s}$ and we compute the closed-loop transfer function for the complete system:

$$\frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{k_{\text{sys}}\left(k_{\text{I}} + k_{\text{P}}s\right)}{k_{\text{sys}}\left(k_{\text{I}} + k_{\text{P}}s\right) + s\left(Ts + 1\right)\left(s\tau + 1\right)} = \frac{k_{\text{sys}}k_{\text{P}}s + k_{\text{sys}}k_{\text{I}}}{\tau T s^3 + \left(T + \tau\right)s^2 + \left(k_{\text{P}}k_{\text{sys}} + 1\right)s + k_{\text{I}}k_{\text{sys}}} \tag{8.24}$$

Hint: This calculation is tedious, but not complex. There are five parameters: τ, T, k_{sys}, k_{P} and k_{I}.

(ii) We consider the situation where we might not know the precise value of T, but we do know that $T \ll \tau$ so that the pole at $-1/T$ is much faster than the one at $-1/\tau$. According to the approximation strategy given in Section 5.5, we decide to neglect the pole at $-1/T$ and consider the simplified dynamics:

$$G_{\text{simplified}}(s) = \frac{k_{\text{sys}}}{s\tau + 1} \tag{8.25}$$

In other words, there is some dynamics in the system that is *unmodeled*. This situation is entirely typical, one cannot expect to model exactly every single phenomenon in the system dynamics.

We now design control gains using the definitions (7.24)-(7.25) given in Section 7.3.3 for the first-order $G_{\text{simplified}}(s) = \frac{k_{\text{sys}}}{s\tau+1}$. Given desirable natural frequency ω_{n} and damping ratio ζ, we design

$$k_{\text{I}} = \frac{\omega_{\text{n}}^2 \tau}{k_{\text{sys}}}, \qquad k_{\text{P}} = \frac{2\omega_{\text{n}}\tau\zeta - 1}{k_{\text{sys}}}. \tag{8.26}$$

Substituting these values into equation (8.24), we compute the closed-loop transfer function:

$$\frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{(2\omega_{\text{n}}\tau\zeta - 1)\, s + \omega_{\text{n}}^2\tau}{\tau T s^3 + (T + \tau)\, s^2 + 2\omega_{\text{n}}\tau\zeta s + \omega_{\text{n}}^2\tau} \tag{8.27}$$

(iii) We now study when this transfer function is stable. The denominator is a third-order polynomial with positive coefficients. To determine if all poles of the transfer functions are stable, we apply the condition (8.54) from the Appendix 8.6. The closed-loop system is stable when

$$(T+\tau)\,2\omega_{\mathsf{n}}\tau\zeta > \tau T\omega_{\mathsf{n}}^2\tau \tag{8.28}$$

$$\iff \qquad \omega_{\mathsf{n}} < 2\zeta\frac{1+T/\tau}{T}. \tag{8.29}$$

This is a remarkable result: The closed-loop system is not always stable! The real system is stable, the simplified system is stable, but the closed-loop system may now be unstable. Given the two time constants and the damping ratio, a sufficiently large value of ω_{n} leads to a closed-loop system that is unstable.

In other words, using the definitions (7.24)-(7.25) in Section 7.3.3 assuming that the plan is $G(s) = \frac{k_{\mathsf{sys}}}{s\tau+1}$ must be done with care when selecting control gains.

(iv) We now provide a numerical example of this instability. We consider

$$\text{system:} \quad \tau = 1, \quad T = .1, \quad k_{\mathsf{sys}} = 1, \tag{8.30}$$

This means that $T = \tau/10$, which is consistent with the assumption $T \ll \tau$. Next, we select $\zeta = 0.4$ (as we discussed in Section 5.4.4, such a value of ζ leads to a fast response with 25% overshoot). Then, the closed-loop system is stable for

$$\omega_{\mathsf{n}} < 2\zeta \frac{1 + T/\tau}{T} = 0.88 \frac{1}{T} = 8.8 \tag{8.31}$$

We design our controller with a natural frequency larger than this threshold:

$$\text{desired controller:} \quad \zeta = 0.4, \quad \omega_{\mathsf{n}} = 10, \tag{8.32}$$

$$\text{PI gains :} \quad k_{\mathsf{I}} = 100, \quad k_{\mathsf{P}} = 7. \tag{8.33}$$

These parameters lead to

$$\frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{7s + 100}{0.1s^3 + 1.1s^2 + 8s + 100} \tag{8.34}$$

which has the three poles (computed numerically) at:

$$s_1 = -11.6$$
$$s_2 = 0.28 \pm \mathrm{i}9.3$$

As predicted, the closed-loop system is unstable!

