UC Santa Barbara, Department of Mechanical Engineering
ME103 Dynamical Systems. Slides by Chapter.

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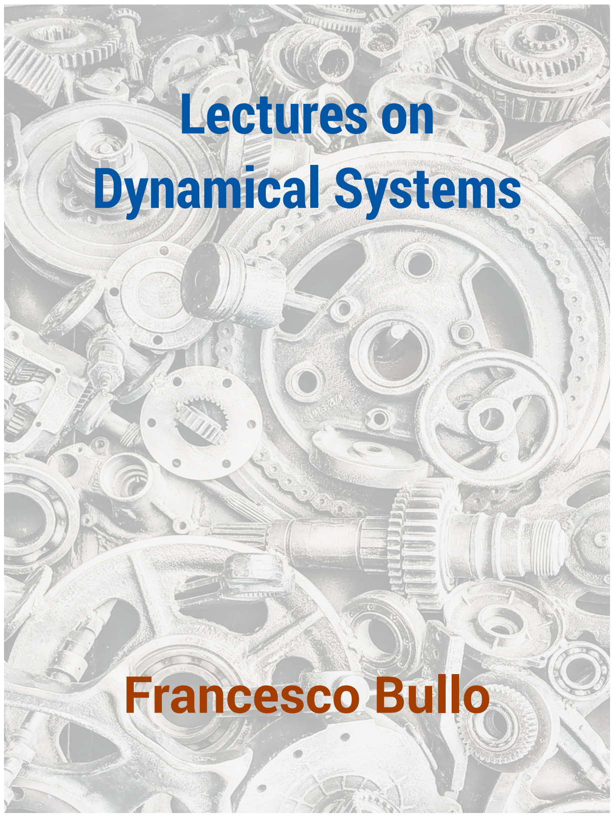
http://motion.me.ucsb.edu/ME103-Fall2024/syllabus.html



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Chapter 5

The Transfer Function and Time Responses of Dynamical Systems

In this chapter we define the transfer function and use it to compute the response of canonical systems (first-order and second-order systems) to canonical inputs (impulse, step, and ramp).

5.1 The transfer function

We consider a dynamical system with state $y(t)$ and input $u(t)$ in the form:

$$a_0 y(t) + a_1 \frac{dy}{dt}(t) + \cdots + a_n \frac{d^n y}{dt^n}(t) \;=\; b_0 u(t) + b_1 \frac{du}{dt}(t) + \cdots + b_m \frac{d^m u}{dt^m}(t) \tag{5.1}$$

where

- $y(t)$ is the *output*, or *response*,
- $u(t)$ is the *input* applied to the system,
- $a_0, \ldots, a_n$ and $b_0, \ldots, b_m$ are constant coefficients.

In this chapter we are mostly interested in the *forced response* where *all initial conditions are zero*. In this case, the response depends only upon the input:

$$\begin{aligned} \frac{d^i y}{dt^i}(0) &= 0 \qquad \text{for } i = 0, 1, \ldots, n-1, \\ \frac{d^j u}{dt^j}(0) &= 0 \qquad \text{for } j = 0, 1, \ldots, m-1. \end{aligned}$$

Since the initial conditions are zero, the derivative property (P2) states $\mathcal{L}\Big[\frac{d}{dt} y(t)\Big] = sY(s)$ and $\mathcal{L}\Big[\frac{d}{dt} u(t)\Big] = sU(s)$. Taking the Laplace transform of left and right hand side of (5.1), we obtain:

$$\Big(a_0 + a_1 s + \cdots + a_n s^n\Big) Y(s) \;=\; \Big(b_0 + b_1 s + \cdots + b_m s^m\Big) U(s). \tag{5.2}$$

The *transfer function* of the control system is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n} = \frac{\mathcal{L}[\text{output}]}{\mathcal{L}[\text{input}]}\Big|_{\text{zero initial conditions}} \tag{5.3}$$

In other words, we have the *multiplication formula*

$$Y(s) = G(s)U(s) \tag{5.4}$$

Note:

- This result is simple to remember: in the Laplace domain,

$$\text{output} = \text{transfer function} \times \text{input} \tag{5.5}$$

- If $G(s)$ and $U(s)$ are *rational* functions, then also $Y(s)$ is a *rational* function.
- If $u(t)$ is exponential-like (as in the Laplace transform Tables 4.1 and 4.2) and the ODE is linear, then also $y(t)$ is exponential-like.
- Here are some simple examples (where k is a constant):

(i) $y(t) = ku(t)$ implies $G(s) = k$,

(ii) $y(t) = k\dot{u}(t)$ implies $G(s) = ks$, and

(iii) $y(t) = k\int_0^t u(\sigma)d\sigma$ implies $G(s) = \frac{k}{s}$.

Remarks 5.1. *Here are some comments and extensions.*

(i) *Systems of the form (5.1) are said to be linear, because the input and state appear linearly, and time-invariant, because the coefficients are assumed constant, that is, time invariant.*

(ii) *The transfer function $G(s)$ is equivalent to the ODE model (5.1), in the sense that $G(s)$ contains the same information as the ODE model, i.e., the coefficients $a_0, \dots, a_n$ and $b_0, \dots, b_m$.*

(iii) *Many different physical systems may have the same transfer function. Therefore, it makes sense to define and study canonical systems, e.g., first-order, second-order, etc.*

Canonical transfer functions and canonical inputs

In this and the next chapter we study the responses of canonical systems (i.e., canonical transfer functions) to canonical inputs.

transfer function:	canonical form	impulse response, step response, and ramp response	frequency response (i.e., response to a sinusoidal input)
first order:	$\dfrac{1}{\tau s + 1}$	Section 5.3	Chapter 6
second order:	$\dfrac{\omega_{\mathrm{n}}^2}{s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2}$	Section 5.4	Chapter 6
higher order:	no typical form	Section 5.5	Chapter 6

Table 5.1: Transfer functions for canonical systems. Their responses to canonical inputs are discussed in this chapter and the next.

Responses to canonical inputs: impulse, step, and ramp

Given a transfer function $G(s)$, we wish to compute how the system responds to *canonical inputs*. Specifically, we consider:

impulse response: the response $y_{\text{impulse}}(t)$ from zero initial condition when the input $u(t) = \delta(t)$ is a unit impulse,

step response: the response $y_{\text{step}}(t)$ from zero initial condition when the input $u(t) = \mathbf{1}(t)$ is a unit step, and

ramp response: the response $y_{\text{ramp}}(t)$ from zero initial condition when the input $u(t) = t \cdot \mathbf{1}(t)$ is a unit ramp.

These canonical input have a very simple physical intuition: in a mechanical example, the impulse corresponds to a hammer hitting a nail, the step corresponds to a constant force applied to a vehicle (like in the car velocity system), and the ramp corresponds to a growing signal with constant (like a thermometer in a tank that is warming up).

From the Laplace transform Table 4.1 recall that $\mathcal{L}[\delta(t)] = 1$, $\mathcal{L}[\mathbf{1}(t)] = \frac{1}{s}$, and $\mathcal{L}[t] = \frac{1}{s^2}$ so that, from $Y(s) = G(s)U(s)$,

$$\boxed{Y_{\text{impulse}}(s)} = \mathcal{L}[y_{\text{impulse}}(t)] = \boxed{G(s)} \tag{5.6}$$

$$\boxed{Y_{\text{step}}(s)} = \mathcal{L}[y_{\text{step}}(t)] = \boxed{\frac{1}{s}G(s)} \tag{5.7}$$

$$\boxed{Y_{\text{ramp}}(s)} = \mathcal{L}[y_{\text{ramp}}(t)] = \boxed{\frac{1}{s^2}G(s)} \tag{5.8}$$

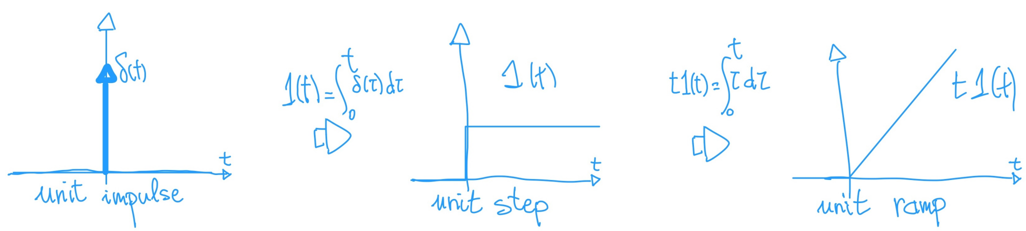


Figure 5.1: Unit impulse, unit step, and unit ramp functions

5.2 The impulse response

Therefore, the impulse response is

$$Y_{\text{impulse}}(s) = \mathcal{L}[y_{\text{impulse}}(t)] = G(s) \tag{5.9}$$

This simple equation has a surprising implication. Taking the inverse Laplace transform of both left and right hand side we obtain:

$$y_{\text{impulse}}(t) = \mathcal{L}^{-1}[G(s)] = g(t) \tag{5.10}$$

where, following our convention, we use $g(t)$ denote the function of time whose Laplace transform is $G(s)$.

We have learned:

(i) *the Laplace transform of the impulse response is the transfer function*,

(ii) to learn the transfer function of an unknown system, *(1) apply an impulse and (2) take the Laplace transform of the response*

(iii) *the impulse response contains all information about the input/output control system*

Note: the following representations are all equivalent:

(i) two vectors of coefficients $a_0, \ldots, a_n$ and $b_0, \ldots, b_m$,

(ii) the differential equation (5.1),

(iii) the transfer function $G(s) = \dfrac{b_0 + b_1 s + \cdots + b_n s^m}{a_0 + a_1 s + \cdots + a_n s^n}$, and

(iv) the impulse response $y_{\text{impulse}}(t) = \mathcal{L}^{-1}[G(s)]$

5.2.1 Detour: The impulse response in vehicle dynamics and audio system analysis

Remark 5.2 (The impulse response in acoustics). *In the field of acoustics and audio engineering, measuring the impulse response of a room (like a concert hall or a living room) or an audio system (like a speaker or a microphone) is very useful. Measuring impulse response is the first step towards optimizing them for audio quality and thereby designing audio-related products and technologies.*

In the context of acoustics, the impulse response is the sound received at a specific location B in response to a brief large-magnitude input signal at location A.

Sound Quality Assessment: By analyzing the impulse response, engineers can determine the reverberation characteristics of a room. This helps in assessing how sound is reflected and absorbed, affecting the quality of audio heard in the space.

Speaker and Microphone Design: Understanding the impulse response of speakers and microphones allows designers to optimize their products for clarity, frequency response, and distortion characteristics.

Audio Mixing and Mastering: In music production, the impulse response of different spaces (like concert halls, studios, etc.) can be used to digitally simulate how music would sound in those environments.

Noise Reduction and Echo Cancellation: In telecommunications, the impulse response of devices and environments helps in developing algorithms for noise reduction and echo cancellation.

5.3 First order systems and their responses

It is useful to consider examples of canonical transfer functions and their responses. We start with first order systems. Examples of first order systems include:

(i) the linear growth/decay model (1.1),
(ii) the car velocity system (2.4),
(iii) the thermal dynamics of a thermometer, and
(iv) the RC circuit.

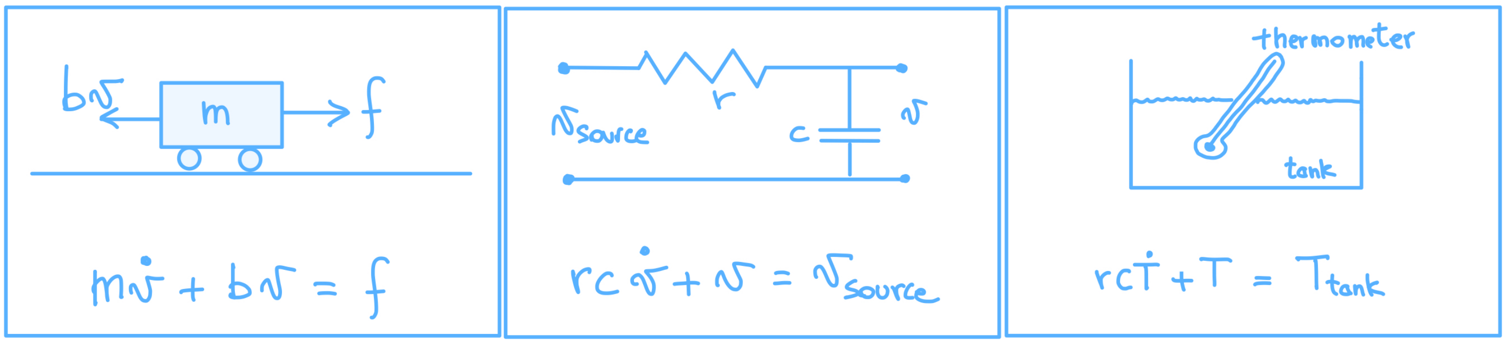


Figure 5.2: Illustrations of first order systems from earlier and later chapters.

