UC Santa Barbara, Department of Mechanical Engineering
ME103 Dynamical Systems. Slides by Chapter.

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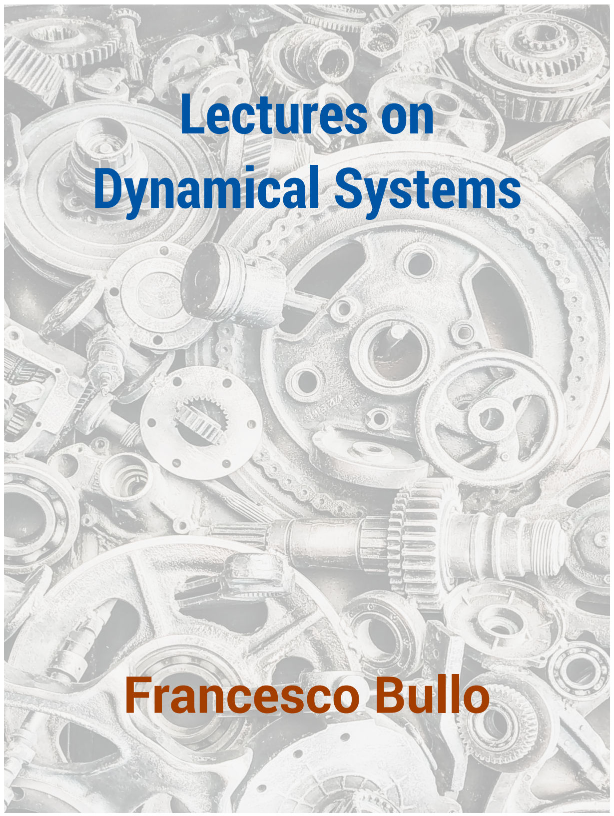
http://motion.me.ucsb.edu/ME103-Fall2024/syllabus.html



Contents

Chapter 6

Frequency Response, Resonances, and Beats in Dynamical Systems

6.1 Introduction

The *frequency response* of a dynamical system describes how the system reacts to sinusoidal inputs of varying frequencies. This concept is central to systems engineering, control theory, and signal processing. *Plotting the magnitude and angular frequency* response helps engineers analyze and design systems subject to oscillatory or periodic inputs.

In mechanical structures like bridges and buildings, frequency response analysis is essential for assessing structural integrity against environmental forces like wind or seismic activities, helping to prevent *destructive resonance phenomena*.

Similarly, in rotating machinery such as turbines and engines, the frequency response helps identify critical speeds to avoid resonance, which can cause *excessive vibrations*, mechanical failures, and reduced efficiency. The frequency response also aids in understanding complex phenomena like *beats*, where interacting oscillations create new vibration patterns.

These videos are strongly recommended:

- resonance in tuning forks,
- beating in tuning forks,
- explanations and useful animations on understanding vibration and resonance (20 min), including a discussion of vibrations in multiple degree of freedom mechanical systems,
- more information about interference beats, for music and guitar lovers.

For further references, see

- the role of aerodynamic flutter in the 1940 Tacoma bridge collapse (less than 1 minute); read about it at wikipedia:Tacoma Narrows Bridge, and
- synchronizing oscillators and metronomes.

6.2 The frequency response and the resonance phenomenon

We consider the problem illustrated in Figure 6.1 below, where a linear time-invariant system with transfer function $G(s)$ is subject to a sinusoidal input with unit magnitude and frequency $\omega > 0$.

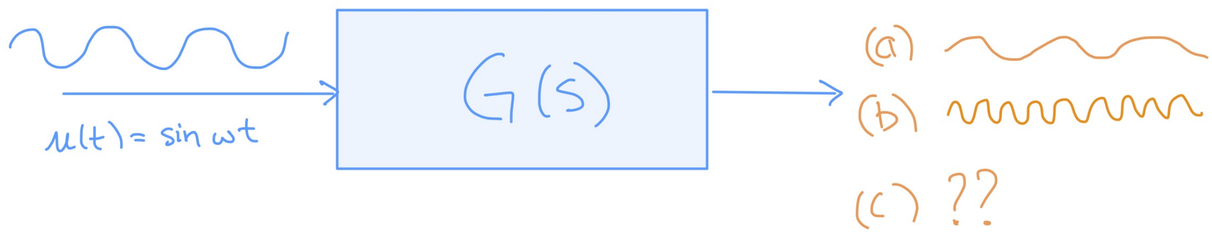


Figure 6.1: System subject to a unit-magnitude sinusoidal input: The main result is that, if the input is a sine wave, so is the output! While the frequency of the output steady state oscillation is the same as the frequency of the input, the magnitude and phase of the output are determined by the frequency response.

6.2.1 The frequency response formula

The main result of this chapter can be stated in one equation.

The steady state response of a stable linear system to a unit-magnitude sinusoidal input satisfies

$$u(t) = \sin(\omega t) \qquad \Longrightarrow \qquad y_{\text{steady-state}}(t) = \big|G(\mathrm{i}\omega)\big| \sin\Big(\omega t + \arg\big(G(\mathrm{i}\omega)\big)\Big) \tag{6.1}$$

where

- given a complex number z, $|z|$ is its magnitude and $\arg(z)$ is its argument or angle,
- $y_{\text{steady-state}}(t)$ is the *steady state solution* of the system, that is, the solution after all exponentially decaying signals have vanished,
- the function $G(\mathrm{i}\omega)$ is called the *frequency response* (also know as *sinusoidal transfer function*) and is equal to the transfer function $G(s)$ evaluated at $s = \mathrm{i}\omega$, that is, evaluated on the imaginary axis.

The correctness of equation (6.1) is studied in Appendix 6.4 via inverse Laplace transforms and partial fraction expansions. The frequency response of the system is a complex number. Given a sinusoidal input at frequency ω,

(i) the *magnitude frequency response* $\big|G(\mathrm{i}\omega)\big|$ determines the amplification (or attenuation) of the output sinusoidal signal as compared with the input; and

(ii) the *angular frequency response* $\arg\big(G(\mathrm{i}\omega)\big)$ determines the phase shift of the output sinusoidal signal as compared with the input.

6.2.2 First-order systems

Recall from Section 5.3 that the transfer function of a first-order system $\tau\dot{y} + y = u$ is

$$\frac{Y(s)}{U(s)} = G_{\text{first-order}}(s) = \frac{1}{\tau s + 1}. \tag{6.2}$$

Hence, the frequency response function of a first-order system is

$$G_{\text{first-order}}(\mathrm{i}\omega) = \frac{1}{\mathrm{i}\tau\omega + 1}. \tag{6.3}$$

The magnitude frequency response is:

$$\left|G_{\text{first-order}}(\mathrm{i}\omega)\right| = \frac{1}{\sqrt{\tau^2\omega^2 + 1}} \tag{6.4}$$

and the angular frequency response is

$$\arg\big(G_{\text{first-order}}(\mathrm{i}\omega)\big) = -\arctan(\tau\omega) \tag{6.5}$$

In summary, the steady-state response of a first-order system to a unit-magnitude sinusoidal input $u(t) = \sin(\omega t)$ is

$$y_{\text{steady-state}}(t) \;=\; \frac{1}{\sqrt{\tau^2\omega^2 + 1}} \sin\big(\omega t - \arctan(\tau\omega)\big). \tag{6.6}$$

(i) For low frequencies $\omega \ll \frac{1}{\tau}$, we have $\sqrt{\tau^2\omega^2 + 1} \approx 1$ and the output amplitude closely approximates the unit input amplitude in the steady-state response; and

(ii) for high frequencies $\omega \gg \frac{1}{\tau}$, we have $\sqrt{\tau^2\omega^2 + 1} \approx \tau\omega$ and the output amplitude is attenuated, approaching approximately $1/(\tau\omega)$ (meaning that the amplitude decreases as $1/\omega$).

We plot the magnitude frequency response of a first-order system in Figure 6.2.

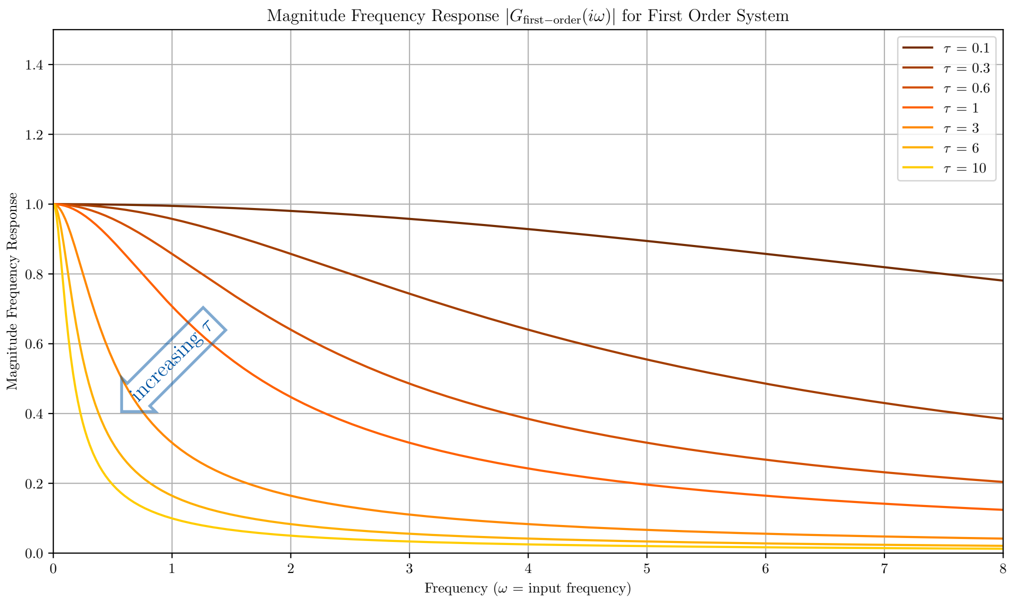


Figure 6.2: The magnitude frequency response $|G_{\text{first-order}}(\mathrm{i}\omega)|$ of a first-order system, as in equation (6.4).
On the horizontal axis, the variable is the frequency ω of the sinusoidal input.
Python code available at `frequencyresponse-firstorder.py`

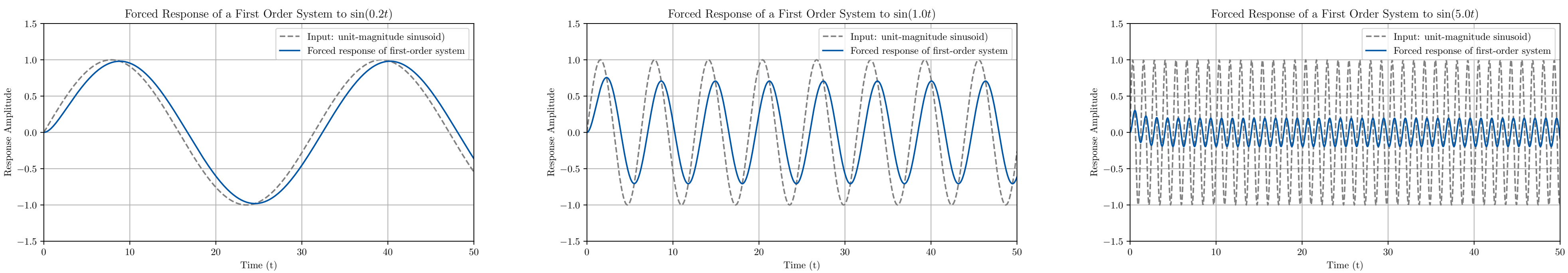


Figure 6.3: The forced response (blue solid line) of a first order system subject to a unit-magnitude sinusoidal forcing (gray dashed lines) for three values of the frequency $\omega = 0.2, 1.0, 5.0$ and $\tau = 1$.
As ω increases, the magnitude of the response decreases and the phase delay increases.