From this simple example we learn these lessons:

The choice of natural frequency ω_n for the closed-loop system need to be informed by an understanding of what are the time constant of unknown or unmodeled dynamics.

8.5 Block diagram of a control system with multiple inputs and outputs

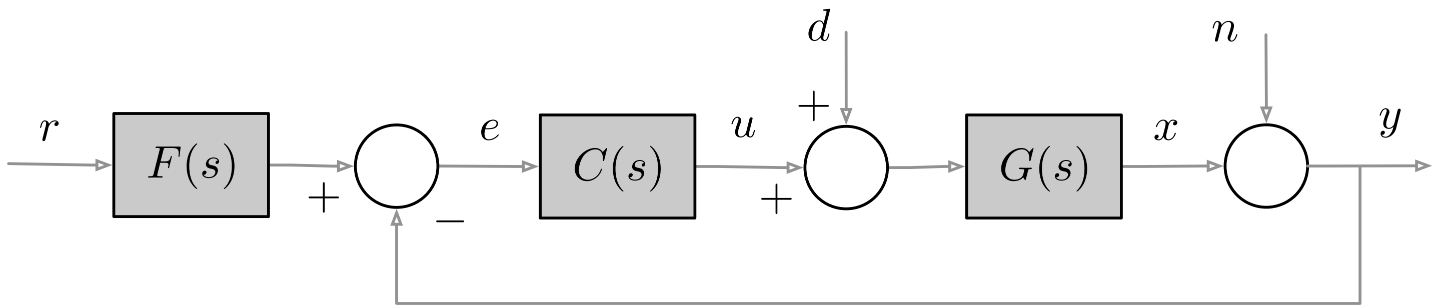

Figure 8.10: The standard block diagram of the canonical/basic control system

Signals:

- r is the reference input,
- $e = r - y$ is the error signal,
- u is the control action (the output of the controller),
- d is the load disturbance signal (pushes state away from desired state),
- x is the system state,
- n is the measurement noise (corrupts information about the state x), and
- y is the system output.

Blocks (described by transfer functions):

- $G(s)$ is the system or process,
- $C(s)$ is the feedback controller
- $F(s)$ is the feedforward controller

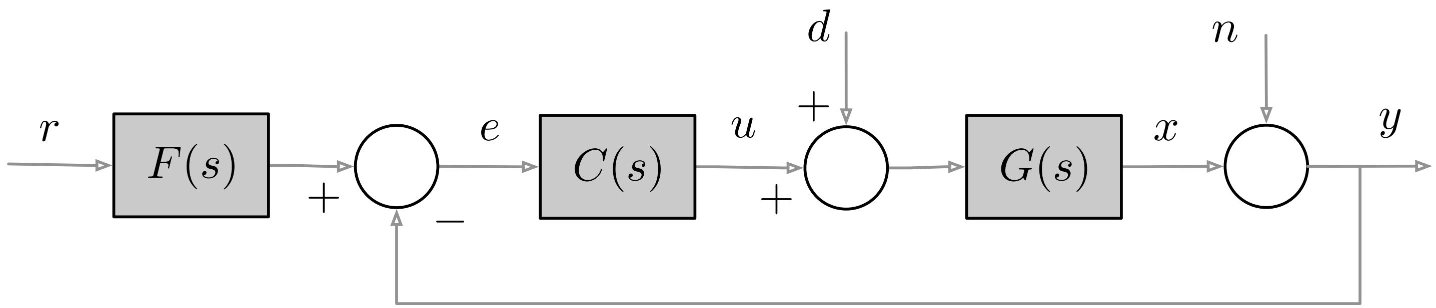

Figure 8.11: Canonical control system

Control objectives: Design the feedforward controller F and the feedback controller C to achieve:

(Objective 1) *reference tracking*: make the state x follow the reference signal r,

(Objective 2) *load disturbance rejection*: reduce the effect of the load disturbance d (on all signals),

(Objective 3) *measurement noise rejection*: reduce the effect of the measurement noise n (on all signals), and

(Objective 4) *sensitivity reduction*: make the closed-loop system insensitive to parameter variations.