As in Section 2.1.1, given a time constant $\tau > 0$, the *canonical form of a first order system* is

$$\tau\dot{y}(t) + y(t) = u(t) \tag{5.11}$$

where, as usual, $u(t)$ and $y(t)$ are the input and output of the system. The transfer function is

$$G_{\text{first-order}}(s) = \frac{Y(s)}{U(s)} = \frac{1}{\tau s + 1} \tag{5.12}$$

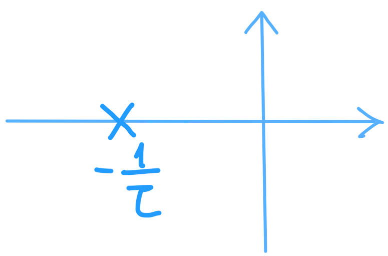


Figure 5.3: The transfer function (5.12) of a first order system has a single real pole at $s = -1/\tau$.
Since $\tau > 0$ is always positive, the pole is always on the strict left half plane.
When the time constant τ increases, the pole $s = -1/\tau$ moves towards the imaginary axis and the system response (both free and forced) becomes slower.

Via the inverse Laplace transform methods, we compute the impulse, step, and ramp response of a first-order system to be:

$$y_{\text{impulse}}(t) = \mathcal{L}^{-1}\left[\frac{1}{s\tau + 1}\right] = \frac{1}{\tau}\,\mathrm{e}^{-t/\tau} \tag{5.13}$$

$$y_{\text{step}}(t) = \mathcal{L}^{-1}\left[\frac{1}{s\,(s\tau + 1)}\right] = 1 - \mathrm{e}^{-t/\tau} \tag{5.14}$$

$$y_{\text{ramp}}(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2\,(s\tau + 1)}\right] = t - \tau(1 - \mathrm{e}^{-t/\tau}) \tag{5.15}$$

These calculations are left to Exercise E5.1.

Impulse, step and ramp responses of first-order systems

```python
import numpy as np; import matplotlib.pyplot as plt; import control as ctrl
plt.rcParams.update({"text.usetex": True, "font.family": "serif", "font.serif": ...
    ["Computer Modern Roman"] })

# Define a range of time constants and time range for the simulation
time_constants = [10, 8, 5, 3, 2, 1, 0.5]
t = np.linspace(0, 25, 1000)

# Define your preferred color vector
colors = ['#752d00', '#a43e00', '#d35000', '#ff6100', '#ff8800', '#ffaf00', '#ffcc00']

# Initialize the figure for impulse, step, and ramp responses
fig, axs = plt.subplots(3, 1, figsize=(10, 10))

# Loop through each time constant and plot the impulse, step, and ramp responses
for idx, tau in enumerate(time_constants):
    # Define the transfer function of the first-order system
    num = [1]
    den = [tau, 1]
    system = ctrl.TransferFunction(num, den)

    # Compute and plot the impulse response
    t_impulse, y_impulse = ctrl.impulse_response(system, T=t)
    axs[0].plot(t_impulse, y_impulse, label=f'$\\tau = {tau}$', color=colors[idx])

    # Compute and plot the step response
    t_step, y_step = ctrl.step_response(system, T=t)
    axs[1].plot(t_step, y_step, label=f'$\\tau = {tau}$', color=colors[idx])

    # Compute and plot the ramp response
    ramp_input = t
    t_ramp, y_ramp = ctrl.forced_response(system, T=t, U=ramp_input)
    axs[2].plot(t_ramp, y_ramp, label=f'$\\tau = {tau}$', color=colors[idx])

# Add labels, legends, grid, and set xlim for all subplots
for ax in axs:
    ax.legend(); ax.grid(True); ax.set_xlim(0, 25); ax.set_ylabel('Response')
axs[0].set_xlim(0, 5);  axs[0].set_xticklabels([]); axs[1].set_xticklabels([]);

# Add arrow with text
arrow_color = '#0055A4'
text_color = arrow_color

axs[0].text(0.5, 0.5, "increasing $\\tau$", ha="center", va="center", rotation=45, ...
    size=15, color=text_color, bbox=dict(boxstyle="larrow,pad=0.3", fc="none", ...
    ec=arrow_color, lw=2, alpha=0.5))
axs[1].text(2.5, 0.75, "increasing $\\tau$", ha="center", va="center", rotation=-45, ...
    size=15, color=text_color, bbox=dict(boxstyle="rarrow,pad=0.3", fc="none", ...
    ec=arrow_color, lw=2, alpha=0.5))
axs[2].text(15, 12, "increasing $\\tau$", ha="center", va="center", rotation=-45, ...
    size=15, color=text_color, bbox=dict(boxstyle="rarrow,pad=0.3", fc="none", ...
    ec=arrow_color, lw=2, alpha=0.5))

# Plot unit step and unit ramp in gray
axs[1].plot(t, np.ones_like(t), label='Unit Step', color='gray', linestyle='--')
axs[2].plot(t, t, label='Unit Ramp', color='gray', linestyle='--')

axs[0].set_title(r'Impulse response of 1st order system, for varying time constant $\tau$');
axs[1].set_title(r'Step response of 1st order system, for varying time constant $\tau$');
axs[2].set_title(r'Ramp response of 1st order system, for varying time constant ...
    $\tau$'); axs[2].set_xlabel('Time (s)')

# Adjust layout and save the plot to a PDF file
plt.tight_layout(); plt.savefig('1storder-responses.pdf', bbox_inches='tight')
```

Listing 5.1: Python script generating Figure 5.4. This script relies upon the Python Control Systems Library (Fuller et al., 2021). Available at 1storder-responses.py

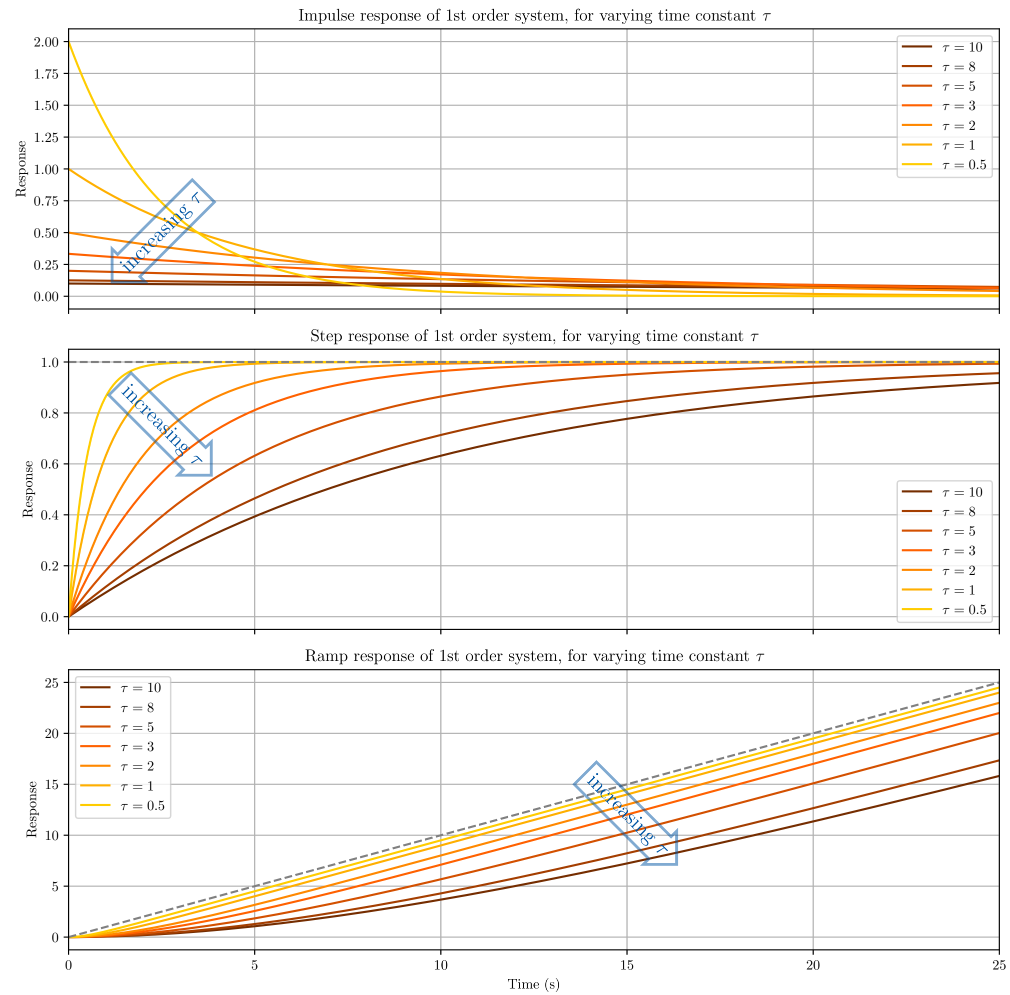


Figure 5.4: Canonical responses of the first order dynamics (5.11), when the input is a unit impulse, a unit step, and a unit ramp.
For increasing time constant τ, the system response become slower for all three inputs and, for the ramp response, the difference between input and output (tracking error) becomes larger.

5.4 Second order systems and their responses

Example of second order systems include:

(i) the forced mass-spring-damper system (2.12),

(ii) the RLC circuit (2.44), and

(iii) the linearized pendulum about either the up or down position (3.30).

(We will study more examples in the third part.[1])

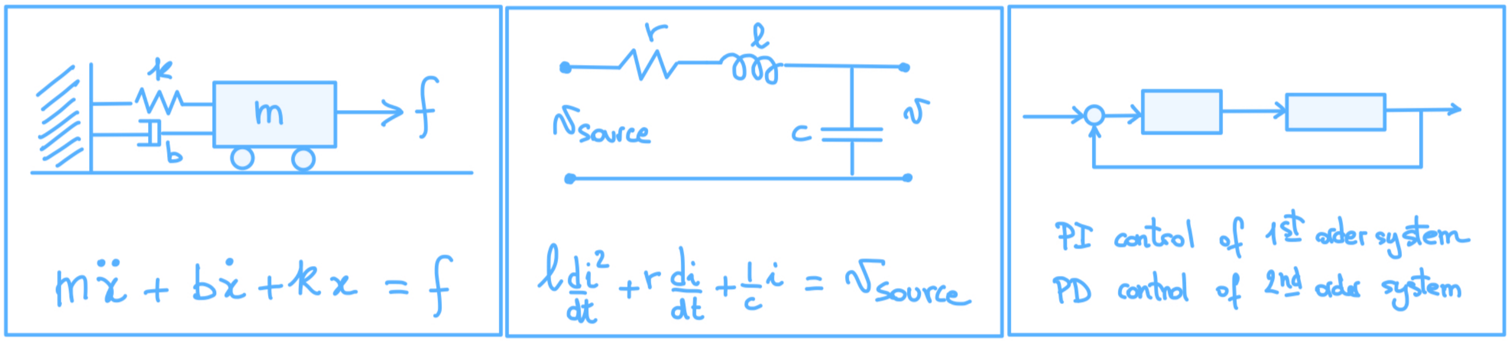


Figure 5.5: Illustrations of second order systems from earlier chapters.

[1]Namely, a first order system subject to proportional + integral control (7.28), and an inverted pendulum subject to proportional + derivative control (7.33).

5.4.1 Transfer function of a mass-spring-damper systems

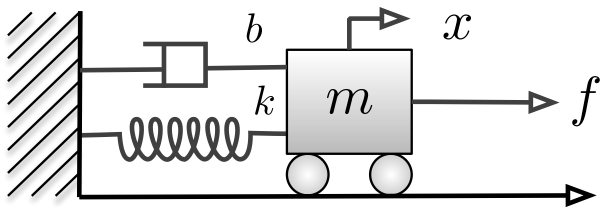


Figure 5.6: Generalizing equation (2.12), a mass-spring-damper system with parameters $m > 0$, $b \geq 0$, and $k > 0$, subject to a force $f(t)$.