6.2.3 Second-order systems and the resonance phenomenon

Recall from Section 5.4 that the transfer function of a second-order system $\ddot{y}(t) + 2\zeta\omega_\mathrm{n}\dot{y}(t) + \omega_\mathrm{n}^2 y(t) = \omega_\mathrm{n}^2 u(t)$ is

$$G_{\text{second-order}}(s) = \frac{Y(s)}{U(s)} = \frac{\omega_\mathrm{n}^2}{s^2 + 2\zeta\omega_\mathrm{n}s + \omega_\mathrm{n}^2}. \tag{6.7}$$

We now introduce the *input frequency* ω (also called *driving frequency*). Note: In the frequency response of second order systems (especially, underdamped systems) there are two relevant frequencies: ω is the frequency of the input sinusoid and ω_n is the natural frequency of the system. We now compute the frequency response:

$$G_{\text{second-order}}(\mathrm{i}\omega) = \frac{\omega_\mathrm{n}^2}{(\omega_\mathrm{n}^2 - \omega^2) + \mathrm{i}(2\zeta\omega_\mathrm{n}\omega)} = \frac{1}{(1 - \omega^2/\omega_\mathrm{n}^2) + \mathrm{i}(2\zeta\omega/\omega_\mathrm{n})} \tag{6.8}$$

and the magnitude frequency response:

$$\left|G_{\text{second-order}}(\mathrm{i}\omega)\right| = \frac{1}{\sqrt{(1 - (\omega/\omega_\mathrm{n})^2)^2 + (2\zeta\omega/\omega_\mathrm{n})^2}} \tag{6.9}$$

(i) For low frequencies $\omega \ll \omega_\mathrm{n}$, we have $\sqrt{(1 - (\omega/\omega_\mathrm{n})^2)^2 + (2\zeta\omega/\omega_\mathrm{n})^2} \approx 1$ and the output amplitude closely approximates the unit input amplitude in the steady-state response;

(ii) for high frequencies $\omega \gg \omega_\mathrm{n}$, we have $\sqrt{(1 - (\omega/\omega_\mathrm{n})^2)^2 + (2\zeta\omega/\omega_\mathrm{n})^2} \approx (\omega/\omega_\mathrm{n})^2$ and the output amplitude is attenuated, approaching approximately $1/(\omega/\omega_\mathrm{n})^2$ (meaning that the amplitude decreases as $1/\omega^2$); and

(iii) for $\omega = \omega_\mathrm{n}$, we have $\sqrt{(1 - (\omega/\omega_\mathrm{n})^2)^2 + (2\zeta\omega/\omega_\mathrm{n})^2} = 2\zeta$ and the output amplitude is $1/(2\zeta)$.

We plot this magnitude frequency response in Figure 6.4.

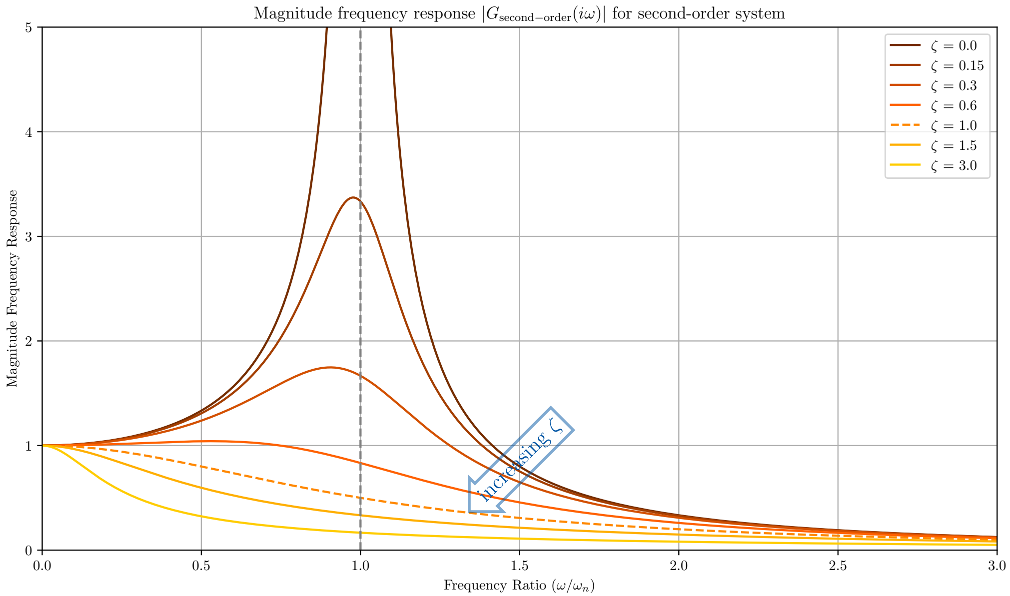


Figure 6.4: The magnitude frequency response of a second-order system, as in equation (6.9).
On the horizontal axis, the variable is the frequency ratio ω/ω_n, where ω is the frequency of the sinusoidal input.
Python code available at `frequencyresponse-secondorder.py`

From equation (6.9) and Figure 6.4, we learn that

- when $\zeta > 1$, the input magnitude is always attenuated,
- when $0 < \zeta < 1$, there is a range of input frequencies for which the input magnitude is amplified, and
- when $0 < \zeta \ll 1$ (the *lightly damped regime*), the amplification can be very large. We call this amplification *resonance*.

We say that *resonance* happens when

(i) the input frequency is very close to the natural frequency $\omega/\omega_{\mathrm{n}} \approx 1$, and

(ii) the second-order system is lightly damped, meaning that the damping ratio ζ is much smaller than 1.

Under these two conditions, the magnitude frequency response is very large: the input sinusoidal signal is *efficiently amplified* to a potentially destructive effect. The physical reason for this efficient amplification is that the input signal adds energy to the system during each oscillation cycle. Even a small periodic driving force can produce large amplitude oscillations due to the constructive interference between external force and natural frequency.

6.2.4 Bode plots

In engineering practice, it is convenient to draw the frequency response in logarithmic coordinates. Specifically, the *Bode magnitude plot* of the frequency response adopts

(i) the ω horizontal axis is logarithmic, and

(ii) the magnitude $|G(\mathrm{i}\omega)|$ is plotted in *decibels*, that is, a value $|G(\mathrm{i}\omega)|$ is plotted at $20\log_{10}|G(\mathrm{i}\omega)|$.

The *Bode phase plot* is plotted on a logarithmic scale for ω and a linear scale for the angle $\arg\big(G(\mathrm{i}\omega)\big)$. Note that, for all frequencies where $G(\mathrm{i}\omega) = 1$, we have $20\log_{10}|G(\mathrm{i}\omega)| = 20\log_{10}(1) = 0$.

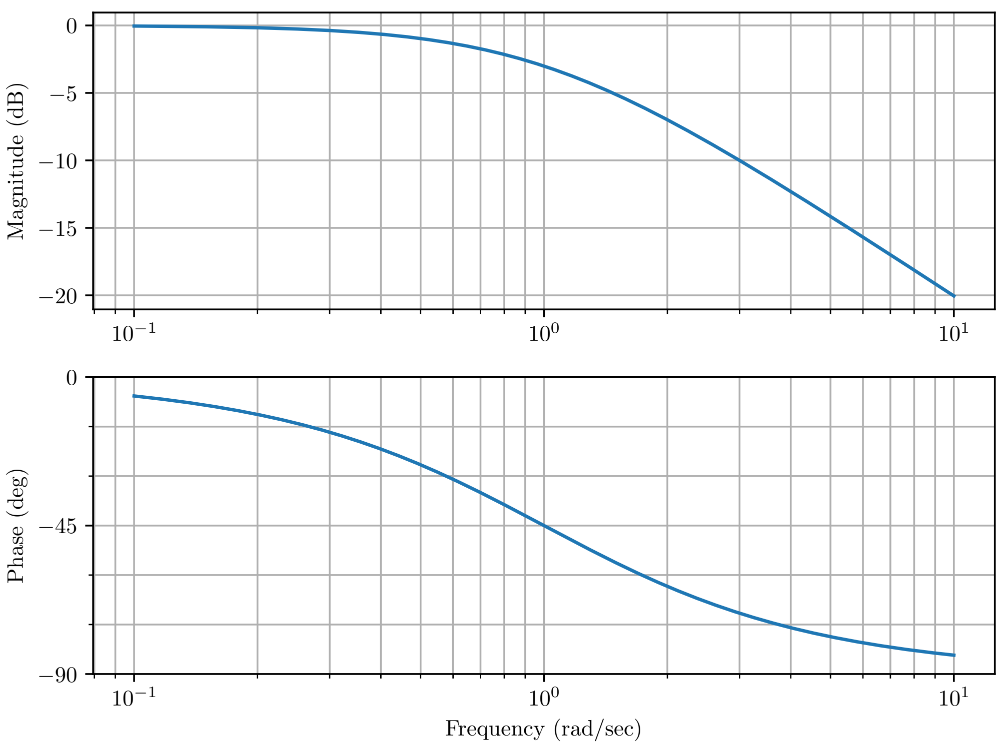


Figure 6.5: The Bode magnitude plot and Bode phase plot for a first-order system, as in equation (6.4), with time constant $\tau = 1$.