Typically, *2 degree-of-freedom (dof) control design*:

- design C to achieve disturbance rejection (Objective 2), noise rejection (Objective 3), and sensitivity reduction (Objective 4).
- design F to achieve reference tracking (Objective 1).

There are many equivalent block diagrams that implement 2 dof control design.

8.5.1 The superposition property of linear systems

Consider a block diagram where two input signals u_1 and u_2 are summed into an input signal u and then fed into a multiplicative block with constant k. Then the output y satisfies:

$$y = ku = k(u_1 + u_2) = ku_1 + ku_2 \tag{8.35}$$

If we were to feed into the block the two inputs separately, we would obtain

$$y_1 = ku_1 \qquad \text{and} \qquad y_2 = ku_2 \tag{8.36}$$

The multiplicative block is linear and therefore it satisfies the *superposition property*, namely:

$$y = y_1 + y_2 \tag{8.37}$$

The interpretation is as follows: the effect due to the sum of two causes is the sum of the two individual isolated effects.

For linear dynamical systems, the superposition property is that the response of a linear system to a sum of inputs is equal to the sum of the individual responses to each input.

8.5.2 The multiple transfer functions relevant in control system

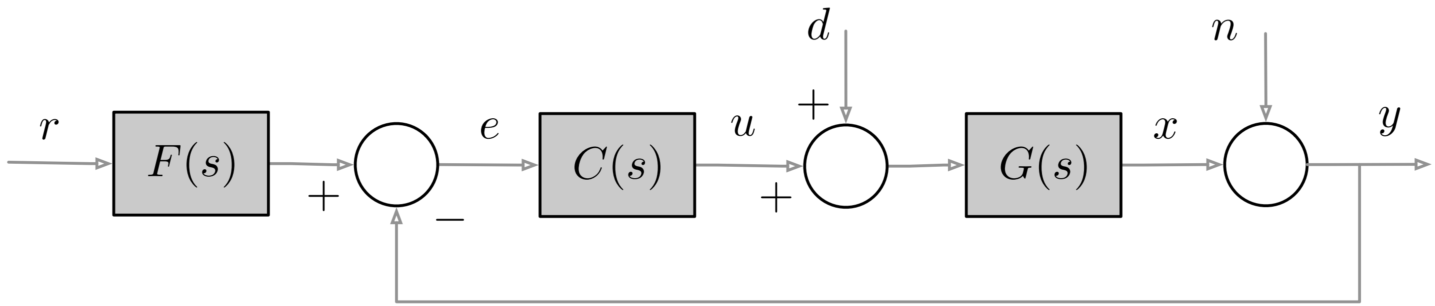

Figure 8.12: Canonical control system

It is possible (and easy) to compute all relations between the inputs signals r, d, n and the signals of interest x, y, u. For example, considering one input at a time (set $r = 0$ and $n = 0$), one can see

$$X(s) \;=\; \frac{G(s)}{1 + G(s)C(s)} D(s) \tag{8.38}$$

Note that

- we call $L(s) = G(s)C(s)$ the *loop transfer function* (or open-loop transfer function),
- for $X(s)/D(s)$, but also for all other possible pairs:

$$\text{closed-loop transfer function from } D \text{ to } X \;=\; \frac{\text{open-loop transfer function from } D \text{ to } X}{1 + L(s)} = \frac{G(s)}{1 + G(s)C(s)}. \tag{8.39}$$