In our discussion, the mass and the spring coefficient are always positive, but we do allow the damper to be present ($b > 0$) or not ($b = 0$).

Consider a forced mass spring damper system:

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = f(t). \tag{5.16}$$

Taking the Laplace transform (at zero initial conditions) we obtain

$$(ms^2 + bs + k)X(s) = F(s) \tag{5.17}$$

and therefore the transfer function is

$$\boxed{\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}} \tag{5.18}$$

Recall the definition of natural frequency $\omega_{\mathrm{n}} = \sqrt{k/m}$.

In class assignment

Recall $m > 0$ and $k > 0$, whereas $b \geq 0$.

Where may the two poles be in the complex plane?
How many qualitatively different cases do there exist?
When is a system fast or slow?

5.4.2 Canonical form of second-order systems with canonical parameters $(\omega_\mathsf{n}, \zeta)$

Before computing the poles of a generic second order system it is convenient to define a *canonical form* with canonical parameters (like the time constant $\tau > 0$ for first order systems).

The *canonical form of a second order system* is

$$\ddot{y}(t) + 2\zeta\omega_\mathsf{n}\dot{y}(t) + \omega_\mathsf{n}^2 y(t) \;=\; \omega_\mathsf{n}^2 u(t) \tag{5.19}$$

with corresponding transfer function

$$G_{\text{second-order}}(s) = \frac{Y(s)}{U(s)} \;=\; \frac{\omega_\mathsf{n}^2}{s^2 + 2\zeta\omega_\mathsf{n} s + \omega_\mathsf{n}^2} \tag{5.20}$$

where, as usual, $u(t)$ and $y(t)$ are the input and output of the system, and where the canonical parameters are:

- $\omega_\mathsf{n} > 0$ is the *natural frequency* of the system, indicating how fast the system oscillates in the absence of damping; and
- $\zeta \geq 0$ is the *damping ratio*, a dimensionless measure of damping in the system.

Remark 5.3 (The canonical form and the mass-spring-damper system). *We compare the canonical form of a second order system with the mass-spring-damper system. The natural frequency ω_n and damping ratio ζ can be computed as functions of mass m, spring stiffness k and damping coefficient b by matching the denominators of (5.20) and (5.18) (divided by m), that is,*

$$\frac{1}{m}(ms^2 + bs + k) \;=\; s^2 + 2\zeta\omega_\mathsf{n} s + \omega_\mathsf{n}^2 \tag{5.21}$$

$$\Longrightarrow \qquad \boxed{\omega_\mathsf{n} = \sqrt{\frac{k}{m}}} \qquad \text{and} \qquad \boxed{\zeta = \frac{b}{2\sqrt{mk}}}\,. \tag{5.22}$$

•

5.4.3 Classification of second-order systems, as a function of the pole placement

We are now ready to compute the two poles of the second-order system in canonical form (5.20), which we report for convenience:

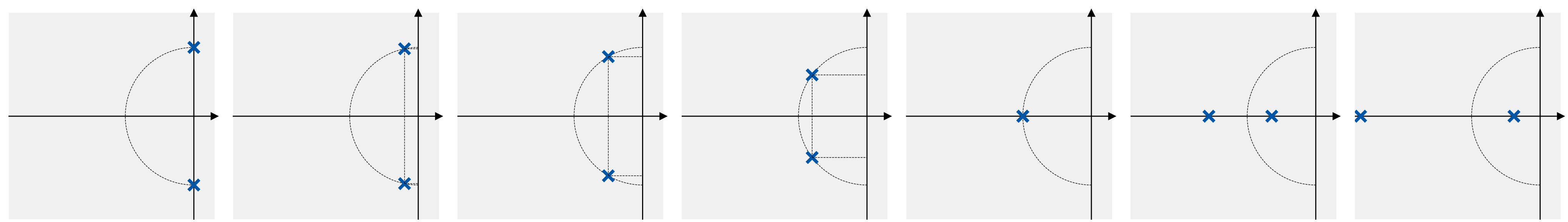
$$G_{\text{second-order}}(s) = \frac{\omega_{\mathsf{n}}^2}{s^2 + 2\zeta\omega_{\mathsf{n}}s + \omega_{\mathsf{n}}^2} \tag{5.23}$$

The poles are

$$\boxed{\text{poles of } G_{\text{second-order}}(s)} = \frac{-2\zeta\omega_{\mathsf{n}} \pm \sqrt{4\zeta^2\omega_{\mathsf{n}}^2 - 4\omega_{\mathsf{n}}^2}}{2} = \boxed{-\omega_{\mathsf{n}}\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)}$$

Depending upon the damping ratio ζ, the poles are purely imaginary, complex conjugate, real equal, or real distinct, see Figure 5.7. When $0 < \zeta < 1$, we write

$$\boxed{\text{complex conjugate poles of } G_{\text{second-order}}(s) = -\zeta\omega_{\mathsf{n}} \pm \mathrm{i}\omega_{\mathsf{d}}}, \quad \text{where the } \textit{damped natural frequency} \text{ is } \quad \boxed{\omega_{\mathsf{d}} = \omega_{\mathsf{n}}\sqrt{1 - \zeta^2}}$$



(a) $\zeta = 0.0 \implies$ two poles $= \pm\mathrm{i}\omega_{\mathsf{n}}$
Case I: undamped system

(b) $\zeta = 0.2$
Case II: underdamped system

(c) $\zeta = 0.5$
Case II: underdamped system

(d) $\zeta = 0.8$
Case II: underdamped system

(e) $\zeta = 1.0 \implies$ two poles $= -\omega_{\mathsf{n}}$
Case III: critically-damped system

(f) $\zeta = 1.1$
Case IV: overdamped system

(g) $\zeta = 1.5$
Case IV: overdamped system

Figure 5.7: Poles of a second order system as a function of the damping ratio ζ, at fixed natural frequency ω_{n}. The dashed semicircle has radius ω_{n}.
At $\zeta = 0$, the two poles are purely imaginary and equal to $\pm\mathrm{i}\omega_{\mathsf{n}}$.
As ζ increases from 0 to 1, the two complex conjugate poles move strictly inside the left half plane, sliding along the semicircle.
When $0 < \zeta < 1$, the two complex conjugate poles have real part $-\zeta\omega_{\mathsf{n}}$ and imaginary part $\pm\mathrm{i}\omega_{\mathsf{d}}$.
At $\zeta = 1$, the two poles are coincident at the real value $-\omega_{\mathsf{n}}$.
For $\zeta > 1$, the two poles split: one moves left towards $-\infty$ (the fast pole) and one moves right towards the imaginary axis (the slow dominant pole).
Available at `2ndorder-poles.py`

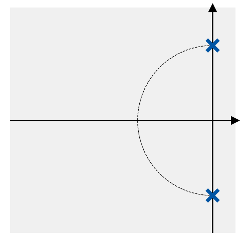
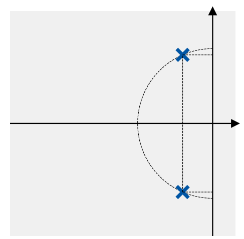
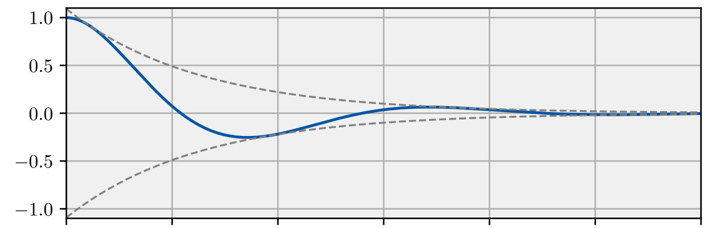
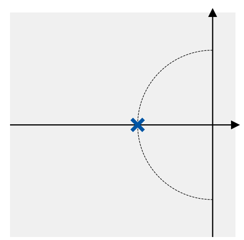
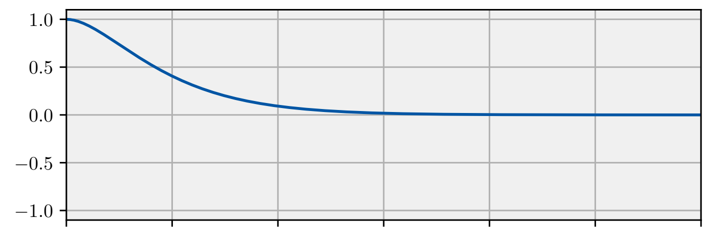
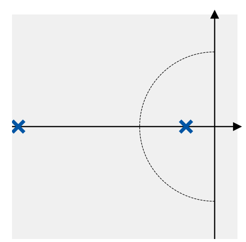
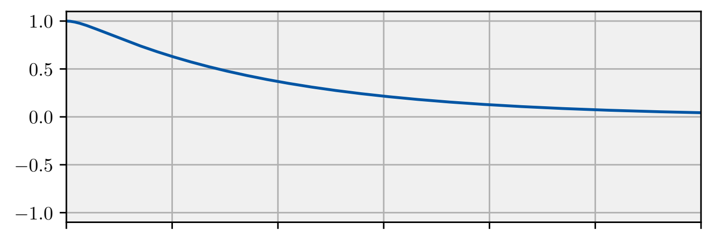
	damping ratio ζ	poles of transfer function (5.23) and corresponding functions of time	poles location (dashed line = circle of radius ω_n)	free response of $\ddot{x} + 2\zeta\omega_\mathrm{n}\dot{x} + \omega_\mathrm{n}^2 x = 0$ with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$
Case I: undamped system	$\zeta = 0$	two poles $= \pm \mathrm{i}\omega_\mathrm{n}$ $\sin(\omega_\mathrm{n} t)$, $\cos(\omega_\mathrm{n} t)$ sinusoidal waves		
Case II: underdamped system	$0 < \zeta < 1$	two poles $= -\omega_\mathrm{n}\zeta \pm \mathrm{i}\omega_\mathrm{n}\sqrt{1-\zeta^2}$ $\mathrm{e}^{-\omega_\mathrm{n}\zeta t}\sin(\omega_\mathrm{d} t)$, $\mathrm{e}^{-\omega_\mathrm{n}\zeta t}\cos(\omega_\mathrm{d} t)$ where $\omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1-\zeta^2}$ damped sinusoidal waves		
Case III: critically-damped system	$\zeta = 1$	two poles $= -\omega_\mathrm{n}$ $\mathrm{e}^{-\omega_\mathrm{n} t}$, $t\,\mathrm{e}^{-\omega_\mathrm{n} t}$ exponential decay (with transient)		
Case IV: overdamped system	$\zeta > 1$	two poles $= -\omega_\mathrm{n}(\zeta \pm \sqrt{\zeta^2 - 1})$ slow pole: $\mathrm{e}^{-\omega_\mathrm{n}(\zeta - \sqrt{\zeta^2-1})t}$ fast pole: $\mathrm{e}^{-\omega_\mathrm{n}(\zeta + \sqrt{\zeta^2-1})t}$ exponential decay		

Table 5.2: Classification of a second order system into 4 classes: undamped, underdamped, critically-damped, and overdamped.
For $0 < \zeta < 1$ (Case II), the *damped frequency* is $\omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1-\zeta^2}$. Note $\omega_\mathrm{d} < \omega_\mathrm{n}$, so the presence of damping diminishes the frequency of oscillations.
In the overdamped case (Case IV), the pole close to the imaginary axis is the *slow pole*, whereas the pole moving towards $-\infty$ is the fast pole.