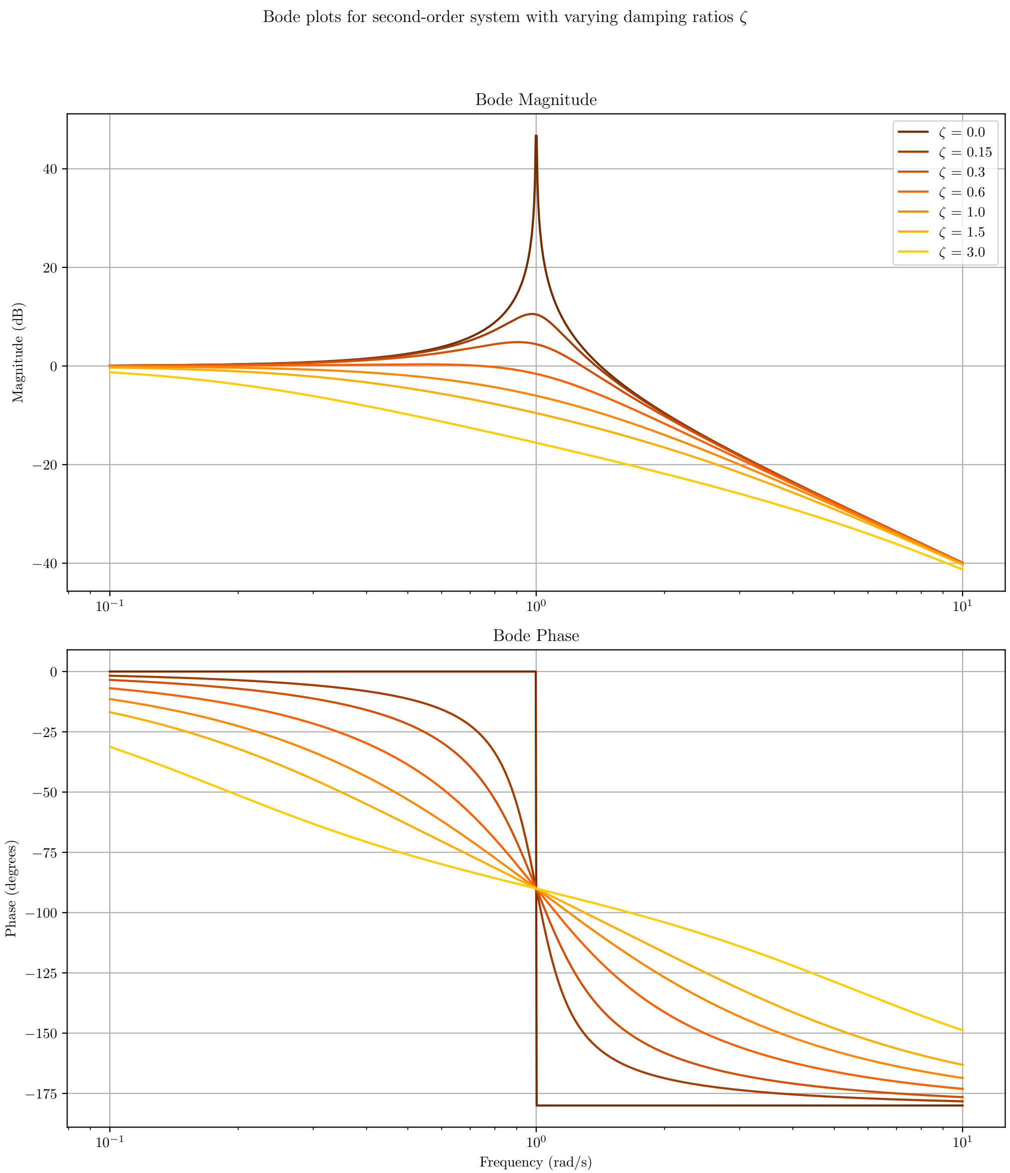


Figure 6.6: The Bode magnitude plot and Bode phase plot for a second-order system with unit natural frequency $\omega_n = 1$ and multiple values of the damping ratio ζ.

6.2.5 Control System Design Software: The Python Control Systems Library

Programming notes

It is useful to compare popular tools for control system design and analysis, including both commercial and open-source domains.

- Leading commercial tools include Matlab's Control System Toolbox and Simulink, which are widely used for their comprehensive features in control design, system simulation, and robust analysis capabilities.
- The Python Control Systems Library (`python-control`) is an open-source library designed for analyzing and designing feedback control systems, see (Fuller et al., 2021). It offers functionality for modeling linear time-invariant (LTI) systems, computing step responses, performing stability and frequency response / Bode analysis, and designing controllers using techniques such as root locus and frequency-domain methods.

  Its tight integration with Python ensures compatibility with other scientific libraries like NumPy and SciPy, making it a strong choice for users who prefer open-source, Python-based workflows, although commercial tools might provide more specialized features and enhanced usability for large-scale projects.

6.3 Lightly damped systems and the beating phenomenon

By *lightly damped system* we mean a second-order system with very small damping ratio $\zeta \ll 1$.

In this section, we study *beating*, a phenomenon that arises when the driving frequency is close to the natural frequency of the lightly damped system, often observed as the system approaches resonance.

To understand the behavior of a lightly-damped second order system, we study the forced response of the *undamped second-order system with* $\zeta = 0$ subject to a unit-magnitude sinusoidal input.

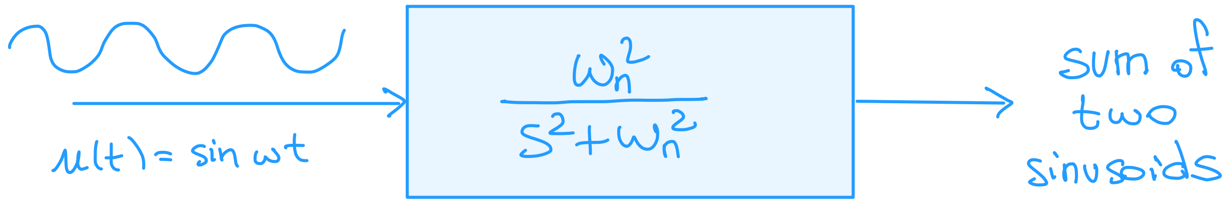


Figure 6.7: A sinusoidal input forcing an undamped second-order system, that is, an harmonic oscillator.

6.3.1 A detour: Constructive and destructive interference of sinusoidal waves

We start by considering the sum of two sinusoidal waves with frequencies ω_1 and ω_2 such that $\omega_1 \approx \omega_2$.

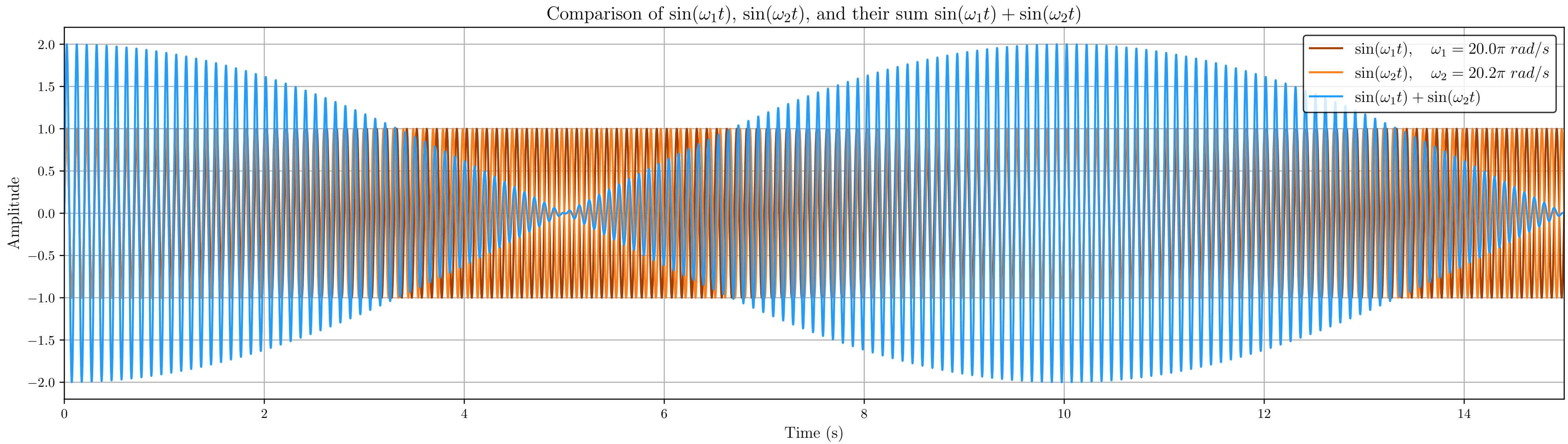


Figure 6.8: This plot illustrates the phenomenon of *constructive* and *destructive interference* between two sine waves, $\sin(\omega_1 t)$ and $\sin(\omega_2 t)$, with close but distinct frequencies ($\omega_1 = 20\pi$ rad/s and $\omega_2 = 20.2\pi$ rad/s). The sum of the two waves exhibits a *beating pattern*, where the amplitude alternates between high (constructive interference) and low (destructive interference), creating the characteristic modulation observed in the combined signal.

To better understand the beating phenomenon for two sinusoidal waves, we use the *sum-to-product* trigonometric formula:[1]

$$\sin\alpha + \sin\beta = 2\sin\Big(\frac{\alpha+\beta}{2}\Big)\cos\Big(\frac{\alpha-\beta}{2}\Big) \qquad \text{(valid for each pair of angles } \alpha,\beta\text{)}$$

and show that the sum (the result of the interference) of two sinusoidal waves satisfies the following equality:

$$\sin(\omega_1 t) + \sin(\omega_2 t) \;=\; \underbrace{2\cos\Big(\frac{\omega_1-\omega_2}{2}\,t\Big)}_{\text{slowly-varying beating amplitude}} \;\cdot\; \underbrace{\sin\Big(\frac{\omega_1+\omega_2}{2}\,t\Big)}_{\frac{\omega_1+\omega_2}{2}\approx\omega_1\approx\omega_2} \tag{6.10}$$

When the two frequencies ω_1 and ω_2 are approximately equal, this expression shows the resulting wave (see the blue curve in Figure 6.8) has

(i) amplitude that slowly varies with the *beat frequency* $\frac{1}{2}|\omega_1-\omega_2| \ll (\omega_1+\omega_2)/2$, and

(ii) frequency equal to the average of the similar frequencies $(\omega_1+\omega_2)/2 \approx \omega_1 \approx \omega_2$.