Keeping $r = 0$ and dropping the s variable, we can compute all transfer functions relevant for the design of C (that is, disturbance rejection (Objective 2), noise rejection (Objective 3), and sensitivity reduction (Objective 4)):

$$X = \frac{G}{1+GC} D - \frac{GC}{1+GC} N \tag{8.40}$$

$$Y = \frac{G}{1+GC} D + \frac{1}{1+GC} N \tag{8.41}$$

$$U = -\frac{GC}{1+GC} D - \frac{C}{1+GC} N \tag{8.42}$$

Note that there are only four distinct transfer functions, since two are repeated. Adopting nomenclature from (Åström and Murray, 2021), the *Gang of Four* transfer functions are:

sensitivity: $$S(s) := \frac{Y(s)}{N(s)} = \frac{1}{1+G(s)C(s)}$$

complementary sensitivity: $$T(s) := -\frac{X(s)}{N(s)} = -\frac{U(s)}{D(s)} = \frac{G(s)C(s)}{1+G(s)C(s)}$$

load→state sensitivity: $$\frac{X(s)}{D(s)} = \frac{Y(s)}{D(s)} = \frac{G(s)}{1+G(s)C(s)}$$

noise→control sensitivity: $$-\frac{U(s)}{N(s)} = \frac{C(s)}{1+G(s)C(s)}.$$

For the design of the feedforward controller F and the system response to a reference input r, we are interested also in the transfer functions from r to x, y, u. We can compute:

$$X = \frac{G}{1+GC}D - \frac{GC}{1+GC}N + \frac{GCF}{1+GC}R \tag{8.43}$$

$$Y = \frac{G}{1+GC}D + \frac{1}{1+GC}N + \frac{GCF}{1+GC}R \tag{8.44}$$

$$U = -\frac{GC}{1+GC}D - \frac{C}{1+GC}N + \frac{CF}{1+GC}R \tag{8.45}$$

Note that there are two additional transfer functions:

reference→control sensitivity: $$\frac{U(s)}{R(s)} = \frac{C(s)F(s)}{1+G(s)C(s)}$$

reference→state sensitivity: $$\frac{X(s)}{R(s)} = \frac{G(s)C(s)F(s)}{1+G(s)C(s)}$$

Remarks 8.1. *Some remarks are in order.*

(i) *A control engineer needs to analyze all* 4 *transfer functions to fully understand a closed-loop system*

(ii) *Additionally, if a feedforward controller is also included, then the transfer functions are* 6.

(iii) *Bottom line: look at step transient response and frequency response of all functions.*

8.5.3 Control design examples: PI control of first-order systems

A first-order process with time constant τ

$$G(s) = \frac{1}{\tau s + 1} \tag{8.46}$$

A PI feedback controller with gains k_{P} and k_{I}

$$C(s) = k_{\mathsf{P}} + \frac{k_{\mathsf{I}}}{s} = \frac{k_{\mathsf{s}} s + k_{\mathsf{I}}}{s} \tag{8.47}$$

Note the loop transfer function is $L(s) = \dfrac{k_{\mathsf{s}} s + k_{\mathsf{I}}}{s(\tau s + 1)}$. We compute the gang of four:

Sensitivity Y/N:
$$\frac{1}{1 + G(s)C(s)} = \frac{1}{1 + \frac{k_{\mathsf{P}} s + k_{\mathsf{I}}}{s(\tau s + 1)}} = \frac{s(\tau s + 1)}{\tau s^2 + (1 + k_{\mathsf{P}})s + k_{\mathsf{I}}}$$

Complementary Sensitivity $-X/N$:
$$\frac{G(s)C(s)}{1 + G(s)C(s)} = \frac{\frac{k_{\mathsf{P}} s + k_{\mathsf{I}}}{s(\tau s + 1)}}{1 + \frac{k_{\mathsf{P}} s + k_{\mathsf{I}}}{s(\tau s + 1)}} = \frac{k_{\mathsf{P}} s + k_{\mathsf{I}}}{\tau s^2 + (1 + k_{\mathsf{P}})s + k_{\mathsf{I}}}$$

Load→State X/D Sensitivity:
$$\frac{G(s)}{1 + G(s)C(s)} = \frac{\frac{1}{\tau s + 1}}{1 + \frac{k_{\mathsf{P}} s + k_{\mathsf{I}}}{s(\tau s + 1)}} = \frac{s}{\tau s^2 + (1 + k_{\mathsf{P}})s + k_{\mathsf{I}}}$$