Poles of an underdamped system in the complex plane

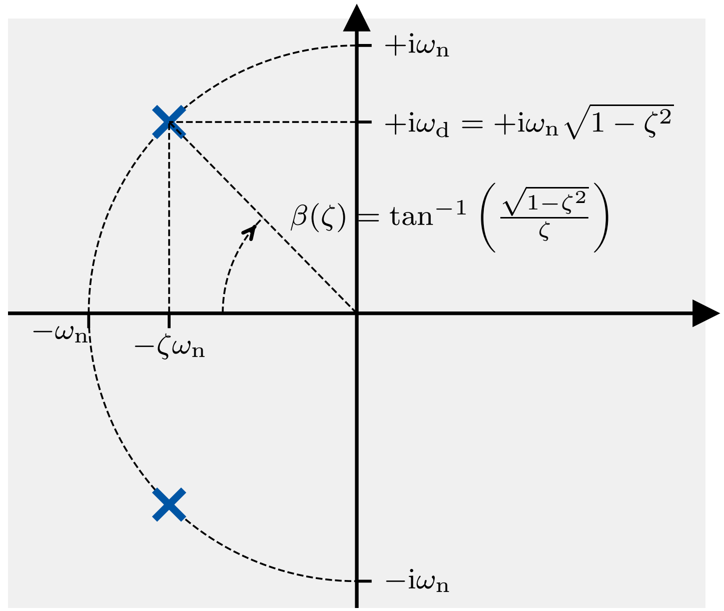


Figure 5.8: Poles of an underdamped second-order system, defined by a natural frequency ω_n and a damping ratio $0 < \zeta < 1$.
Note the damped natural frequency ω_d and the damping angle $\beta(\zeta)$.
To verify that the complex conjugate poles of $G_{\text{second-order}}(s)$ move on the circle of radius ω_n, it suffices to show that $|-\zeta\omega_\mathrm{n} \pm \mathrm{i}\omega_\mathrm{d}| = \omega_\mathrm{n}$.
Available at `2ndorder-pole-beta.py`

Case I: *Undamped systems*. When $\zeta = 0$, the system is *undamped* and exhibits persistent oscillatory behavior.

Case I: *Underdamped systems*. When $0 < \zeta < 1$, the system is *underdamped* and exhibits damped oscillatory behavior.

Case III: *Critically-damped systems*. When $\zeta = 1$, the system is *critically damped* and returns to equilibrium as quickly as possible without oscillating.

Case IV: *Overdamped systems*. When $\zeta > 1$, the system is *overdamped* and returns to equilibrium without oscillating, but more slowly than in the critically damped case.

Regarding the natural frequency: this parameter determines the speed of the response in each of the four cases.

Remark 5.4 (The canonical form and the mass-spring-damper system: continued). *For a mass-spring-damper system, the characteristic equation is* $ms^2 + bs + k = 0$ *and its solutions are* $\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$*. Therefore, the two roots are equal and real when* $b^2 = 4mk$*. We define the critical damping parameter as* $b_{\text{critical}} = 2\sqrt{mk}$*. Then*

- *the system is underdamped for* $b < b_{\text{critical}} = 2\sqrt{mk}$*,*
- *the system is critically damped for* $b = b_{\text{critical}} = 2\sqrt{mk}$*, and*
- *the system is overdamped for* $b > b_{\text{critical}} = 2\sqrt{mk}$*.*
- *It is now clear why* ζ *is called the damping ratio: for mass-spring-dampers systems,* ζ *is indeed a ratio:*

$$\zeta = \frac{b}{b_{\text{critical}}} = \frac{b}{2\sqrt{mk}}.$$

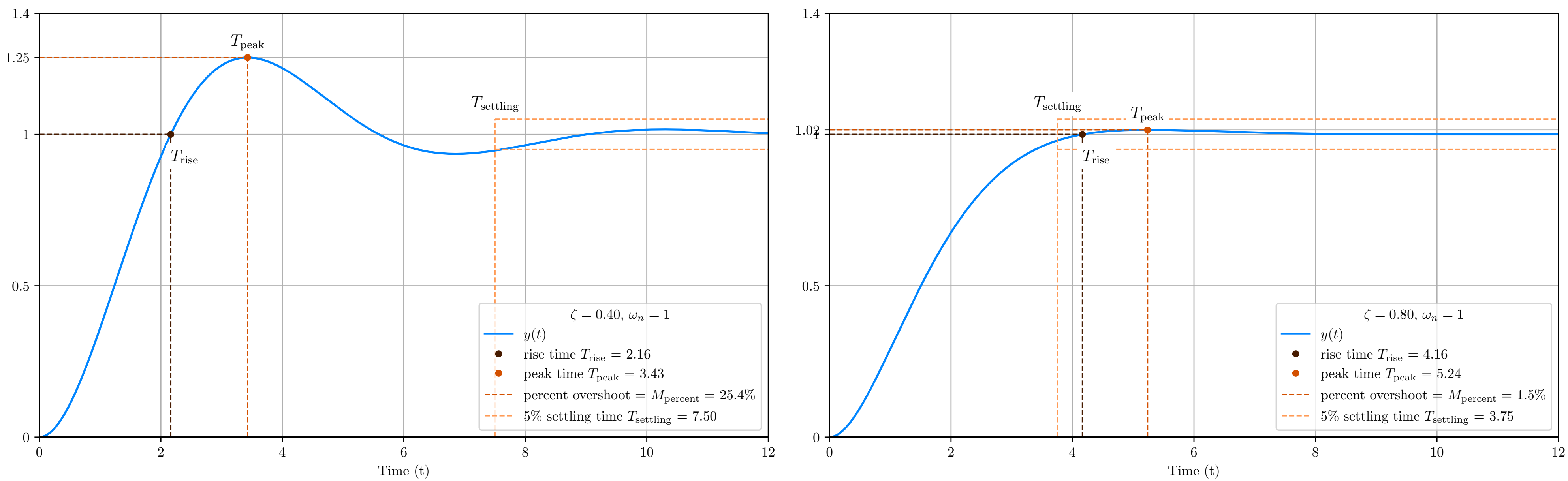
•

5.4.4 Step response of an underdamped system

For an underdamped second-order system with damping ratio $0 < \zeta < 1$, and arbitrary natural frequency ω_n, the step response is

$$y(t) \; = \; 1 - \mathrm{e}^{-\zeta\omega_\mathrm{n} t}\left(\cos(\omega_\mathrm{d} t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_\mathrm{d} t)\right) \tag{5.24}$$

where $\omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1-\zeta^2}$ is the *damped natural frequency*. (We refer to Appendix 5.6 for inverse Laplace transform calculations.)



(a) $\zeta = .4$ and $\omega_\mathrm{n} = 1$

(b) $\zeta = .8$ and $\omega_\mathrm{n} = 1$

Figure 5.9: Step response of an underdamped second order system from zero initial position and initial velocity, for varying damping ratios ζ.
The step response shows how different values of ζ affect key characteristics such as rise time, peak time, percent overshoot, and settling time.

A low damping ratio $\zeta = .4$ leads to fast response times, but also high overshoot and prolonged oscillations before settling.
A high damping ratio $\zeta = .8$ provides a smooth slower response with minimal overshoot, but also slow reaction times.

Time domain specifications

- The *rise time* T_{rise} is the time required for the response to rise from 0% to 100% of the final value. Some calculations show:

$$T_{\text{rise}} = \frac{\pi - \beta(\zeta)}{\omega_{\text{d}}} \qquad \text{where} \qquad \beta(\zeta) = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \tag{5.25}$$

- The *peak time* T_{peak} is the time it takes for the response to reach the maximum overshoot value (this is the first peak in the oscillatory response, at which the overshoot is maximum). Some calculations show:

$$T_{\text{peak}} = \frac{\pi}{\omega_{\text{d}}} = \frac{\pi}{\omega_{\text{n}}\sqrt{1-\zeta^2}} \tag{5.26}$$

- The *percent overshoot* M_{percent} is the maximum amount the system response overshoots its final value, devided by its final value. Some calculations show:

$$M_{\text{percent}} = \mathrm{e}^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \tag{5.27}$$

- The *settling time* T_{settling} is the time it takes for the response to remain within a certain range (typically 1% or 5%) of the steady-state value. For the 1% ad 5% criteria, approximate formulas are:

$$T_{\text{settling }1\%} \approx \frac{5}{\zeta\omega_{\text{n}}} \qquad \text{and} \qquad T_{\text{settling }5\%} \approx \frac{3}{\zeta\omega_{\text{n}}} \tag{5.28}$$

  On a related note, the *time constant* of the underdamped system is

$$\tau = \frac{1}{\zeta\omega_{\text{n}}} \tag{5.29}$$

It is a useful simple exercise to verify that the values of T_{rise}, T_{peak}, T_{settling} and M_{percent} in Figure 5.9 are correct, for values of $\omega = 1$ and $\zeta \in \{0.4, 0.8\}$.

5.4.5 Impulse, step, and ramp responses of second-order systems

```python
import numpy as np; import matplotlib.pyplot as plt; import control as ctrl
plt.rcParams.update({"text.usetex": True, "font.family": "serif", "font.serif": ...
    ["Computer Modern Roman"] })

# Define the parameters of the system
natural_frequency = 1.0  # Natural frequency, omega_n
damping_ratios = [0.0, 0.2, 0.4, 0.8, 1.0, 1.5, 3.0]  # Damping ratios, zeta

# Define time range for the simulation
t = np.linspace(0, 12, 1000); ramp_input = t  # Unit ramp input
colors = ['#752d00', '#a43e00', '#d35000', '#ff6100', '#ff8800', '#ffaf00', '#ffcc00']

# Initialize the figure for impulse, step, and ramp response
fig, axs = plt.subplots(3, 1, figsize=(10, 11))

# Loop through each damping ratio and plot impulse, step, and ramp responses
for idx, zeta in enumerate(damping_ratios):
    # Define the transfer function of the second-order system
    num = [natural_frequency**2]; den = [1, 2 * zeta * natural_frequency, ...
        natural_frequency**2]
    system = ctrl.TransferFunction(num, den)

    # Compute and plot the impulse response
    t_impulse, y_impulse = ctrl.impulse_response(system, T=t)
    line_style = '--' if idx == 4 else '-'
    axs[0].plot(t_impulse, y_impulse, line_style, label=f'$\\zeta$ = {zeta}', ...
        color=colors[idx])

    # Compute and plot the step response
    t_step, y_step = ctrl.step_response(system, T=t)
    axs[1].plot(t_step, y_step, line_style, label=f'$\\zeta$ = {zeta}', color=colors[idx])

    # Compute and plot the ramp response
    t_ramp, y_ramp = ctrl.forced_response(system, T=t, U=ramp_input)
    axs[2].plot(t_ramp, y_ramp, line_style, label=f'$\\zeta = {zeta}$', color=colors[idx])

# Add labels, legends, grid, and set xlim for all subplots
for ax in axs:
    ax.legend(loc='lower right'); ax.grid(True); ax.set_xlim(0, 25); ...
        ax.set_ylabel('Response')

# Set plot properties
axs[0].set_xlim(0, 12); axs[0].set_ylim(-1.1, 1.1)
axs[0].set_title('Impulse response of 2nd order system, for varying damping ratio $\zeta$')
axs[1].set_xlim(0, 12); axs[1].set_ylim(-0.1, 2.1)
axs[1].set_title('Step response of 2nd order system, for varying damping ratio $\zeta$')
axs[2].set_xlim(0, 12); axs[2].set_ylim(0, 12); axs[2].set_xlabel('Time (s)');
axs[2].set_title('Ramp response of 2nd order system, for varying damping ratio $\zeta$')

# Add arrow with text
arrow_color = '#0055A4'
text_color = arrow_color  # Set text color same as arrow color

axs[0].text(1, 0.5, "increasing $\\zeta$", ha="center", va="center", rotation=-45, ...
    size=15, color=text_color, bbox=dict(boxstyle="rarrow,pad=0.3", fc="none", ...
    ec=arrow_color, lw=2, alpha=0.5))
axs[1].text(2, 0.75, "increasing $\\zeta$", ha="center", va="center", rotation=-45, ...
    size=15, color=text_color, bbox=dict(boxstyle="rarrow,pad=0.3", fc="none", ...
    ec=arrow_color, lw=2, alpha=0.5))
axs[2].text(7, 5, "increasing $\\zeta$", ha="center", va="center", rotation=-45, ...
    size=15, color=text_color, bbox=dict(boxstyle="rarrow,pad=0.3", fc="none", ...
    ec=arrow_color, lw=2, alpha=0.5))

# Save the plot to a PDF file
plt.tight_layout(); plt.savefig('2ndorder-responses.pdf', bbox_inches='tight')
```

Listing 5.2: Python script generating Figure 5.10. This script relies upon the Python Control Systems Library (Fuller et al., 2021). Available at 2ndorder-responses.py