[1]More sum-to-product formulas are reviewed in Appendix 6.5

6.3.2 The response of an harmonic oscillator to a sinusoidal input

At zero damping $\zeta = 0$ and *natural frequency* $\omega_n > 0$, the transfer function (in canonical form) is

$$G_{\text{undamped}}(s) = \frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2} \tag{6.11}$$

and its two poles are purely imaginary so that the system is *marginally stable* (but not stable). Therefore, even when we study the forced response from zero initial conditions, there will be non-vanishing terms due to the marginally stable system dynamics.

We now take $u(t) = \sin(\omega t)$ at some *input frequency* ω, and compute the response in the Laplace domain:

$$Y(s) = G_{\text{undamped}}(s) \cdot \mathcal{L}[\sin(\omega t)] = \frac{\omega_n^2}{s^2 + \omega_n^2} \cdot \frac{\omega}{s^2 + \omega^2} \tag{6.12}$$

Using the inverse Laplace transform (see Exercise E6.2), one can verify that, for $\omega \neq \omega_n$, the forced response is the weighted sum of two sinusoidal signals:

$$y_{\text{forced}}(t) = \mathcal{L}^{-1}\left[\frac{\omega\omega_n^2}{(\omega^2 + s^2)(\omega_n^2 + s^2)}\right] = \frac{\omega_n}{\omega^2 - \omega_n^2}\big(\omega \sin(\omega_n t) - \omega_n \sin(\omega t)\big) \tag{6.13}$$

In class assignment

Why are there two sinusoids in $y_{\text{forced}}(t)$?

6.3.3 Approximating the solution at approximately-equal frequencies

Next, it is important to study the case when input and natural frequencies are approximately equal:

$$\omega \approx \omega_{\mathrm{n}} \quad \Longrightarrow \quad \omega + \omega_{\mathrm{n}} \approx 2\omega_{\mathrm{n}} \quad \text{and} \quad |\omega - \omega_{\mathrm{n}}| \ll \omega_{\mathrm{n}}.$$

Using the trigonometric analysis in Appendix 6.5, the forced response $y_{\mathrm{forced}}(t)$ in equation (6.13) can be approximated as:

$$y_{\mathrm{forced}}(t) = \frac{\omega_{\mathrm{n}}}{\omega^2 - \omega_{\mathrm{n}}^2}\left(\omega \sin(\omega_{\mathrm{n}} t) - \omega_{\mathrm{n}} \sin(\omega t)\right) \approx \underbrace{\frac{\omega_{\mathrm{n}}}{\omega - \omega_{\mathrm{n}}} \cdot \sin\Big(\frac{\omega_{\mathrm{n}} - \omega}{2} t\Big)}_{\text{large slowly-varying beating amplitude}} \cdot \cos\Big(\frac{\omega_{\mathrm{n}} + \omega}{2} t\Big) \tag{6.14}$$

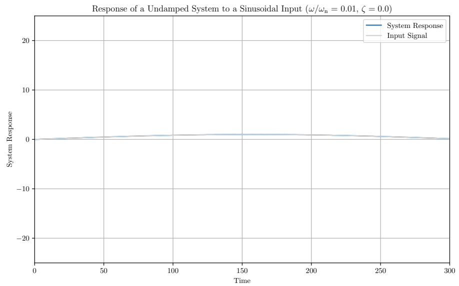
where

- $\dfrac{\omega_{\mathrm{n}}}{\omega - \omega_{\mathrm{n}}}$ is a *large amplitude* proportional to $1/|\omega - \omega_{\mathrm{n}}|$,
- $\sin\Big(\dfrac{\omega_{\mathrm{n}} - \omega}{2} t\Big)$ is a *slow oscillatory enclosing envelope* with *beat frequency* $|\omega - \omega_{\mathrm{n}}|/2 \ll \omega_{\mathrm{n}}$,
- $\cos\Big(\dfrac{\omega_{\mathrm{n}} + \omega}{2} t\Big)$ is a cosine wave at high frequency $(\omega + \omega_{\mathrm{n}})/2 \approx \omega_{\mathrm{n}}$.

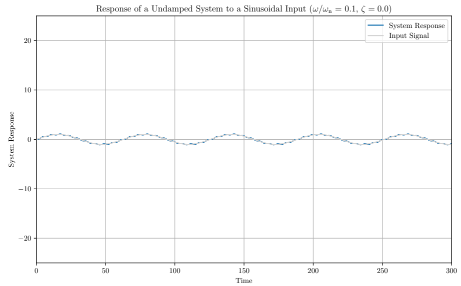
The response alternates between constructive and destructive interference, causing the amplitude of the oscillation to slowly rise and fall periodically. In other words,

> When input and natural frequency are approximately equal $\omega \approx \omega_{\mathrm{n}}$, the response is an oscillation at the natural frequency ω_{n} with a large amplitude that rises and falls periodically with a slow beat frequency. This is called the *beating phenomenon*.

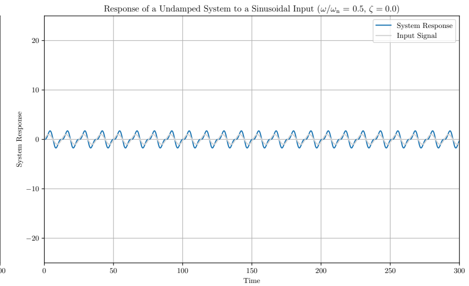
Response of an undamped system to a sinusoidal forcing ($\zeta = 0$)



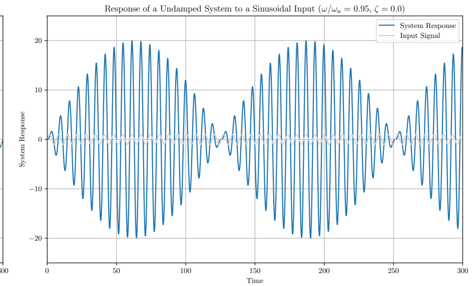
(a) $\omega/\omega_\mathrm{n} = 0.01$



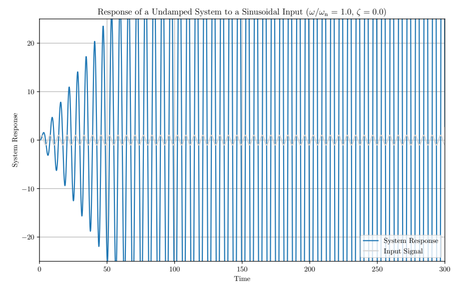
(b) $\omega/\omega_\mathrm{n} = 0.1$



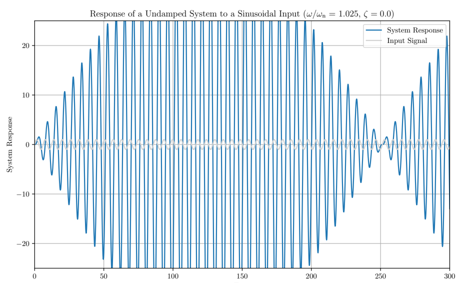
(c) $\omega/\omega_\mathrm{n} = 0.5$



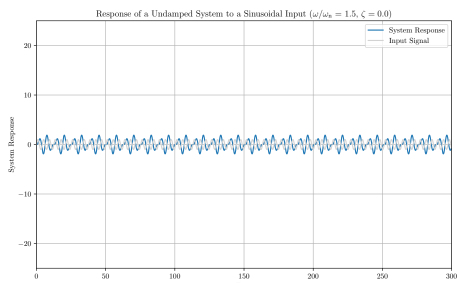
(d) $\omega/\omega_\mathrm{n} = 0.95$ – *beats*



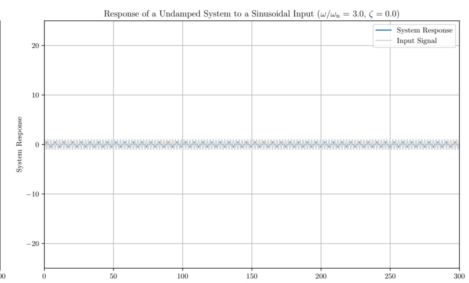
(e) $\omega/\omega_\mathrm{n} = 1.0$ – *resonance*



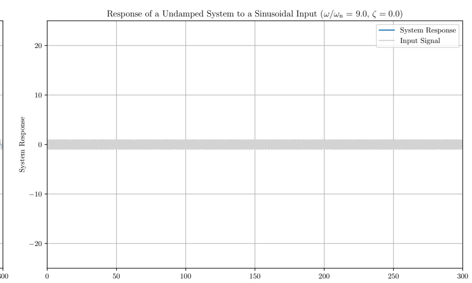
(f) $\omega/\omega_\mathrm{n} = 1.025$ – *beats*



(g) $\omega/\omega_\mathrm{n} = 1.5$



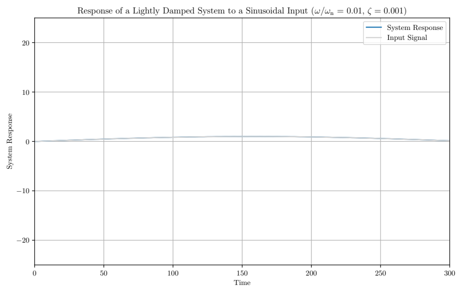
(h) $\omega/\omega_\mathrm{n} = 3.0$



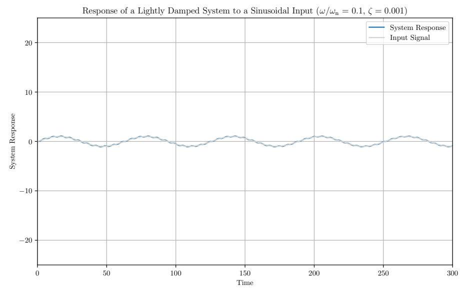
(i) $\omega/\omega_\mathrm{n} = 9.0$

Figure 6.9: Forced response of an *undamped second-order system* ($\zeta = 0$ and $\omega_\mathrm{n} = 1$) to a unit-magnitude sinusoidal input, plotted for different values of the ratio ω/ω_n. Python code available at `frequencyresponse-undamped.py`
For $\omega \approx \omega_\mathrm{n}$, the forced response consists of an oscillation at nearly the driving frequency ω, whose amplitude rises and falls periodically with a *beat frequency* $(1/2)|\omega - \omega_\mathrm{n}|$.