Noise→Control U/N Sensitivity:
$$\frac{C(s)}{1 + G(s)C(s)} = \frac{\frac{k_{\mathsf{P}} s + k_{\mathsf{I}}}{s}}{1 + \frac{k_{\mathsf{P}} s + k_{\mathsf{I}}}{s(\tau s + 1)}} = \frac{(\tau s + 1)(k_{\mathsf{P}} s + k_{\mathsf{I}})}{\tau s^2 + (1 + k_{\mathsf{P}})s + k_{\mathsf{I}}}$$

After simplifications, it can be seen that the denominators are identical and of second order.

In the plots, we select $\tau = 2.0$, $k_{\mathsf{I}} = 0.5$, and $k_{\mathsf{P}} = \{0.01, 0.1, 1, 10, 100\}$.

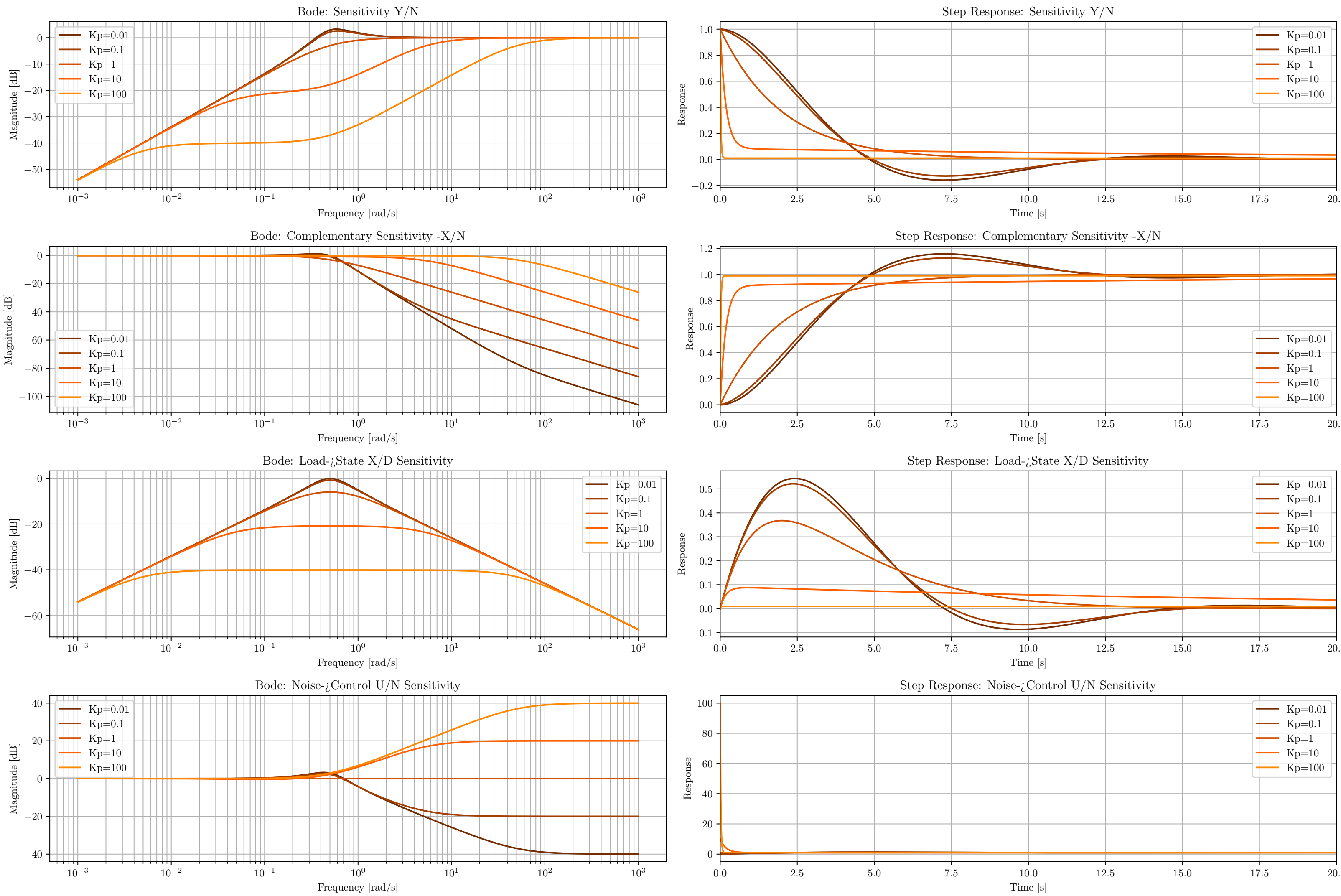

Figure 8.13: A first order system controlled by a PI controller: the step response (left four panels) and the Bode magnitude frequency (right four panels) for each transfer function in the gang of four. For the selected parameter values, too low proportional gain k_P leads to large oscillations in the top three step responses and too low k_P leads to high-frequency noise amplification in the control action.

8.5.4 Control design examples: Right-halfplane zero-pole cancellation (some more lurking danger)

Consider an unstable first-order process:

$$G(s) = \frac{1}{s-1} \tag{8.48}$$

feedback controller