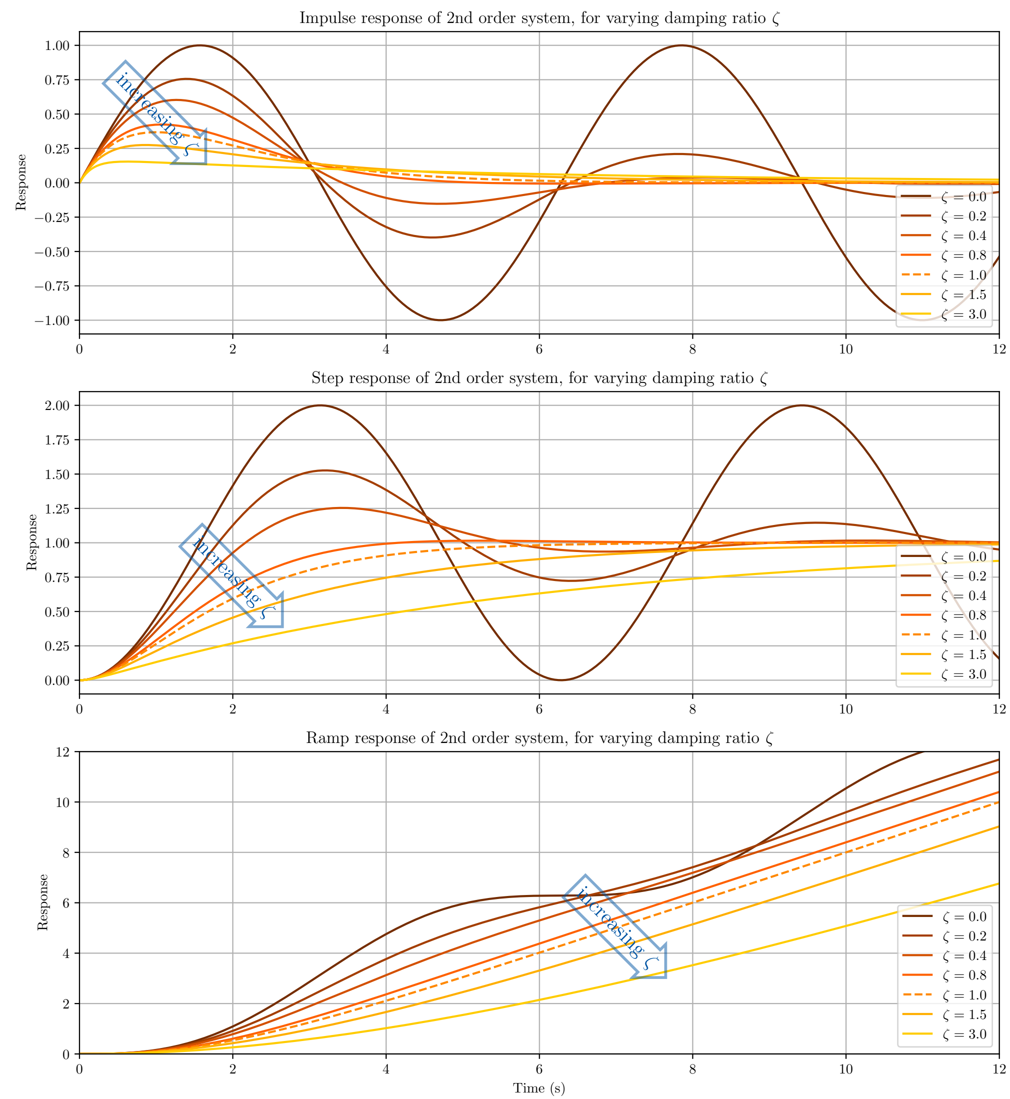


Figure 5.10: Solutions of the 2nd order dynamics (5.19).

5.4.6 Free response of an underdamped system

In the undamped and underdamped regime, when $0 \leq \zeta < 1$, consider

$$\ddot{x} + 2\zeta\omega_{\mathsf{n}}\dot{x} + \omega_{\mathsf{n}}^2 x = 0$$

with positive initial position $x(0) = x_0 > 0$ and zero initial velocity $\dot{x}(0) = 0$. Via the inverse Laplace transform calculations in Appendix 5.6, the free response of an underdamped system

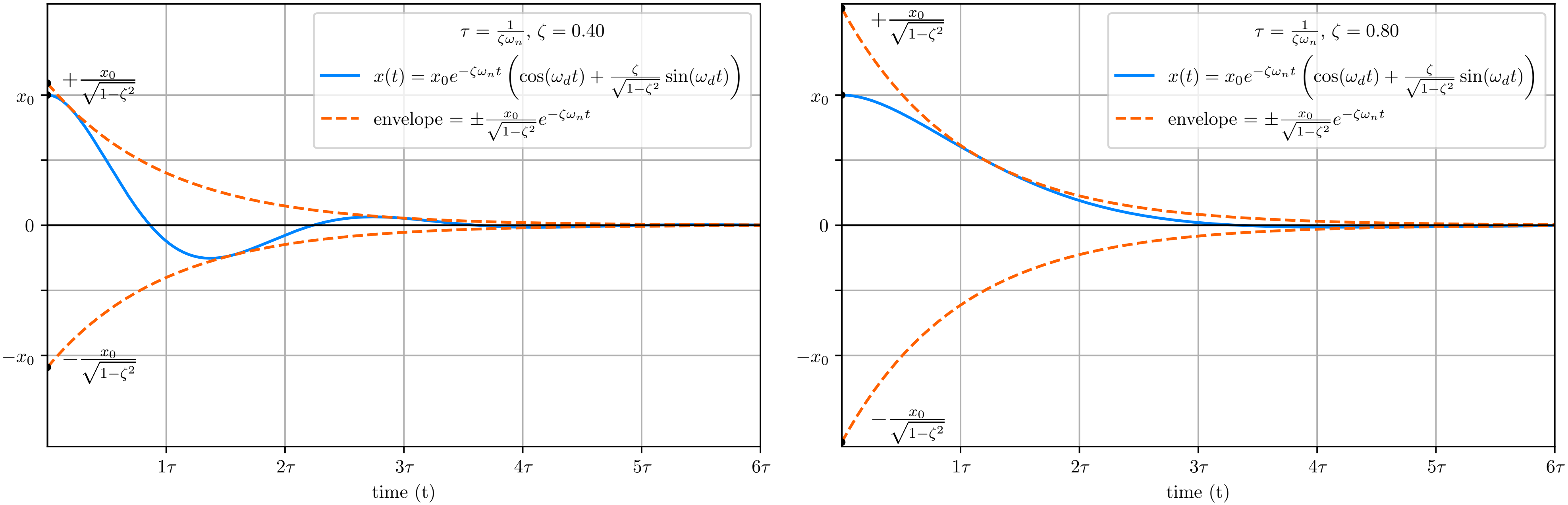
$$\boxed{x(t) = x_0 \, \mathrm{e}^{-\zeta\omega_{\mathsf{n}}t}\Big(\cos(\omega_{\mathsf{d}}t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_{\mathsf{d}}t)\Big)} \tag{5.30}$$

where the *damped frequency* is $\omega_{\mathsf{d}} = \omega_{\mathsf{n}}\sqrt{1-\zeta^2}$. Using trigonometric equalities, we can rewrite the solution as

$$x(t) = \underbrace{\frac{x_0}{\sqrt{1-\zeta^2}}\,\mathrm{e}^{-\zeta\omega_{\mathsf{n}}t}}_{\text{exponentially-decaying envelope}} \cdot \cos\Big(\omega_{\mathsf{d}}t + \arctan\frac{\zeta}{\sqrt{1-\zeta^2}}\Big) \tag{5.31}$$

The expression (5.31) is useful because the precise expression of the exponentially-decaying envelope is now clear.

As for first order systems, after time equal to $5 \cdot \tau$, the free response is guaranteed to be below 1% of the initial value $\dfrac{x_0}{\sqrt{1-\zeta^2}}$.



(a) $\zeta = .4$ (b) $\zeta = .8$

Figure 5.11: Free response of an underdamped second order system from initial position $x_0 > 0$ and zero initial velocity.
Note: the the exponentially-decaying envelope starts at $\pm\dfrac{x_0}{\sqrt{1-\zeta^2}}$.

Note: after time equal to $5 \cdot \tau = 5/(\zeta\omega_\mathrm{n})$, the solution is guaranteed to be below 1% of the initial value $\dfrac{x_0}{\sqrt{1-\zeta^2}}$.

Note however: for $0 < \zeta < 1$, the factor $\frac{1}{\sqrt{1-\zeta^2}}$ is always greater than 1 and approximately 2.3, 7.1 and 22.4 at $\zeta = .9, .99, .999$, respectively.

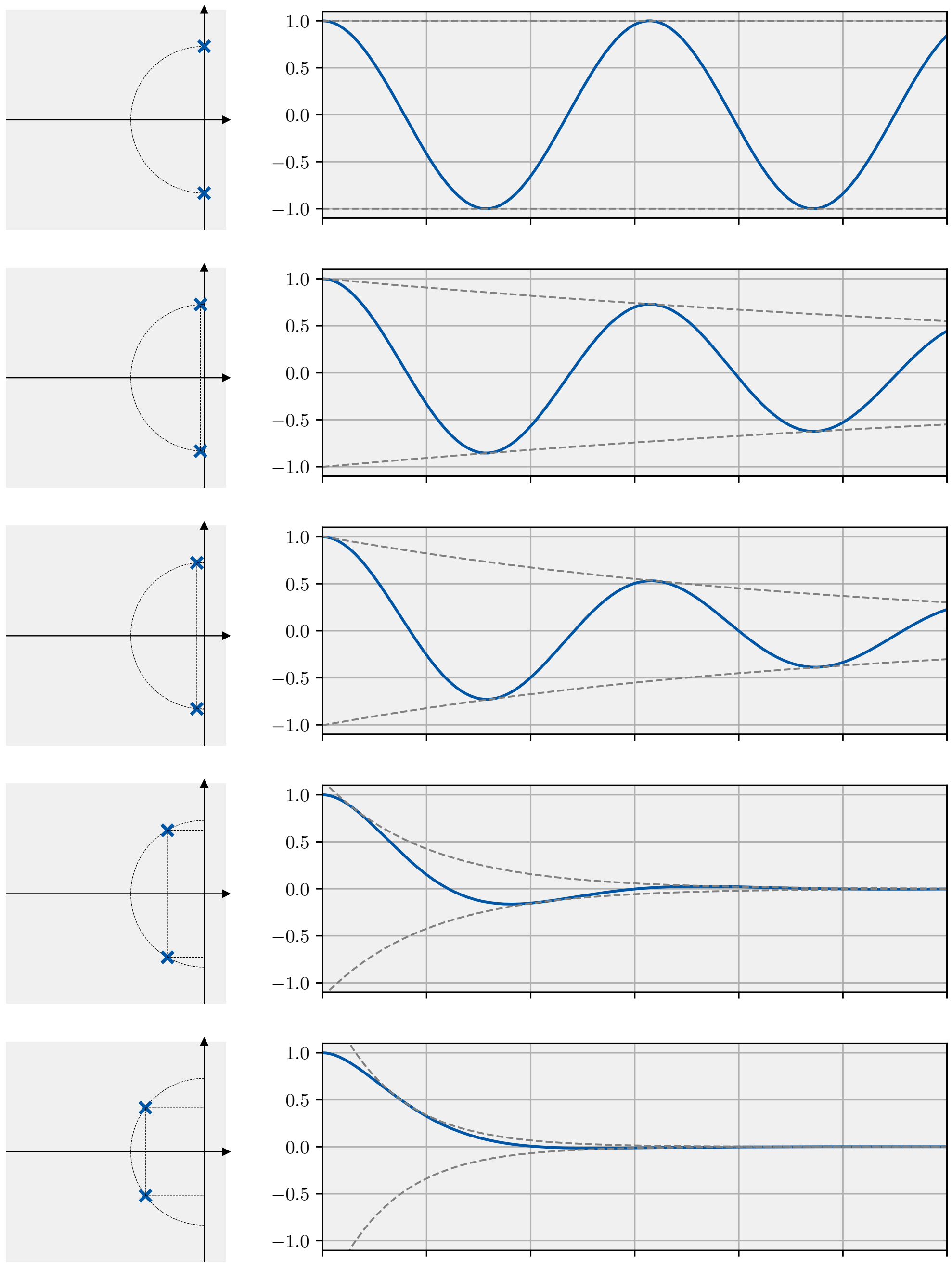


Figure 5.12: Illustrations of the free response of undamped and underdamped second-order systems.
Left panels: location of the two poles and semicircle of radius ω_n, we let $\omega_n = 1$.
Right panels: the free response from zero initial velocity (solid blue line) and the exponentially-decaying envelope (dashed gray lines): $\pm \frac{x_0}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n t}$.