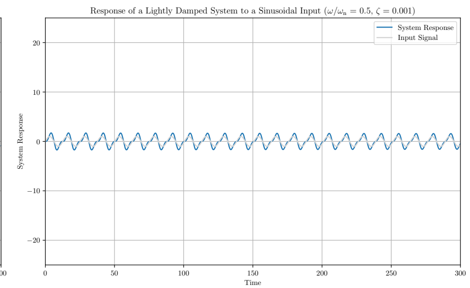
Response of a lightly damped system to a sinusoidal forcing ($\zeta = 0.001$)



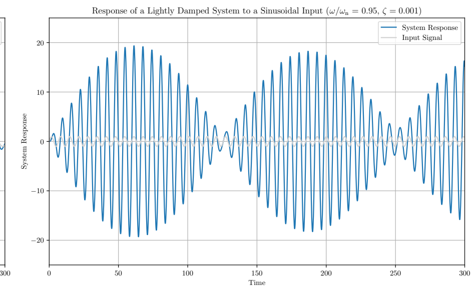
(a) $\omega/\omega_\text{n} = 0.01$



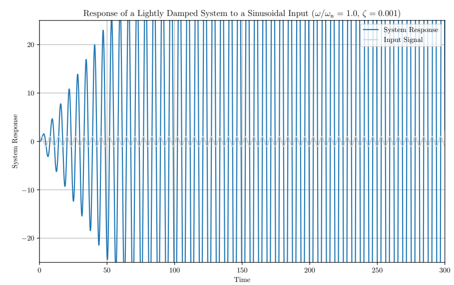
(b) $\omega/\omega_\text{n} = 0.1$



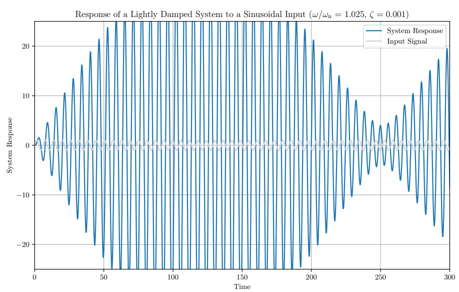
(c) $\omega/\omega_\text{n} = 0.5$



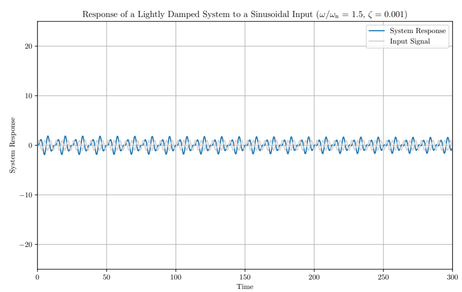
(d) $\omega/\omega_\text{n} = 0.95$ – *beats*



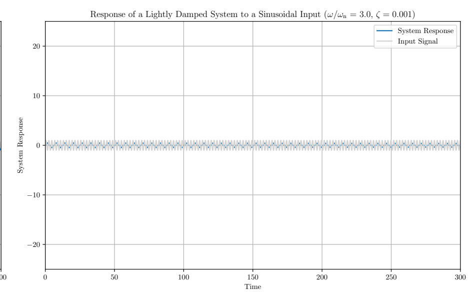
(e) $\omega/\omega_\text{n} = 1.0$ – *resonance*



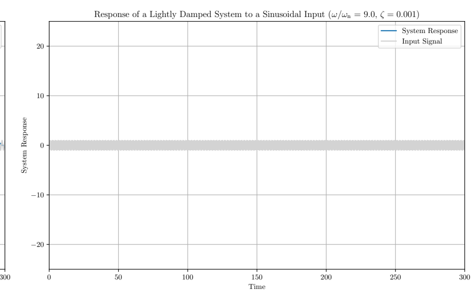
(f) $\omega/\omega_\text{n} = 1.025$ – *beats*



(g) $\omega/\omega_\text{n} = 1.5$



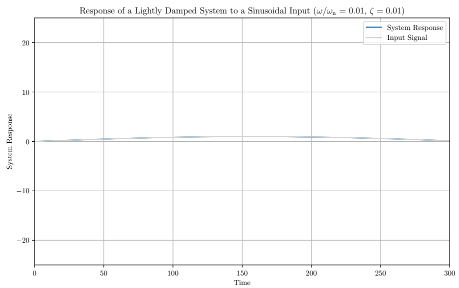
(h) $\omega/\omega_\text{n} = 3.0$



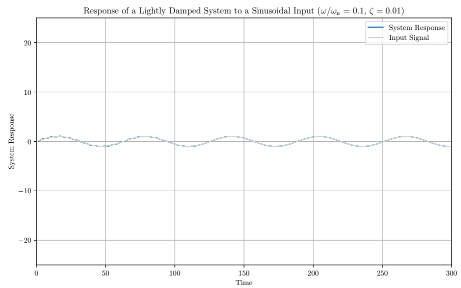
(i) $\omega/\omega_\text{n} = 9.0$

Figure 6.10: Forced response of an *lightly-damped second-order system* ($\zeta = 0.001$ and $\omega_\text{n} = 1$) to a unit-magnitude sinusoidal input, plotted for different values of the ratio ω/ω_n.
The lesson is that: for very small damping and short duration of time, the solution is very similar to that for the undamped solution.

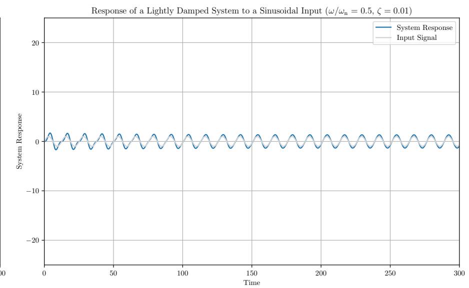
Response of a damped system to a sinusoidal forcing ($\zeta = 0.01$)



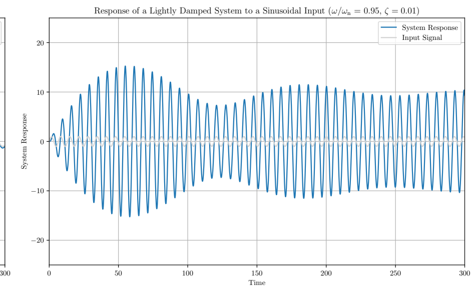
(a) $\omega/\omega_\mathrm{n} = 0.01$



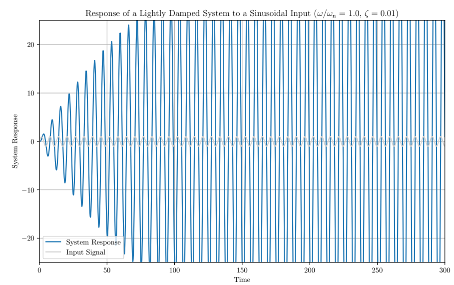
(b) $\omega/\omega_\mathrm{n} = 0.1$



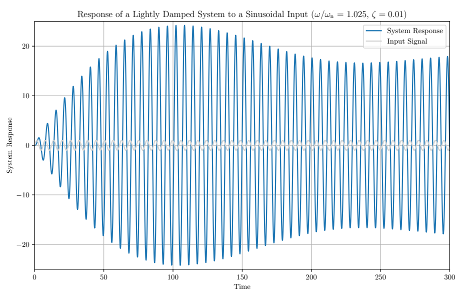
(c) $\omega/\omega_\mathrm{n} = 0.5$



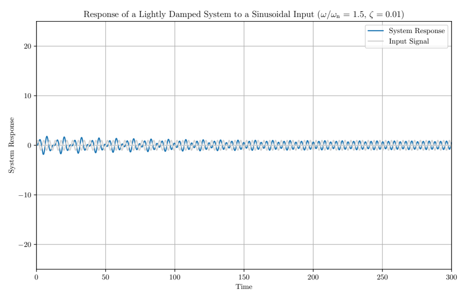
(d) $\omega/\omega_\mathrm{n} = 0.95$ – *beats*



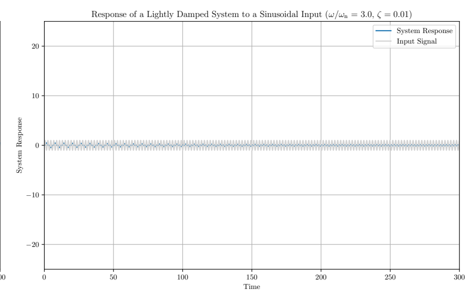
(e) $\omega/\omega_\mathrm{n} = 1.0$ – *resonance*



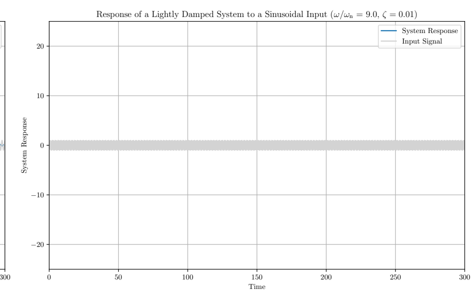
(f) $\omega/\omega_\mathrm{n} = 1.025$ – *beats*



(g) $\omega/\omega_\mathrm{n} = 1.5$



(h) $\omega/\omega_\mathrm{n} = 3.0$



(i) $\omega/\omega_\mathrm{n} = 9.0$

Figure 6.11: Forced response of an *lightly-damped second-order system* ($\zeta = 0.01$ and $\omega_\mathrm{n} = 1$) to a unit-magnitude sinusoidal input, plotted for different values of the ratio ω/ω_n.
The lesson is that: as ζ increases, the beating phenomenon disappears.

Comments on the beats phenomenon According to wikipedia:beats: "In acoustics, a *beat* is an interference pattern between two sounds of slightly different frequencies, perceived as a periodic variation in volume." Beats are often used to tune musical instruments to the correct pitch by comparing the instrument's frequency with a reference frequency.

In *mechanical systems*, beats can be observed when two oscillating components (like springs or pendulums) with similar but not identical natural frequencies are coupled. Beats can be an indicator of potential resonance problems. For instance, in a machine with rotating parts, the presence of a beat frequency might indicate a close match in the rotational frequencies, potentially leading to resonance.

6.4 Appendix: Steady-state response of stable systems to sinusoidal inputs

As for the step response in Section 5.5, we consider only *stable* dynamical systems, that is, all poles of the transfer function are in the (strict) left half plane. The formula (6.1) can be then derived in three steps.