$$C(s) = 1 - \frac{1}{s} = \frac{s-1}{s} \tag{8.49}$$

With such a design, a zero-pole cancellation occurs in the loop function:

$$L(s) = G(s)C(s) = \frac{1}{s} \tag{8.50}$$

Therefore, it appears that we get very nice behavior. Take for example $F = 1$ and calculate

$$\frac{Y(s)}{R(s)} = \frac{G(s)C(s)F(s)}{1+G(s)C(s)} = \frac{1/s}{1+1/s} = \frac{1}{1+s} \tag{8.51}$$

Specifically, since $\frac{1}{1+s}\Big|_{s=0} = 1$, a reference signal $r(t) = \mathbf{1}(t)$, causes a steady state response $y_{\text{steady-state}}(t) = \mathbf{1}(t)$.

However, a more careful analysis is well warranted and revealing. Consider the load→state sensitivity:

$$\frac{X(s)}{D(s)} = \frac{G(s)}{1+G(s)C(s)} = \frac{\frac{1}{s-1}}{1+\frac{1}{s}} = \frac{s}{(s+1)(s-1)} \tag{8.52}$$

This means that this transfer function is unstable!

To explain a second potentially-surprising result in the next slide, we compute also the noise→control sensitivity:

$$\frac{C(s)}{1+G(s)C(s)} = \frac{\frac{s-1}{s}}{1+\frac{1}{s}} = \frac{s-1}{s+1} \quad \Longrightarrow \quad \left|\frac{C(\mathrm{i}\omega)}{1+G(\mathrm{i}\omega)C(\mathrm{i}\omega)}\right| = 1. \tag{8.53}$$