5.5 Higher order systems and their step response

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n} \tag{5.32}$$

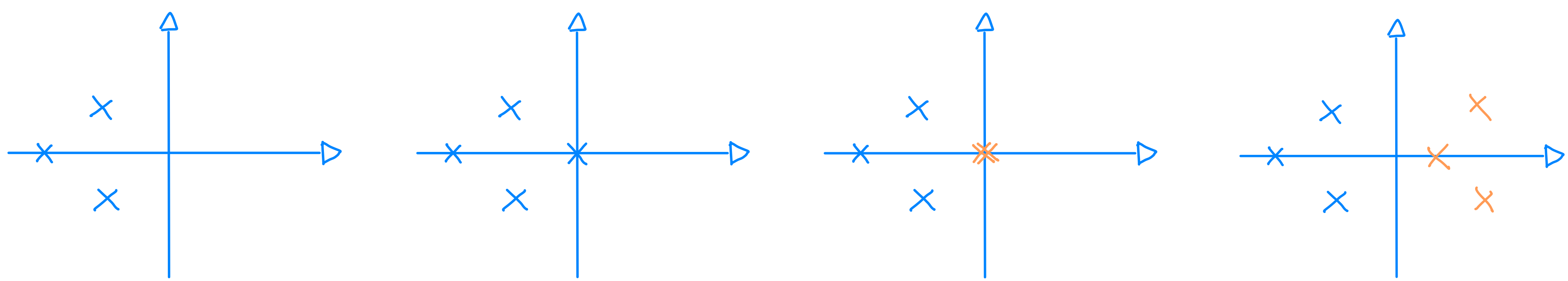
Assume $G(s)$ has distinct real poles $-p_1, \ldots, -p_n$, meaning that the denominator of $G(s)$ can be factored as $(s+p_1)(s+p_2)\ldots(s+p_n)$. We assume the poles are in the strict left half plane, that is, all p_i are strictly positive.

In class assignment

Why do we assume that the poles are in the left half plane?

A linear time-invariant system is

- *stable* when all poles are in the strict left half plane,
- *unstable* when at least one pole lies in the strict right half plane,
- *marginally stable* when
  - all poles are in the strict half plane or on the imaginary axis,
  - the poles on the imaginary axis (if any) are not repeated.



(a) stable system (b) marginally stable system (c) marginally unstable system (d) unstable system

For a stable transfer function $G(s)$,

(i) if the input is a unit step, then the steady-state output (the output after all decaying signals have decayed) is a step of magnitude $G(0)$:

$$u(t) = \mathbf{1}(t) \quad \Longrightarrow \quad y_{\text{steady-state}}(t) = G(0)\mathbf{1}(t), \tag{5.33}$$

(ii) $G(0)$ is the *steady-state gain* (or *DC gain*) since it is the amplification (or attenuation) of the input signal at the output.

Note: $G(0) = 1$ for the canonical forms of first and second-order systems.

We now verify these statements. When $u(t) = \mathbf{1}(t)$ and $U(s) = \frac{1}{s}$, the partial fraction expansion of $Y(s) = G(s) \cdot \frac{1}{s}$ is

$$Y(s) = \frac{r}{s} + \sum_{i=1}^{n} \frac{r_i}{s + p_i} \tag{5.34}$$

for appropriate residues $r, r_1, \dots, r_n$. Therefore, the output is the sum of a step function and n exponentially decaying terms:

$$y(t) = r + \sum_{i=1}^{n} r_i \, \mathrm{e}^{-p_i t} \tag{5.35}$$

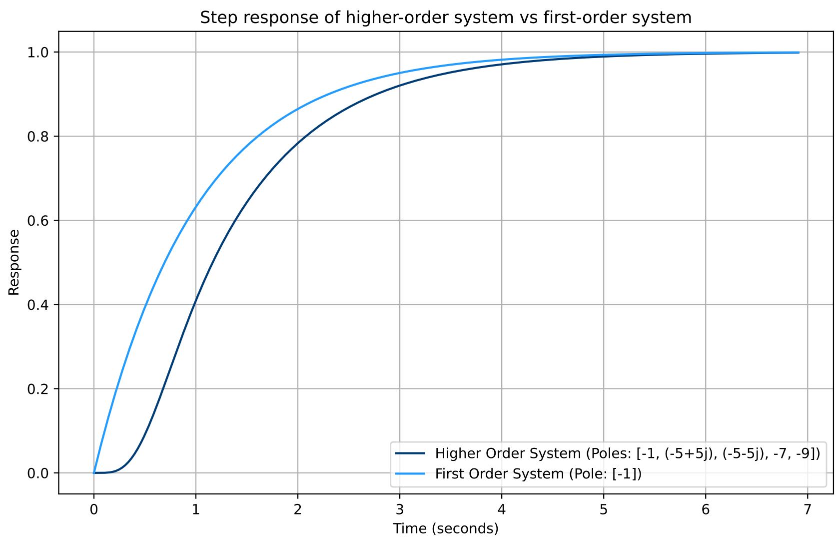
We are particularly interested in the behavior for large times t, when the exponentially decaying terms are below 1% of their initial value. To study this asymptotic behavior, we compute r using the single-pole residue formula:

$$r \;=\; sY(s)\Big|_{s=0} \;=\; s \cdot \frac{1}{s} G(s)\Big|_{s=0} \;=\; G(0) \tag{5.36}$$

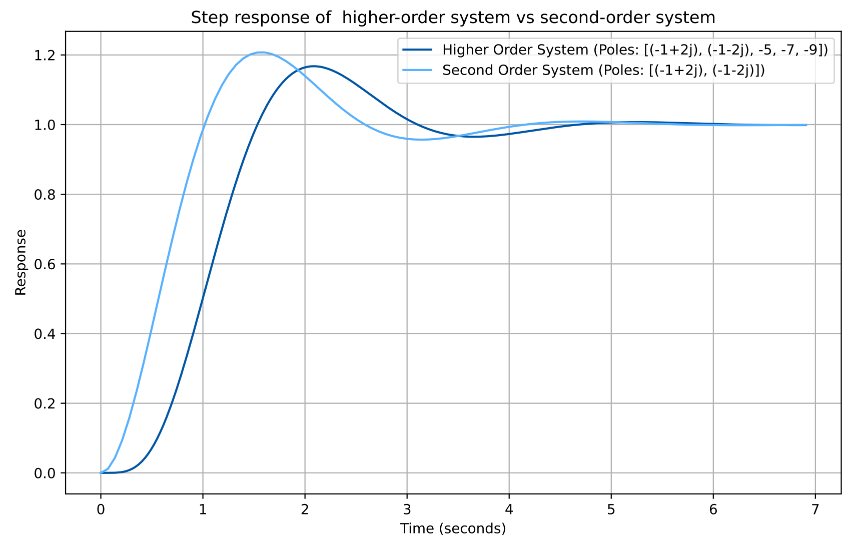
In systems with multiple poles, one or a few *dominant poles* might primarily determine the system's transient response. The *dominant poles are the ones closest to the imaginary axis* (i.e., with the smallest real parts) which decay more slowly:

- if the dominant pole is a single real pole, the system's response resembles that of a first-order system, characterized by a single exponential decay, and
- if the dominant poles are a pair of complex conjugate poles, the response resembles that of a second-order system, featuring oscillatory behavior with a decay rate governed by the real part of the dominant poles.

This approximation is accurate when the dominant pole(s) are significantly slower (e.g., 5x slower) than the remaining poles.



(a) one dominant pole



(b) complex conjugate poles

Figure 5.13: Step responses of higher-order systems with either a single dominant pole or a pair of dominant complex conjugate poles.
In both cases, the dominant pole approximation has numerator set to have the same DC gain as the original system (unit DC gain in these examples).

5.6 Appendix: Free and step response for second order systems via Laplace calculations

In this appendix we report some useful calculations that explain some of the formulas and plots presented earlier.

5.6.1 Step response for underdamped system

We consider an underdamped second-order system with zero initial conditions ($x(0) = \dot{x}(0) = 0$) subject to a step input:

$$\ddot{y} + 2\zeta\omega_{\mathsf{n}}\dot{y} + \omega_{\mathsf{n}}^2 y = \omega_{\mathsf{n}}^2 \mathbf{1}(t) \tag{5.37}$$

with natural frequency ω_{n} and damping ratio ζ. Since $U(s) = 1/s$, we compute

$$Y(s) = \frac{\omega_{\mathsf{n}}^2}{s(s^2 + 2\omega_{\mathsf{n}}\zeta s + \omega_{\mathsf{n}}^2)} \tag{5.38}$$

Since $s^2 + 2\omega_{\mathsf{n}}\zeta s + \omega_{\mathsf{n}}^2 = (s + \omega_{\mathsf{n}}\zeta)^2 + \omega_{\mathsf{d}}^2$ for $\omega_{\mathsf{d}} = \omega_{\mathsf{n}}\sqrt{1-\zeta^2}$, we now expand this rational function in a partial fraction expansion using the terms corresponding to unit step and damped sine and cosine waves:

$$Y(s) = \frac{\alpha}{s} + \beta\frac{\omega_{\mathsf{d}}}{(s + \omega_{\mathsf{n}}\zeta)^2 + \omega_{\mathsf{d}}^2} + \gamma\frac{s + \zeta\omega_{\mathsf{n}}}{(s + \omega_{\mathsf{n}}\zeta)^2 + \omega_{\mathsf{d}}^2} \tag{5.39}$$

To compute α, we can use the residue's formula:

$$\alpha = sY(s)\Big|_{s=0} = 1. \tag{5.40}$$

Using the numerators matching method, we can compute the coefficients β and γ and obtain

$$\boxed{y(t) \;=\; 1 - \mathrm{e}^{-\zeta\omega_{\mathsf{n}}t}\left(\cos(\omega_{\mathsf{d}}t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_{\mathsf{d}}t)\right)} \tag{5.41}$$

As this response is equal to the one given in equation (5.24).

5.6.2 Free response for underdamped system

For $0 < \zeta < 1$, we consider

$$\ddot{x} + 2\zeta\omega_{\mathsf{n}}\dot{x} + \omega_{\mathsf{n}}^2 x = 0$$

with initial position $x(0)$ and initial velocity $\dot{x}(0)$. We take the Laplace transform to obtain:

$$\Big(s^2X(s) - sx(0) - \dot{x}(0)\Big) + 2\zeta\omega_{\mathsf{n}}\Big(sX(s) - x(0)\Big) + \omega_{\mathsf{n}}^2 X(s) = 0. \tag{5.42}$$

From here we compute $X(s)$ as follows

$$X(s) = \frac{(s + 2\zeta\omega_{\mathsf{n}})x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_{\mathsf{n}}s + \omega_{\mathsf{n}}^2} \tag{5.43}$$

Since the system is underdamped, we define the damped frequency by $\omega_{\mathsf{d}} = \omega_{\mathsf{n}}\sqrt{1-\zeta^2}$ and we note

$$s^2 + 2\zeta\omega_{\mathsf{n}}s + \omega_{\mathsf{n}}^2 = (s + \zeta\omega_{\mathsf{n}})^2 + (\omega_{\mathsf{n}}\sqrt{1-\zeta})^2 \quad \overset{\text{by definition}}{=} \quad (s+\zeta\omega_{\mathsf{n}})^2 + \omega_{\mathsf{d}}^2 \tag{5.44}$$

With this denominator, recalling rows (7) and (8) of Table 4.1, we compute the partial fraction expansion:

$$X(s) = \frac{\zeta\omega_{\mathsf{n}}x(0) + \dot{x}(0)}{\omega_{\mathsf{d}}} \cdot \frac{\omega_{\mathsf{d}}}{(s+\zeta\omega_{\mathsf{n}})^2 + \omega_{\mathsf{d}}^2} + x(0) \cdot \frac{s + \zeta\omega_{\mathsf{n}}}{(s+\zeta\omega_{\mathsf{n}})^2 + \omega_{\mathsf{d}}^2} \tag{5.45}$$

so that the inverse Laplace transform is immediate:

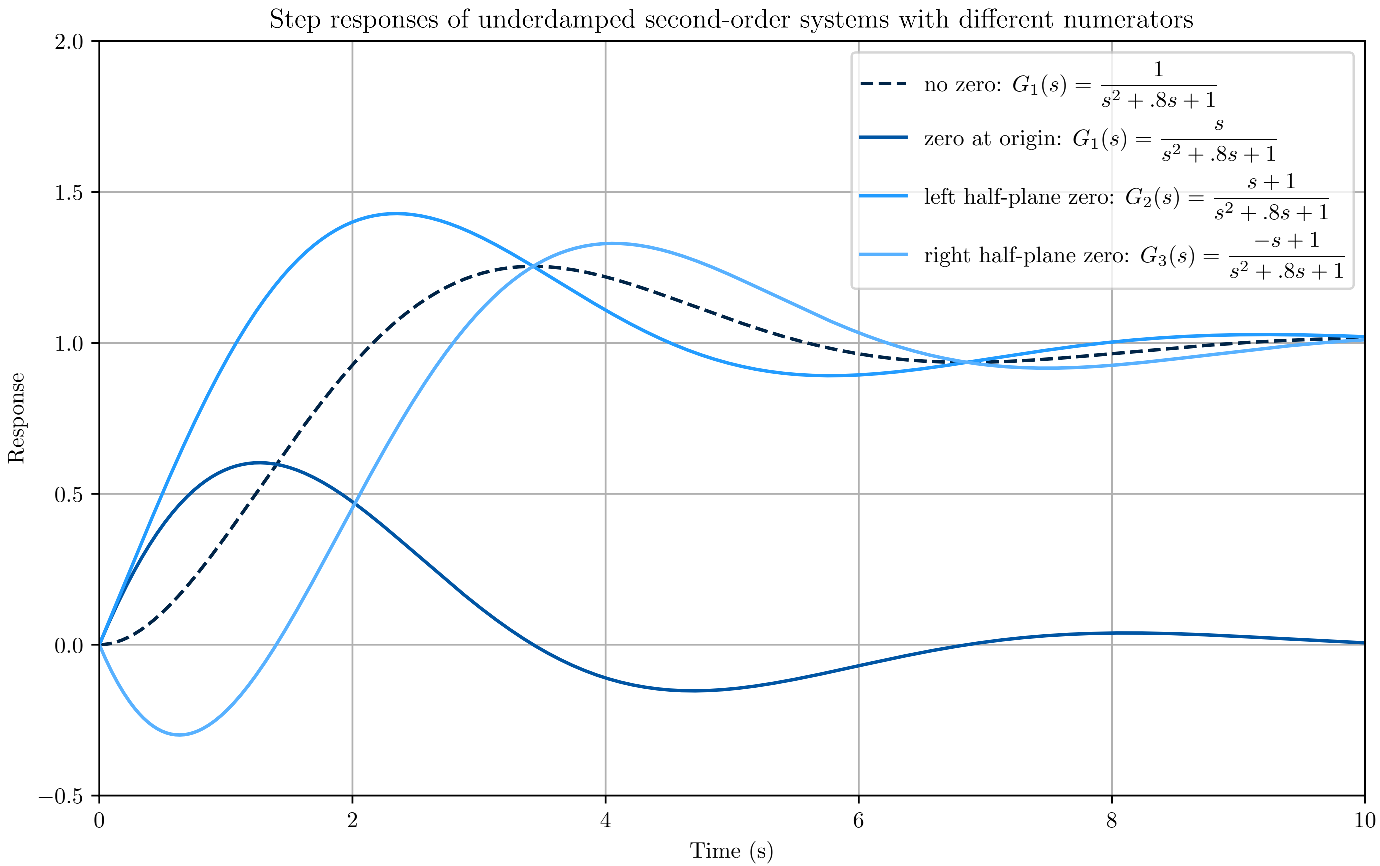
$$x(t) \;=\; \frac{\zeta\omega_{\mathsf{n}}x(0) + \dot{x}(0)}{\omega_{\mathsf{d}}} \cdot \mathrm{e}^{-\zeta\omega_{\mathsf{n}}t}\sin(\omega_{\mathsf{d}}t) \;+\; x(0)\cdot \mathrm{e}^{-\zeta\omega_{\mathsf{n}}t}\cos(\omega_{\mathsf{d}}t) \tag{5.46}$$

When $\dot{x}(0) = 0$, we simplify this expression to

$$x(t) = x(0)\,\mathrm{e}^{-\zeta\omega_{\mathsf{n}}t}\Big(\cos(\omega_{\mathsf{d}}t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_{\mathsf{d}}t)\Big). \tag{5.47}$$

This solution is shown in Figure 5.12, for varying values of the damping ratio ζ.

5.7 Appendix: Underdamped systems with zeros in the left and right half plane



5.8 Exercises

E5.1 **Inverse Laplace transforms appearing in the responses of first-order systems).** Using the Tables 4.1 and 4.2 of Laplace transforms and the partial fraction expansion method, verify:

$$\mathcal{L}^{-1}\left[\frac{1}{s\tau+1}\right] = \frac{1}{\tau}\,\mathrm{e}^{-t/\tau}, \tag{E5.1}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s\,(s\tau+1)}\right] = 1-\mathrm{e}^{-t/\tau}, \tag{E5.2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2\,(s\tau+1)}\right] = t-\tau(1-\mathrm{e}^{-t/\tau}) \tag{E5.3}$$

Answer:

(i) Using the property from row (4) in Table 4.1 that

$$\frac{1}{s+a} \;\mapsto\; \mathrm{e}^{-at}$$

we can multiply the numerator and denominator by $1/\tau$ to then find

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{\tau}\left(\frac{1}{s+1/\tau}\right)\right] = \frac{1}{\tau}\,\mathrm{e}^{-t/\tau}}$$

(ii) The first step is to select the following partial fraction expansion:

$$\frac{1}{s(s\tau+1)} = \frac{\alpha}{s}+\frac{\beta}{s\tau+1}$$

Multiplying both sides of the equation by $s(s\tau+1)$ we then have

$$1 = \alpha s\tau+\alpha+\beta s$$

which yields the set of equations

$$\begin{aligned} &\alpha = 1 \\ &\alpha s+\beta s = 0s \qquad\Longrightarrow\qquad \alpha+\beta = 0 \end{aligned}$$

We can solve for α and β to find $(\alpha,\beta)=(1,-\tau)$. Finally, using rows (2) and (4) of Table 4.1 we find

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{s}-\tau\frac{1}{s\tau+1}\right] = 1-\mathrm{e}^{-t/\tau}}$$

(iii) Performing a partial fraction decomposition, we find that

$$\frac{1}{s^2(s\tau+1)} = \frac{\alpha}{s} + \frac{\beta}{s^2} + \frac{\gamma}{s\tau+1}$$

Multiplying both sides by $s(s^2)(s\tau+1)$ yields

$$\begin{aligned} & s = \alpha(s\tau+1)(s^2) + \beta(s\tau+1)(s) + \gamma(s)(s^2) \\ \implies & s = \alpha\tau s^3 + \alpha s^2 + \beta\tau s^2 + \beta s + \gamma s^3 \end{aligned}$$

This in turn yields the set of equations

$$\begin{aligned} \alpha\tau s^3 + \gamma s^3 = 0s^3 \quad & \implies \quad \alpha\tau + \gamma = 0 \\ \alpha s^2 + \beta\tau s^2 = 0s^2 \quad & \implies \quad \alpha + \beta\tau = 0 \\ \beta = 1 & \end{aligned}$$

Solving for α, β, and γ, we find $(\alpha, \beta, \gamma) = (-\tau, 1, \tau^2)$. Finally, using rows (2), (3), and (4) of Table 4.1, we obtain

$$\mathcal{L}^{-1}\left[\frac{-\tau}{s} + \frac{1}{s^2} + \frac{\tau^2}{s\tau+1}\right] = -\tau + t + \tau\,\mathrm{e}^{-t/\tau} = t - \tau\left(1 - \mathrm{e}^{-t/\tau}\right)$$

■

E5.2 **Free response of undamped harmonic oscillator.** Consider the undamped harmonic oscillator $m\ddot{x} + kx = 0$ without any input, where $m > 0$ and $k > 0$.

(i) Compute the free response in the Laplace domain $X(s)$ from initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$

(ii) Compute the free response $x(t)$ by performing the inverse Laplace transform of $X(s)$.

Note: *We studied the undamped harmonic oscillator in Section 2.1.2. The results in this exercises are consistent with that previous analysis.*

Answer:

(i) We take the Laplace transform to get

$$m(s^2X(s) - sx_0 - v_0) + kX(s) \;=\; 0.$$

We obtain the free response by solving for $X(s)$:

$$\boxed{X(s) \;=\; \frac{m(sx_0 + v_0)}{ms^2 + k} \;=\; \frac{sx_0 + v_0}{s^2 + \omega_{\mathsf{n}}^2}}$$

where $\omega_{\mathsf{n}} = \sqrt{k/m}$.

(ii) First, we rewrite the free response in partial fraction expansion as:

$$X(s) \;=\; x_0 \frac{s}{s^2 + \omega_{\mathsf{n}}^2} \;+\; \frac{v_0}{\omega_{\mathsf{n}}} \frac{\omega_{\mathsf{n}}}{s^2 + \omega_{\mathsf{n}}^2}$$

Then, taking the inverse Laplace transform and using rows (5) and (6) of Table 4.1, we obtain

$$\boxed{x(t) \;=\; x_0 \cos(\omega_{\mathsf{n}} t) \;+\; \frac{v_0}{\omega_{\mathsf{n}}} \sin(\omega_{\mathsf{n}} t)}$$

■

E5.3 **Transfer function of building system.** Recall the dynamics of the building system (without air conditioner) studied in Section 3.1:

$$\begin{aligned} c_1\dot{\theta}_1 &= \frac{1}{r_{12}}(\theta_2-\theta_1)+\frac{1}{r_{1,\text{ext}}}(\theta_{\text{ext}}-\theta_1) \\ c_2\dot{\theta}_2 &= \frac{1}{r_{12}}(\theta_1-\theta_2)+\frac{1}{r_{23}}(\theta_3-\theta_2) \\ c_3\dot{\theta}_3 &= \frac{1}{r_{23}}(\theta_2-\theta_3). \end{aligned}$$

Note that we changed notation: we let $\theta_i(t)$ denote the temperature in room i and $\Theta_i(s)=\mathcal{L}[\theta_i(t)]$ be its Laplace transform. Similarly, we let $\Theta_{\text{ext}}(s)=\mathcal{L}[\theta_{\text{ext}}(t)]$.

We aim to compute the transfer function of the building system (without air conditioner) from the external temperature to the temperature in room 3.

(i) Take the Laplace transforms of the three equations, assuming zero initial conditions.

(ii) Explain, in words, how to find the overall transfer function from $\Theta_{\text{ext}}(s)$ to $\Theta_3(s)$.

(iii) Find the overall transfer function from $\Theta_{\text{ext}}(s)$ to $\Theta_3(s)$. You may use Matlab or Python for this if you wish, but be sure to include your code if you choose to do so.

(iv) What is the order of this transfer function?

Answer:

(i) Taking the Laplace transforms of the three equations, we get

$$c_1 s\Theta_1(s) = \frac{1}{r_{12}}(\Theta_2(s)-\Theta_1(s))+\frac{1}{r_{1,\text{ext}}}(\Theta_{\text{ext}}(s)-\Theta_1(s))$$

$$c_2 s\Theta_2(s) = \frac{1}{r_{12}}(\Theta_1(s)-\Theta_2(s))+\frac{1}{r_{23}}(\Theta_3(s)-\Theta_2(s))$$

$$c_3 s\Theta_3(s) = \frac{1}{r_{23}}(\Theta_2(s)-\Theta_3(s))$$

(ii) The three Laplace transforms above give three equations for four unknowns. Use your favorite method to eliminate the intermediate variables $\Theta_1(s)$ and $\Theta_2(s)$ to get one equation relating $\Theta_{\text{ext}}(s)$ and $\Theta_3(s)$. Solve for the ratio $\Theta_3(s)/\Theta_{\text{ext}}(s)$. This ratio is the transfer function.

(iii) After several tedious steps of algebra, the overall transfer function is found to be

$$\frac{\Theta_3(s)}{\Theta_{\text{ext}}(s)} = \frac{r_{12}r_{23}}{\Big((c_2r_{12}r_{23}s+r_{12}+r_{23})(c_1r_{12}r_{1,\text{ext}}s+r_{12}+r_{1,\text{ext}})-r_{23}r_{1,\text{ext}}\Big)(c_3r_{23}s+1)\;-\;r_{12}(c_1r_{12}r_{1,\text{ext}}s+r_{12}+r_{1,\text{ext}})} \tag{E5.4}$$

(iv) This is a third-order transfer function, as the highest power of s appearing in the denominator is 3.