Step 1: Computing the Laplace transform. Assuming $G(s)$ has distinct stable poles $-p_1, \dots, -p_n$, since $s^2 + \omega^2 = (s - \mathrm{i}\omega)(s + \mathrm{i}\omega)$

$$Y(s) \overset{u(t)=\sin(\omega t)}{=} G(s)\frac{\omega}{s^2+\omega^2} \overset{\text{partial fraction expansion}}{=} \frac{r_-}{s-\mathrm{i}\omega} + \frac{r_+}{s+\mathrm{i}\omega} + \sum_{i=1}^{n} \frac{r_i}{s+p_i} \tag{6.15}$$

where the residues r_- and r_+ may be complex. Since the poles $-p_i$ are stable, each term $\frac{r_i}{s+p_i}$ gives rise to an exponentially decaying term. Therefore, the steady state response is

$$y_{\text{steady-state}}(t) = r_- \,\mathrm{e}^{+\mathrm{i}\omega t} + r_+ \,\mathrm{e}^{-\mathrm{i}\omega t} \tag{6.16}$$

Step 2: Using the single-pole residue formula on both complex poles. From the residue formula (4.40), we compute

$$\begin{aligned}
r_- &= (s-\mathrm{i}\omega)G(s)\frac{\omega}{s^2+\omega^2}\Big|_{s=+\mathrm{i}\omega} = G(s)\frac{\omega}{s+\omega}\Big|_{s=\mathrm{i}\omega} = \frac{1}{2\mathrm{i}}G(\mathrm{i}\omega) \quad = \frac{1}{2\mathrm{i}}|G(\mathrm{i}\omega)|\,\mathrm{e}^{\mathrm{i}\phi} \qquad (\phi = \arg\big(G(\mathrm{i}\omega)\big)) \\
r_+ &= (s+\mathrm{i}\omega)G(s)\frac{\omega}{s^2+\omega^2}\Big|_{s=-\mathrm{i}\omega} = G(s)\frac{\omega}{s-\omega}\Big|_{s=-\mathrm{i}\omega} = -\frac{1}{2\mathrm{i}}G(-\mathrm{i}\omega) \overset{(*)}{=} -\frac{1}{2\mathrm{i}}|G(\mathrm{i}\omega)|\,\mathrm{e}^{-\mathrm{i}\phi},
\end{aligned} \tag{6.17}$$

where the equality $(*)$ follows from the property $\overline{G(s)} = G(\bar{s})$ for any complex number s and rational function G.

Step 3: Using the inverse Euler formula. Plugging the expressions for r_- and r_+ into formula (6.16), we obtain

$$y_{\text{steady-state}}(t) = |G(\mathrm{i}\omega)|\frac{\mathrm{e}^{\mathrm{i}(\omega t+\phi)} - \mathrm{e}^{-\mathrm{i}(\omega t+\phi)}}{2\mathrm{i}} = |G(\mathrm{i}\omega)|\sin(\omega t+\phi), \tag{6.18}$$

where we used the inverse Euler formula $\sin(\theta) = \frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2\mathrm{i}}$ from Figure 4.1.

6.5 Appendix: Trigonometric explanation of the beating phenomenon

The sum-to-product formulas in trigonometry The *sum-to-product formulas* are trigonometric identities that transform the sum or difference of trigonometric functions into a product of trigonometric functions. The four main formulas are:

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{6.19}$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right), \tag{6.20}$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right), \tag{6.21}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right). \tag{6.22}$$

But the general concept holds even more generally.

Approximate frequency response of an undamped system In this appendix, we study the expression $\frac{1}{\omega^2-\omega_n^2}\big(\omega\sin(\omega_n t)-\omega_n\sin(\omega t)\big)$ and obtain an approximate equality when $\omega\approx\omega_n$. Summing and subtracting the sum-to-product formulas (6.19) and (6.20), we obtain:

$$\sin\alpha = \cos\Big(\frac{\alpha+\beta}{2}\Big)\sin\Big(\frac{\alpha-\beta}{2}\Big)+\sin\Big(\frac{\alpha+\beta}{2}\Big)\cos\Big(\frac{\alpha-\beta}{2}\Big) \tag{6.23}$$

$$\sin\beta = \sin\Big(\frac{\alpha+\beta}{2}\Big)\cos\Big(\frac{\alpha-\beta}{2}\Big)-\cos\Big(\frac{\alpha+\beta}{2}\Big)\sin\Big(\frac{\alpha-\beta}{2}\Big) \tag{6.24}$$

We let $\alpha=\omega_n t$, $\beta=\omega t$ and scale the two equations with appropriate coefficients to obtain:

$$\omega\sin(\omega_n t) = \omega\cos\big(\frac{\omega_n+\omega}{2}t\big)\sin\big(\frac{\omega_n-\omega}{2}t\big)+\omega\sin\big(\frac{\omega_n+\omega}{2}t\big)\cos\big(\frac{\omega_n-\omega}{2}t\big), \tag{6.25}$$

$$\omega_n\sin(\omega t) = \omega_n\sin\big(\frac{\omega_n+\omega}{2}t\big)\cos\big(\frac{\omega_n-\omega}{2}t\big)-\omega_n\cos\big(\frac{\omega_n+\omega}{2}t\big)\sin\big(\frac{\omega_n-\omega}{2}t\big). \tag{6.26}$$

Next, we subtract the second equation from the first

$$\omega\sin(\omega_n t)-\omega_n\sin(\omega t) = (\omega+\omega_n)\cos\Big(\frac{\omega_n+\omega}{2}t\Big)\sin\Big(\frac{\omega_n-\omega}{2}t\Big)+(\omega-\omega_n)\sin\Big(\frac{\omega_n+\omega}{2}t\Big)\cos\Big(\frac{\omega_n-\omega}{2}t\Big) \tag{6.27}$$

and we scale the result to obtain

$$\frac{1}{\omega^2-\omega_n^2}\Big(\omega\sin(\omega_n t)-\omega_n\sin(\omega t)\Big) = \frac{1}{\omega-\omega_n}\cos\Big(\frac{\omega_n+\omega}{2}t\Big)\sin\Big(\frac{\omega_n-\omega}{2}t\Big)+\frac{1}{\omega+\omega_n}\sin\Big(\frac{\omega_n+\omega}{2}t\Big)\cos\Big(\frac{\omega_n-\omega}{2}t\Big). \tag{6.28}$$

Finally, when $\omega\approx\omega_n$, we note $\Big|\frac{1}{\omega-\omega_n}\Big| \gg \frac{1}{\omega+\omega_n}$ so that

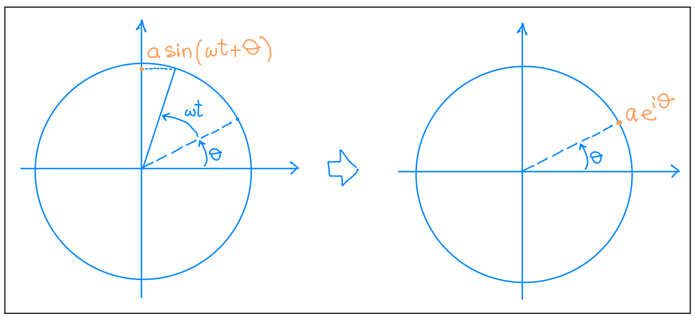
$$\frac{1}{\omega^2-\omega_n^2}\Big(\omega\sin(\omega_n t)-\omega_n\sin(\omega t)\Big) \approx \frac{1}{\omega-\omega_n}\sin\Big(\frac{\omega_n-\omega}{2}t\Big)\cos\Big(\frac{\omega_n+\omega}{2}t\Big) \tag{6.29}$$

Physical interpretation of beating via phasors The *phasor representation* of a sinusoidal signal is a complex number that encodes the amplitude and phase of the sinusoid. Specifically,

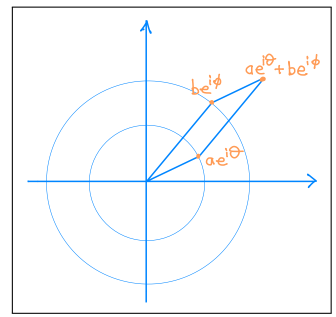
$$a\sin(\omega t+\theta) \qquad \mapsto \qquad \underbrace{a\,\mathrm{e}^{\mathrm{i}\theta}}_{\text{phasor representation}} \tag{6.30}$$

Phasors simplify the analysis of sinusoidal signals by expressing them as a constant magnitude and phase angle, ignoring the explicit time dependence. A key property is the *sum property* for sinusoidal signals with *equal frequency*:

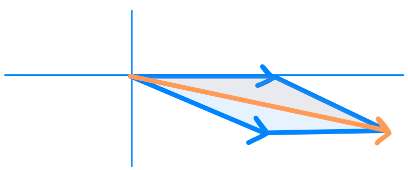
$$a\sin(\omega t+\theta)+b\sin(\omega t+\phi) \qquad \mapsto \qquad \underbrace{a\,\mathrm{e}^{\mathrm{i}\theta}+b\,\mathrm{e}^{\mathrm{i}\phi}}_{\text{sum of the phasors in complex plane}} \tag{6.31}$$



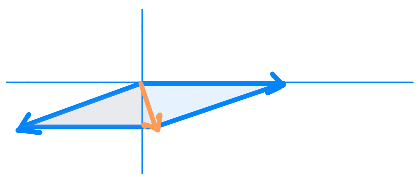
(a) definition of phasor



(b) sum of two phasors (with equal frequency)



(c) constructive



(d) destructive

Figure 6.12: Sum of phasors: constructive and destructive interference

6.6 Historical notes, further reading, and online resources

A classic reference on vibrations is the famous textbook by Den Hartog (1956).

6.7 Exercises

E6.1 **Transfer function and frequency response of an RC circuit.** In the diagram below of an RC circuit, let $v_{\text{input}}(t)$ be the voltage at the input, $r > 0$ be a resistance in Ohms, $c > 0$ be a capacitance in Farads, and $v_{\text{output}}(t)$ be the voltage at the output.

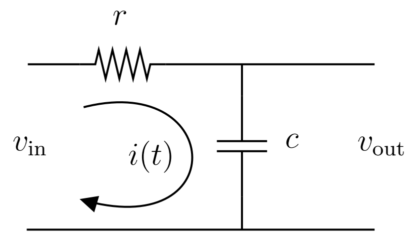


Figure E6.1: First-order RC circuit

(i) Find the governing equation of the circuit in terms of the variables v_{input} and v_{output}.
(ii) Compute the Laplace transform of the governing equation assuming $v_{\text{output}}(0) = 0$.
(iii) Obtain the input-to-output transfer function $G(s) = \frac{V_{\text{output}}(s)}{V_{\text{input}}(s)}$ for the system.
(iv) Show that the pole of the system is stable.
(v) Compute the magnitude frequency response of the transfer function.
(vi) In applications where signal quality is crucial (e.g., telecommunications and music), circuits such as the above RC circuit are often employed. These circuits serve as either high-pass filters (i.e., removing low-frequency noise) or low-pass filters (i.e., removing high-frequency noise). Is the above circuit a high-pass filter or a low-pass filter? Explain your reasoning using the magnitude-frequency response of the transfer function.