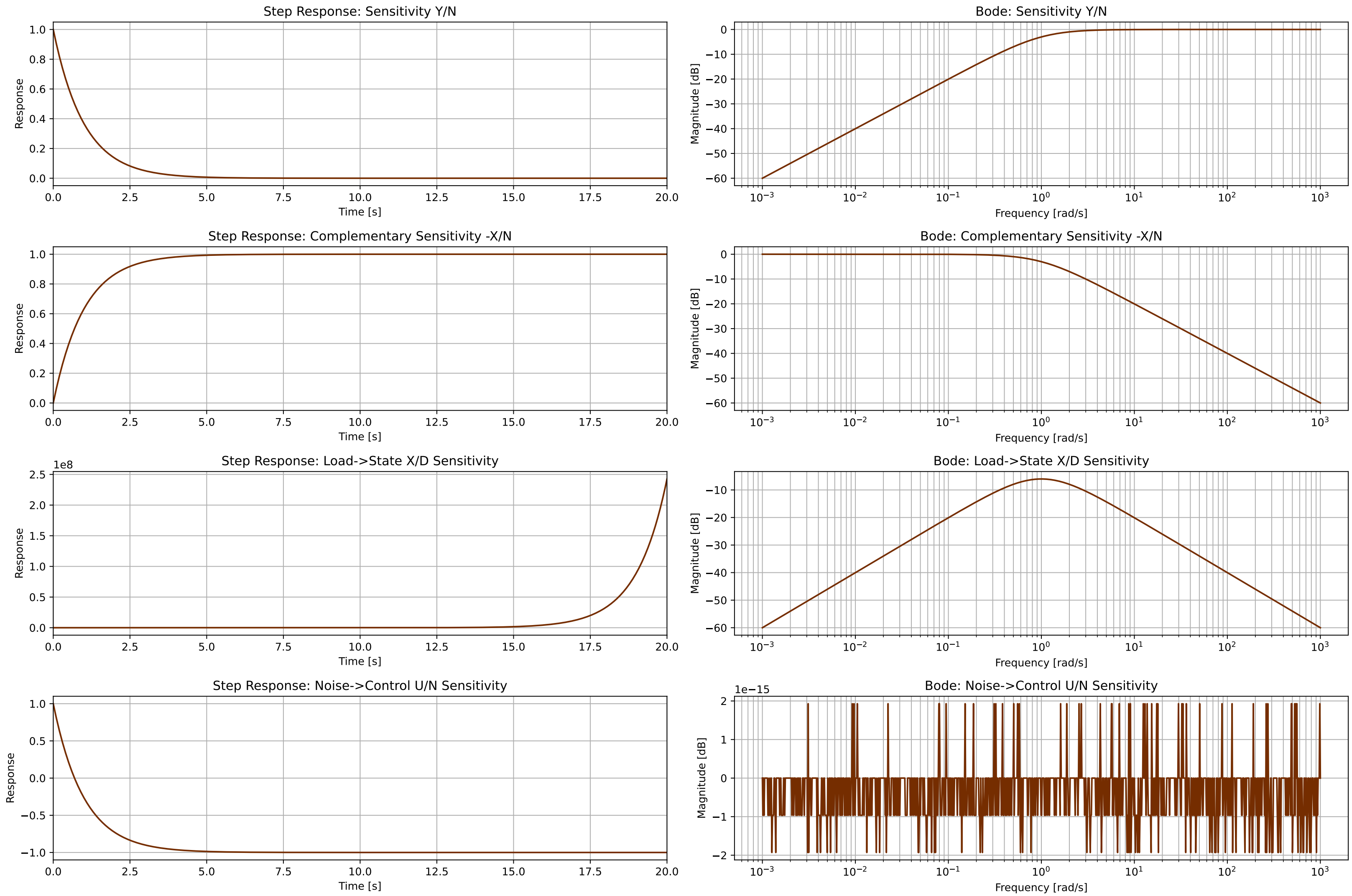

Figure 8.14: An unstable first order system controlled by a PI controller (with negative k_I gain) with parameters achieving an unstable zero-pole cancellation: the Bode magnitude frequency response (right four panels) and the step response (left four panels) for each transfer function in the gang of four. Note that the state $x(t)$ response to a step input in the load disturbance (plotted in the third panel on the right) is unbounded!

8.6 Appendix: Stability tests for low-order transfer functions

The *Routh-Hurwitz stability criterion* provides a method to determine the stability of a transfer function $G(s)$ by examining the signs and values of the coefficients of the denominator of $G(s)$, that is, its characteristic polynomial. We refer to (DiStefano et al., 1997) for a complete treatment and here we focus on low-order transfer functions.

The criterion (which ensures that all poles of $G(s)$ are in the left-half plane) is summarized for first, second, and third-order polynomials as follows:

(i) A first-order polynomial

$$P(s) = a_1 s + a_0,$$

has a zero with strictly negative real part if $a_0 > 0$ and $a_1 > 0$.

(ii) A second-order polynomial

$$P(s) = a_2 s^2 + a_1 s + a_0$$

has zeros with strictly negative real part if $a_0 > 0$, $a_1 > 0$, and $a_2 > 0$.

(iii) A third-order polynomial

$$P(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

has zeros with strictly negative real part if $a_0 > 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, and

$$a_2 a_1 - a_3 a_0 > 0. \tag{8.54}$$

Bibliography

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