■

E5.4 **Transfer function of DC motor.** In this exercise, we compute the transfer function of the DC motor in Section 2.5. We recall the governing equations (2.46):

$$I_{\mathrm{m}}\ddot{\theta}_{\mathrm{m}}(t) + b\dot{\theta}_{\mathrm{m}}(t) = K_{\mathrm{torque}} i_{\mathrm{cond}}(t) \tag{E5.5a}$$

$$\ell \frac{d}{dt} i_{\mathrm{cond}}(t) + r i_{\mathrm{cond}}(t) = v_{\mathrm{source}}(t) - K_{\mathrm{velocity}}\dot{\theta}_{\mathrm{m}}(t) \tag{E5.5b}$$

and refer to Section 2.5 for the definition of all terms.

Let $\omega_{\mathrm{m}} = \dot{\theta}_{\mathrm{m}}$ be the shaft angular velocity. Use the following notation: $V_{\mathrm{source}}(s) = \mathcal{L}[v_{\mathrm{source}}(t)]$, $\Omega_{\mathrm{m}}(s) = \mathcal{L}[\omega_{\mathrm{m}}(t)]$, and $I_{\mathrm{cond}}(s) = \mathcal{L}[i_{\mathrm{cond}}(t)]$.

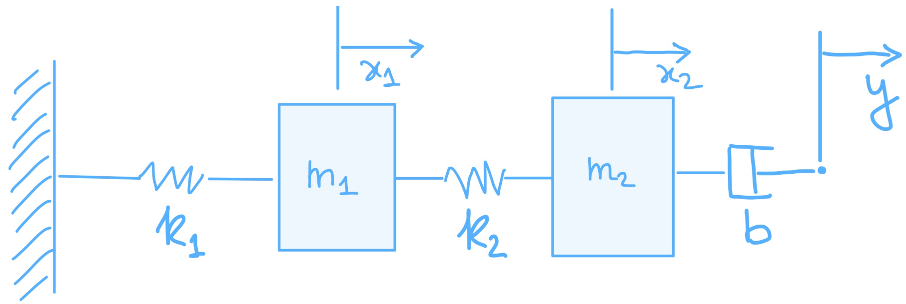
(i) Take the Laplace transforms of the two equations, assuming zero initial conditions and using only the shaft angular velocity (and not the shaft angle).

(ii) Compute the transfer function from the voltage source $V_{\mathrm{source}}(s)$ to the angular velocity $\Omega_{\mathrm{m}}(s)$.

(iii) What is the order of this transfer function?

(iv) Explain why the system is underdamped for large values of K_{velocity} and K_{torque}.

E5.5 **Transfer function of a (spring-mass)^2-damper system.** Consider the following system composed of two masses, two springs and a damper. As usual, let $X_1(s) = \mathcal{L}[x_1(t)]$, $X_2(s) = \mathcal{L}[x_2(t)]$, and $Y(s) = \mathcal{L}[y(t)]$.



(i) Derive the equations of motion for this system.
(ii) Take the Laplace transforms of the equations you derived in part (i), assuming zero initial conditions.
(iii) Compute the transfer function from $Y(s)$ to $X_1(s)$.
(iv) What is the order of this transfer function?

Answer:

(i) Using free-body diagrams, the equations of motion for this system are

$$\begin{aligned} m_1\ddot{x}_1 + (k_1+k_2)x_1 &= k_2x_2 \\ m_2\ddot{x}_2 + b\dot{x}_2 + k_2x_2 &= k_2x_1 + b\dot{y}\,. \end{aligned}$$

(ii) Taking the Laplace transforms of the two equations yields

$$\begin{aligned} (m_1s^2 + k_1 + k_2)X_1(s) &= k_2X_2(s) \\ (m_2s^2 + bs + k_2)X_2(s) &= k_2X_1(s) + bsY(s)\,. \end{aligned}$$

(iii) Solving the first equation for $X_2(s)$, substituting into the second equation, and solving for the transfer function $G(s) = X_1(s)/Y(s)$ yields

$$G(s) = \frac{k_2bs}{(m_2s^2+bs+k_2)(m_1s^2+k_1+k_2) - k_2^2}$$

(iv) The order of this transfer function is 4.

■

E5.6 **From complex conjugate poles to canonical parameters and functions of time.** Consider a function of time $x(t)$ with Laplace transform $X(s)$. The rational function $X(s)$ has two poles drawn in the attached figure.

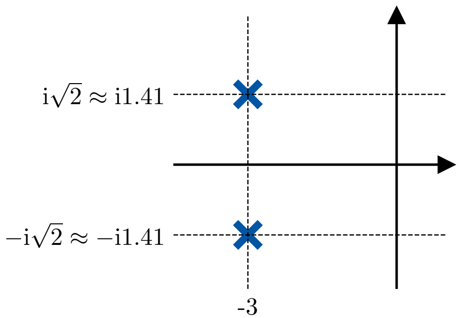


Figure E5.1: Complex plane with two complex conjugate poles.
Recall that $s_{1,2} = -\omega_n\big(\zeta \pm \sqrt{\zeta^2 - 1}\big)$ and that that the poles belong to a semicircle of radius ω_n.

(i) Compute the damping ratio ζ, natural frequency ω_n, damped natural frequency ω_d, and time constant τ for these poles.

(ii) What are the two functions of time $f_1(t)$ and $f_2(t)$ associated to the two poles? Substitute in the vaues of the ζ and ω_n.

(iii) Assume $x(t) = \alpha f_1(t) + \beta f_2(t)$, assume $x(0) = 0$ and $\dot{x}(0) = 10$, and write a formula for $x(t)$.

E5.7 **From poles to transfer function and differential equation.** The poles of a transfer function $G(s)$ are drawn in the attached figure.

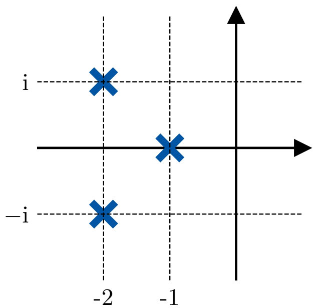


Figure E5.2: Complex plane with poles of a transfer function

Assume $G(0) = 1/5$.

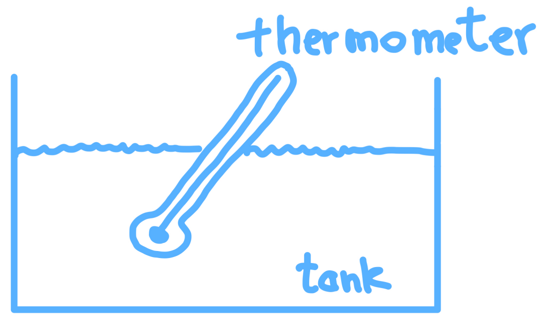
Recall the canonical denominator of a second order system is of the form $s^2 + 2\zeta\omega_n s + \omega_n^2$.

(i) Compute the transfer function $G(s)$ such that $G(0) = 1/5$.

(ii) Compute the damping ratio ζ and natural frequency ω_n for the two complex poles.

(iii) Let $X(s) = G(s)U(s)$ and compute the differential equation associated to these poles, governing $x(t)$ as a function of $u(t)$.

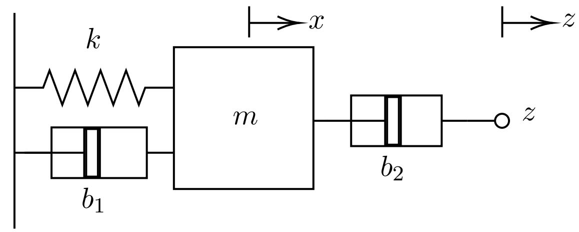
E5.8 **Thermometer transfer function and ramp response in a warming tank.** Consider a thermometer with temperature $\theta(t)$ immersed in a water tank with temperature $\theta_{\text{tank}}(t)$. Let c and r denote the thermal capacity of the thermometer and the tank-thermometer thermal resistance, respectively.



(i) Derive the governing equation for the system.
(ii) Take the Laplace transform of the equation you found in part (i), assuming zero initial conditions.
(iii) Compute the transfer function from $\Theta_{\text{tank}}(s) = \mathcal{L}[\theta_{\text{tank}}(t)]$ to $\Theta(s) = \mathcal{L}[\theta(t)]$.
(iv) Is this a first-order or a second-order system? If it is first order, compute the time constant. Otherwise, if it is a second-order system, compute the natural frequency and the damping ratio.
(v) Compute the step response of this system in the time domain.
(vi) Finally, assume $\theta_{\text{tank}}(t) = t$ is the unit ramp function. Compute the asymptotic value $e_{\text{steady-state}} = \lim_{t\to\infty} e(t)$, where the error $e(t) := \theta(t) - \theta_{\text{tank}}(t)$.

Hint: In exercise E6.1, we computed $\dfrac{1}{s^2(s\tau+1)} = -\dfrac{\tau}{s} + \dfrac{1}{s^2} + \dfrac{\tau^2}{s\tau+1}$.

E5.9 **Transfer function, step response, and final value of a mass-spring-damper plus extra damper.** Consider a mass-spring-damper system (with parameters m, b_1 and k) connected to an additional damper (with parameter b_2), as illustrated in Figure. Let $z(t)$ be the position of the right-most point connected to the additional damper. At $t = 0$, a unit-step input is applied to position $z(t)$. Assume the initial conditions are $x(0) = \dot{x}(0) = z(0) = 0$.



(i) Write the governing equation for the position $x(t)$ with input $z(t)$.
(ii) Write the transfer function from $Z(s) = \mathcal{L}[z(t)]$ to $X(s) = \mathcal{L}[x(t)]$.
(iii) Compute the step response $X(s)$ in the Laplace domain, using the following parameter values: $m = 1$kg, $b_1 = b_2 = 5$Ns/m, $k = 50$N/m.
(iv) Compute the inverse Laplace transform $x(t) = \mathcal{L}^{-1}[X(s)]$.
(v) What is the *final value* of $x(t)$, that is, $x(\infty) = \lim_{t\to+\infty} x(t)$?
(vi) In the Appendix 4.4 to Chapter 4, there is a property called Final Value Theorem. Feel free to assume that the limit of $x(t)$ exists and apply the Final Value Theorem to $X(s)$ as computed in point (iii). Verify that you obtain the same result as in point (v).

Note: *Does this final value behavior of this mechanical system makes physical sense to you?*

Answer:

(i) The equation of motion is

$$m\ddot{x} + b_1\dot{x} + b_2(\dot{x} - \dot{z}) + kx = 0$$

so that

$$\boxed{m\ddot{x} + (b_1 + b_2)\dot{x} + kx = b_2\dot{z}}$$

(ii) We compute the Laplace transform with zero initial conditions:

$$\big(ms^2 + (b_1 + b_2)s + k\big)X(s) = b_2 s Z(s)$$

so that

$$\boxed{\frac{X(s)}{Z(s)} = \frac{b_2 s}{ms^2 + (b_1 + b_2)s + k}}$$

(iii) We now apply a unit step $Z(s) = \frac{1}{s}$ to compute

$$X(s) = \frac{b_2}{ms^2 + (b_1 + b_2)s + k}$$

and substitute in the parameter values:

$$\boxed{X(s) = \frac{5}{s^2 + 10s + 50} = \frac{5}{(s+5)^2 + 5^2}}$$

(iv) The inverse Laplace transform of $X(s)$ is precisely (no need to perform the partial fraction expansion in this case):

$$\boxed{x(t) = \mathrm{e}^{-5t} \sin(5t)}$$

(v) Since $x(t)$ is a damped sinusoidal wave, $\boxed{\lim_{t\to+\infty} x(t) = 0}$.

(vi) It is immediate to see that, from the Final Value Theorem,

$$\boxed{\lim_{t\to+\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{b_2 s}{ms^2 + (b_1 + b_2)s + k} = 0}$$

■

Bibliography

S. Fuller, B. Greiner, J. Moore, R. M. Murray, R. van Paassen, and R. Yorke. The Python Control Systems library (python-control). In *IEEE Conf. on Decision and Control*, pages 4875–4881, 2021. doi.