Answer:

(i) Applying Kirchoff's laws, the differential equation governing the system is given by

$$v_{\text{output}}(t) = v_{\text{input}}(t) - rc\frac{\mathrm{d}v_{\text{output}}}{\mathrm{d}t}$$

(ii) The Laplace transform is given by

$$\begin{aligned} &\mathcal{L}[v_{\text{output}}] = V_{\text{input}}(s) - rc(sV_{\text{output}}(s) - v_{\text{output}}(0)) \\ \implies &V_{\text{output}}(s) = v_{\text{input}}(s) - srcV_{\text{output}}(s) \end{aligned}$$

(iii) The input-to-output transfer function is found by rearranging the equation found in the solution of part (ii). It is as follows:

$$\begin{aligned} &V_{\text{output}}(s) = v_{\text{input}}(s) - srcV_{\text{output}}(s) \\ \implies &V_{\text{output}}(s)(src+1) = v_{\text{input}}(s) \\ \implies &G(s) = \frac{V_{\text{output}}(s)}{v_{\text{input}}(s)} = \frac{1}{src+1} \end{aligned}$$

(iv) By inspecting the transfer function, we find a pole at $s = -1/rc$. Because $r > 0$, $c > 0$, then $rc > 0$ and this pole must be in the left half of the complex plane and is therefore stable.

(v) The magnitude-frequency response is found by

$$\begin{aligned} |G(s)| = |G(\mathrm{i}\omega)| &= \left|\frac{1}{\mathrm{i}\omega rc + 1}\right| \\ &= \frac{1}{\sqrt{1+(\omega rc)^2}} \end{aligned}$$

(vi) By inspection of the magnitude-frequency response, as the frequency ω becomes larger (i.e., high-frequency) then $|G(\mathrm{i}\omega)| \to 0$. Therefore, this would eliminate high-frequency noise indicating that the circuit behaves as a low-pass filter.

■

E6.2 **Sinusoidal forcing of an undamped second-order system.** Given damping ratio ζ and natural frequency ω_n, the canonical form of a second order system is

$$\ddot{y}(t) + 2\zeta\omega_\mathrm{n}\dot{y}(t) + \omega_\mathrm{n}^2 y(t) = \omega_\mathrm{n}^2 u(t)$$

In this exercise, we set the damping ratio to zero: $\zeta = 0$.

(i) Compute the transfer function $G(s) = Y(s)/U(s)$.

(ii) Compute the poles of $G(s)$.

(iii) Is the system stable, marginally stable, or unstable?

(iv) Assume the input is a unit-magnitued sinusoidal signal $u(t) = \sin(\omega t)$ and use the partial fraction expansion to compute the forced response $Y(s)$ from zero initial conditions.

Hint: A correct answer needs to have the correct expansion, with all potential terms (even those that, in the end, have zero coefficient). There should be 4 terms and 4 free coefficients.

(v) Compute the inverse Laplace transform of $Y(s)$ to obtain $y(t)$.

Answer:

(i) The undamped second order system is $\ddot{y} + \omega_\mathrm{n}^2 y = \omega_\mathrm{n}^2 u(t)$, so that

$$s^2 Y(s) + \omega_\mathrm{n}^2 Y(s) = \omega_\mathrm{n}^2 U(s) \qquad \Longrightarrow \qquad \boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{\omega_\mathrm{n}^2}{s^2 + \omega_\mathrm{n}^2}}$$

(ii) We compute

$$s^2 + \omega_\mathrm{n}^2 = 0 \qquad \Longrightarrow \; s^2 = -\omega_\mathrm{n}^2$$

so that

$$s = \pm\sqrt{-\omega_\mathrm{n}^2} \qquad \Longrightarrow \qquad \boxed{s = \pm \mathrm{i}\omega_\mathrm{n}}$$

(iii) The system is marginally stable, because the poles are on the imaginary axis and not repeated.

(iv) We compute

$$Y(s) = G(s) \cdot \mathcal{L}[\sin(\omega t)] = \frac{\omega\omega_\mathrm{n}^2}{(s^2 + \omega_\mathrm{n}^2)(s^2 + \omega^2)} \tag{E6.1}$$

Therefore, we setup the partial fraction expansion:

$$\frac{\omega\omega_\mathrm{n}^2}{(s^2 + \omega_\mathrm{n}^2)(s^2 + \omega^2)} = A_1 \frac{\omega_\mathrm{n}}{(s^2 + \omega_\mathrm{n}^2)} + A_2 \frac{s}{(s^2 + \omega_\mathrm{n}^2)} \quad + \quad B_1 \frac{\omega}{(s^2 + \omega^2)} + B_2 \frac{s}{(s^2 + \omega^2)} \tag{E6.2}$$

Next, we compute

$$\begin{aligned} \omega\omega_n^2 &= A_1\omega_n(s^2+\omega^2) + A_2 s(s^2+\omega^2) + B_1\omega(s^2+\omega_n^2) + B_2 s(s^2+\omega_n^2) &\text{(E6.3)}\\ &= A_1\omega_n s^2 + A_1\omega_n\omega^2 + A_2 s^3 + A_2\omega^2 s + B_1\omega s^2 + B_1\omega\omega_n^2 + B_2 s^3 + B_2\omega_n^2 s &\text{(E6.4)}\\ &= (A_2+B_2)s^3 + (A_1\omega_n + B_1\omega)s^2 + (A_2\omega^2 + B_2\omega_n^2)s + A_1\omega_n\omega^2 + B_1\omega\omega_n^2 &\text{(E6.5)} \end{aligned}$$

We now setup 4 linear equations in 4 unknowns (A_1, A_2, B_1, and B_2):

$$\begin{aligned} A_2 + B_2 &= 0 &\text{(E6.6)}\\ A_1\omega_n + B_1\omega &= 0 &\text{(E6.7)}\\ A_2\omega^2 + B_2\omega_n^2 &= 0 &\text{(E6.8)}\\ A_1\omega_n^2\omega^2 + B_1\omega\omega_n^2 &= \omega\omega_n^2 &\text{(E6.9)} \end{aligned}$$

After some calculations, we obtain

$$A_1 = \frac{\omega\omega_n}{\omega^2-\omega_n^2}, \quad A_2 = 0, \quad B_1 = \frac{-\omega_n^2}{\omega^2-\omega_n^2}, \quad B_2 = 0,$$

and, finally:

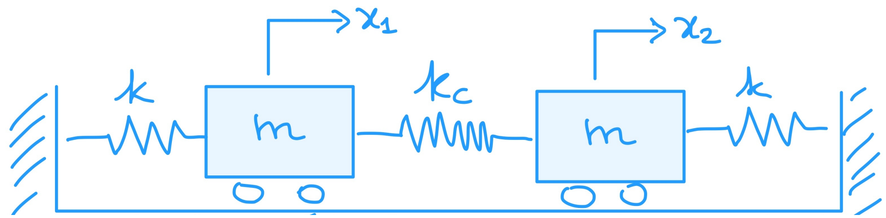
$$Y(s) = \frac{\omega_n}{\omega^2-\omega_n^2}\left(\frac{\omega\omega_n}{s^2+\omega_n^2} - \frac{\omega\omega_n}{s^2+\omega^2}\right)$$

(v) We are now ready to compute the inverse Laplace transform

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{\omega_n}{\omega^2-\omega_n^2}\Big(\omega\sin(\omega_n t) - \omega_n\sin(\omega t)\Big)$$

■

E6.3 **Systems with two degrees of freedom: Beating in weakly-coupled identical harmonic oscillators.** Consider two identical harmonic oscillators with mass m and spring stiffness k interconnected by a spring with stiffness k_c. (We assume the system has no dampers and no friction).



(i) Write the equations of motion for $x_1(t)$ and $x_2(t)$, possibly using a free body diagram.

(ii) Define the *sum position* $x_{\text{sum}}(t) = x_1(t) + x_2(t)$. Summing the equations of motion for $x_1(t)$ and $x_2(t)$, obtain a differential equation for x_{sum}.
Define the *difference position* $x_{\text{diff}}(t) = x_1(t) - x_2(t)$. Subtracting the equations for $x_2(t)$ from the equation for $x_1(t)$, obtain a differential equation for x_{diff}.

(iii) What is the natural frequency of the second order dynamics of x_{sum}? What is the natural frequency of the second order dynamics of x_{diff}?

Next, assume $x_1(0) = 1, \dot{x}_1(0) = 0$ and $x_2(0) = \dot{x}_2(0) = 0$, that is, only the first mass is displaced and zero initial velocities.

(iv) What are corresponding initial conditions for $x_{\text{sum}}(0), \dot{x}_{\text{sum}}(0)$ and $x_{\text{diff}}(0), \dot{x}_{\text{diff}}(0)$? Write the solutions for $x_{\text{sum}}(t)$ and $x_{\text{diff}}(t)$.
Hint: Recall the solutions to the harmonic oscillator from Section 2.1.2.

(v) Write the solutions for $x_1(t)$ and $x_2(t)$.

Note: *We have learned that*
(1) in a mechanical system with two degrees of freedom there exist two natural frequencies,
(2) x_1 and x_2 are the sum and difference of sinusoidal functions, and
(3) when $k_c \ll k$, the two frequency satisfy $\omega_{\text{sum}} \approx \omega_{\text{diff}}$ and the system exhibits the beating phenomenon.

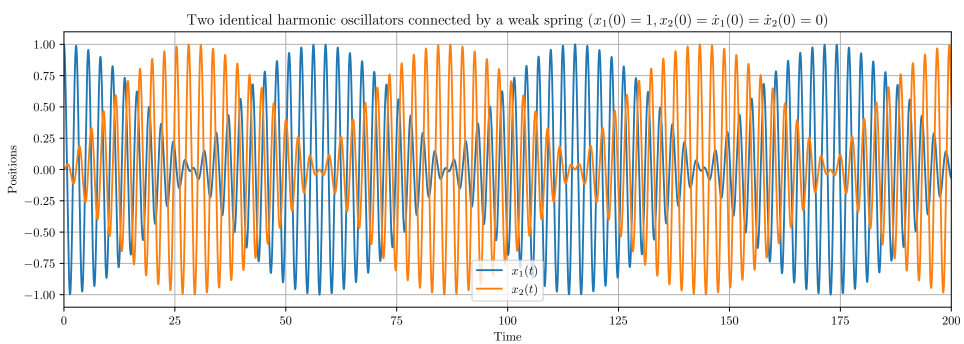


Figure E6.2: Two identical harmonic oscillators coupled by a weak spring ($m = 1$, $k = 5$, and $k_c = 0.25$) display the beating phenomenon.
(1) From physics viewpoint, the potential energy in the initial displacement of the first mass leaks into the second mass, in the sense that at each time t^* when $x_2(t^*) = 1$, energy conservation implies $\dot{x}_2(t^*) = 0$ and $x_1(t^*) = \dot{x}_1(t^*) = 0$.
(2) Even if the coupling between the two masses is weak, the cumulative effect of the dynamics is not weak!
Python code available at `ex-coupled-oscillators.py`

Answer:

(i) Using Newton's law, we obtain

$$m\ddot{x}_1 + kx_1 = k_{\mathsf{c}}(x_2 - x_1) \tag{E6.10}$$
$$m\ddot{x}_2 + kx_2 = k_{\mathsf{c}}(x_1 - x_2) \tag{E6.11}$$

(ii) Summing the equations, we obtain

$$m(\ddot{x}_1 + \ddot{x}_2) + k(x_1 + x_2) = 0 \qquad \Longrightarrow \qquad m\ddot{x}_{\mathsf{sum}} + kx_{\mathsf{sum}} = 0 \tag{E6.12}$$

Subtracting the second equation from the first, we obtain

$$m(\ddot{x}_1 - \ddot{x}_2) + k(x_1 - x_2) = 2k_{\mathsf{c}}(x_2 - x_1) \qquad \Longrightarrow \qquad m\ddot{x}_{\mathsf{diff}} + (k + 2k_{\mathsf{c}})x_{\mathsf{diff}} = 0 \tag{E6.13}$$

(iii) The natural frequency for x_{sum} is $\omega_{\mathsf{sum}} = \sqrt{k/m}$.
The natural frequency for x_{diff} is $\omega_{\mathsf{diff}} = \sqrt{(k + 2k_{\mathsf{c}})/m}$.

(iv) The initial conditions are $x_{\mathsf{sum}}(0) = 1$, $\dot{x}_{\mathsf{sum}}(0) = 0$ and $x_{\mathsf{diff}}(0) = 1$, $\dot{x}_{\mathsf{diff}}(0) = 0$.
Given these initial conditions, recall from Section 2.1.3 that the harmonic oscillator $\ddot{y} + \omega_{\mathsf{n}} y = 0$ has a solution of the form $y(t) = a\sin(\omega_{\mathsf{n}} t) + b\cos(\omega_{\mathsf{n}} t)$. Since $y(0) = 1$ and $\dot{y}(0) = 0$, one can calculate that the solution is $y(t) = \cos(\omega_{\mathsf{n}} t)$.
Since the two dynamics are simple harmonic oscillators, starting with a unit displacement and zero initial velocity, we have

$$x_{\mathsf{sum}}(t) = \cos(\omega_{\mathsf{sum}} t) \qquad \text{and} \qquad x_{\mathsf{diff}}(t) = \cos(\omega_{\mathsf{diff}} t) \tag{E6.14}$$

(v) Summing and subtracting:

$$x_1(t) = \frac{1}{2}\big(x_{\mathsf{sum}}(t) + x_{\mathsf{diff}}(t)\big) = \frac{1}{2}\big(\cos(\omega_{\mathsf{sum}} t) + \cos(\omega_{\mathsf{diff}} t)\big) \tag{E6.15}$$
$$x_2(t) = \frac{1}{2}\big(x_{\mathsf{sum}}(t) - x_{\mathsf{diff}}(t)\big) = \frac{1}{2}\big(\cos(\omega_{\mathsf{sum}} t) - \cos(\omega_{\mathsf{diff}} t)\big) \tag{E6.16}$$

■

E6.4 **High-pass and low-pass filters.** Consider a first-order system with the following transfer function:

$$G(s) = \frac{\tau s}{\tau s + 1}$$

with time constant $\tau > 0$.

(i) Write a formula for the magnitude frequency response $|G(\mathrm{i}\omega)|$ and the angular frequency response $\arg(G(\mathrm{i}\omega))$.

(ii) Use your answer from part (i) to write down the steady-state response of the system $y_{\mathrm{ss}}(t)$ to a unit-magnitude sinusoidal input $u = \sin(\omega t)$.

(iii) Use your answer from part (ii) to determine

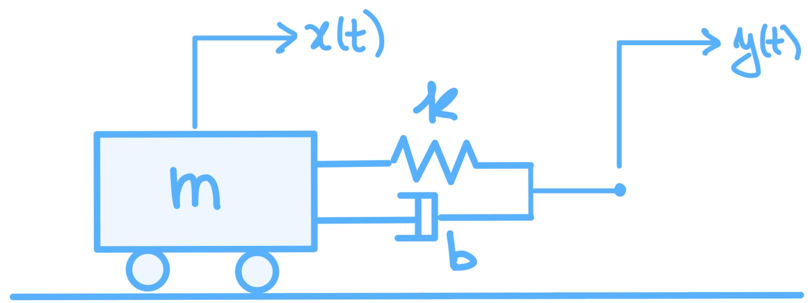
(a) the approximate steady-state response of the system to a low-frequency input $\omega \ll 1/\tau$, and

(b) the approximate steady-state response of the system to a high-frequency input $\omega \gg 1/\tau$.

(iv) A *low-pass filter* is a system that preserves low-frequency sinusoidal inputs and attenuates high-frequency sinusoidal inputs. A *high-pass filter* preserves high-frequency sinusoidal inputs and attenuates low-frequency sinusoidal inputs. Comparing your results in this exercise and the results from Section 6.2.2, identify which transfer function ($G(s) = \frac{1}{\tau s+1}$ and $G(s) = \frac{\tau s}{\tau s+1}$) represents a low-pass filter and which represents a high-pass filter.

E6.5 **Mass-spring-damper system connected to a moving point.** In this exercise we study a second-order transfer function that does not match the canonical form because it has a polynomial of first order in the numerator. This polynomial has a root, called a zero of the transfer function.

Given positive parameters m, b, k, consider the following mass-spring-damper system with position $x(t)$ connected to a moving point $y(t)$:



(i) Derive the equation of motion for the system.

(ii) Take the Laplace transform of the equation you derived in part (i), assuming zero initial conditions.

(iii) Write down the transfer function $G(s)$ from $Y(s)$ to $X(s)$.

(iv) Compute the magnitude frequency response $|G(\mathrm{i}\omega)|$.

(v) Plot the magnitude frequency response using Python or your software of choice. Use the parameter values $m = 1$, $b = 0.5$, and $k = 1$.

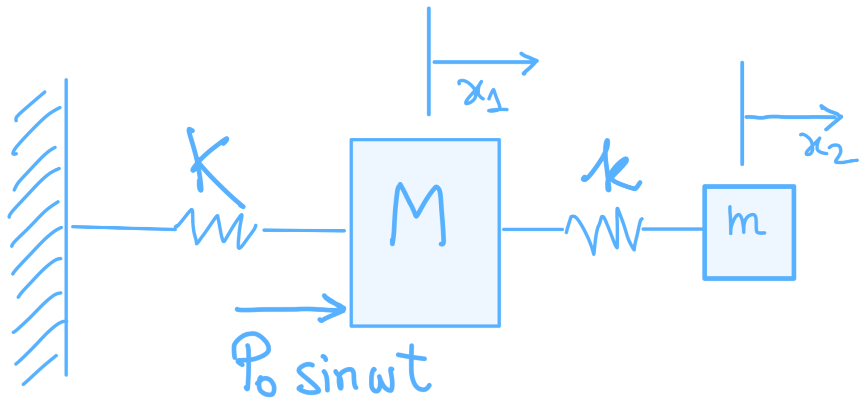
(vi) Where (i.e., for what value of ω) does the resonant peak occur, approximately? What is the approximate value of the magnitude frequency response at this peak?

(vii) Where (i.e., for what value(s) of ω) does the value of the magnitude frequency response equal 1?

(viii) What happens to the value of the magnitude frequency response as ω gets very large? Provide a physical interpretation of what this means for our system.

E6.6 **The undamped vibration absorber.** During operation, mechanical systems are often subjected to vibrations (think of your car engine when you turn it on). Often, these vibrations can pose a potential problem for performance, and as such it can be desirable to eliminate them. Although it is desirable to mitigate vibrations, the process of doing so is non-trivial. One potential solution to address the problem of vibrations in mechanical systems is the *dynamic vibration absorber* designed by Frahn in 1909.

Consider a main mass M connected to a wall with spring stiffness K and subject to an oscillatory force $P_0 \sin(\omega t)$. To dampen the undesirable oscillations, we attach to the mass M a second smaller mass m with a spring with stiffness k. (For clarity, the two springs do not need to have the same stiffness: $K \neq k$.)



(i) Compute the equations of motion for the system.

(ii) We claim that one solution to the dynamical system is of the form

$$x_1(t) = a_1 \sin(\omega t) \qquad \text{and} \qquad x_2(t) = a_2 \sin(\omega t) \tag{E6.17}$$

for some frequency ω and constant amplitudes a_1 and a_2. Substitute the solution in equations (E6.17) into the differential equation found in part (i) and obtain an algebraic equation for (a_1, a_2) and the system parameters.

(iii) Show that, when the forcing frequency $\omega = \sqrt{k/m}$, the large mass M will not vibrate because $a_1 = 0$.
In other words, the small oscillator (m, k) vibrates in such a manner that its spring force on M is equal and opposite to $P_0 \sin(\omega t)$ at all times.

(iv) Compute the initial conditions for $x_1(0)$, $\dot{x}_1(0)$, $x_2(0)$, and $\dot{x}_2(0)$ such that $x_1(t) = 0$, and $x_2(t) = a_2 \sin(\omega t)$ is a solution to the differential equation in part (i).

Bibliography

J. Den Hartog. *Vibrations*. McGraw-Hill, 1956.

S. Fuller, B. Greiner, J. Moore, R. M. Murray, R. van Paassen, and R. Yorke. The Python Control Systems library (python-control). In *IEEE Conf. on Decision and Control*, pages 4875–4881, 2021. doi.