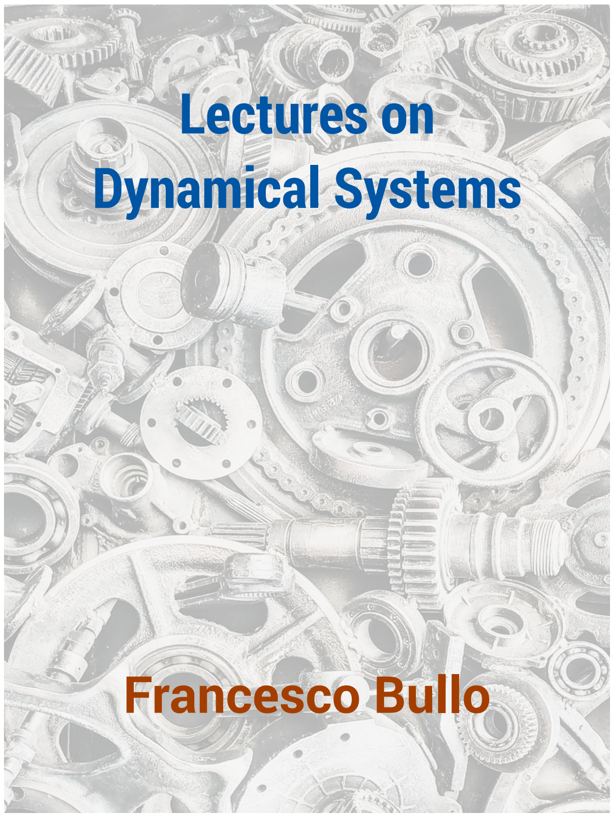
UC Santa Barbara, Department of Mechanical Engineering
ME103 Dynamical Systems. Slides by Chapter.

Francesco Bullo
http://motion.me.ucsb.edu/ME103-Fall2024/syllabus.html



Contents

Chapter 4

The Laplace Transform

The Laplace transform is a powerful mathematical tool used to simplify the modeling and analysis of linear time-invariant systems and differential equations. It converts a function of time t into a function of a complex variable s, making it easier to solve and analyze differential equations.

Illustrating the content of this chapter

Here is an introduction to the magical power of Laplace transforms. Given two positive constants a, b, consider the ODE:

$$\ddot{x} + (a+b)\dot{x} + abx = 0 \tag{4.1}$$

The Laplace transform simplifies the understanding of this ODE by converting it into the simpler algebraic equation:

$$s^2 + (a+b)s + ab = 0. \tag{4.2}$$

This algebraic equation can be rewritten as

$$s^2 + (a+b)s + ab = (s+a)(s+b) = 0, \tag{4.3}$$

and so it has two solutions $s_1 = -a$ and $s_2 = -b$.

As we will study in this chapter, the properties of the Laplace transform ensure that each solution to the ODE (4.1) is of the form

$$x(t) = c_1 \mathrm{e}^{-at} + c_2 \mathrm{e}^{-bt} \tag{4.4}$$

where the constants c_1 and c_2 are determined by the initial conditions. In other words

a zero $-a$ of the algebraic equation (4.2) $\quad\rightarrow\quad$ a term e^{-at} in the solution to the differential equation (4.1).

This chapter is dedicated to understanding concepts and methods to generalize this result to arbitrary ODEs with inputs.

A brief review of complex numbers

We let i denote the *imaginary unit*, that is, $\mathrm{i}^2 = -1$. Any *complex number* z is of the form

$$z = x + \mathrm{i}y, \tag{4.5}$$

where x and $\mathrm{i}y$ are the real and imaginary parts. When useful, we let $\mathbb{C}$ denote the set of complex number, also known as the *complex plane*. The *conjugate* of z is

$$\bar{z} = x - \mathrm{i}y. \tag{4.6}$$

The *magnitude* (or *modulus*) of a complex number z is

$$|z| = \sqrt{x^2 + y^2}. \tag{4.7}$$

(Recall $|z_1 z_2| = |z_1| \cdot |z_2|$ and $|1/z| = 1/|z|$.) The *argument* of a complex number z is the angle θ formed by the line representing the complex number in the complex plane with the positive real axis, measured counterclockwise. The argument[1] is denoted by

$$\arg(z) = \theta. \tag{4.8}$$

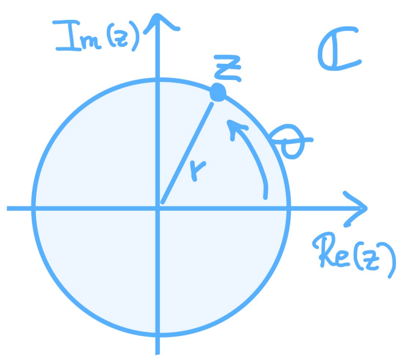


Figure 4.1: The *Euler formula* is $\mathrm{e}^{\mathrm{i}\theta} = \cos(\theta) + \mathrm{i}\sin(\theta)$. A complex number can be represented in *polar form* as $r(\cos\theta + \mathrm{i}\sin\theta)$, where r is the magnitude of the complex number and θ is its argument:

$$z = r\big(\cos(\theta) + \mathrm{i}\sin(\theta)\big).$$

The inverse Euler formulas are:

$$\sin(\theta) = \frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2\mathrm{i}} \qquad \text{and} \qquad \cos(\theta) = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}.$$

[1] The argument of a complex number $z = a + bi$ can be computed using the two-argument arctangent function, also known as atan2(b, a), which takes into account the signs of a and b to return the correct quadrant for the angle.

4.1 The Laplace transform

The *Laplace transform* of a function $f(t)$ is a function $F(s)$ formally defined by

$$F(s) = \mathcal{L}[f(t)] = \int_0^{+\infty} \mathrm{e}^{-st} f(t)\, dt, \tag{4.9}$$

where

- $f(t)$ is a function of time, such that $f(t) = 0$ for all $t < 0$,
- s is a complex variable,
- $\mathcal{L}[\cdot]$ is the symbol indicating the Laplace transform of its argument.

Because $f(t)$ can be discontinuous at $t = 0$, we interpret $f(0)$ in the formula (4.9) as the limit from the right: $f(0) = \lim_{t\to 0^+} f(t)$.

The reverse process of finding the function of time $f(t)$ from its Laplace transform $F(s)$ is called the *inverse Laplace transform* and is denoted by

$$\mathcal{L}^{-1}[F(s)] = f(t). \tag{4.10}$$

Note: Not all functions admit a well-defined Laplace transform (e.g., the integral could be unbounded or not exist), but all functions we will encounter are. There exists an integral formula[2] for the inverse Laplace transform, but we will not need it.

[2] For example, see https://en.wikipedia.org/wiki/Inverse_Laplace_transform

Figure 4.2: Left image: Laplace transforms are defined for functions that are zero for negative time. Right image: the unit step function.

In what follows, every function is to be understood as being zero for negative time. Hence the function $f(t) = 1$ is understood to be the *unit step function* $\mathbf{1}(t)$ defined by

$$\mathbf{1}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0. \end{cases} \tag{4.11}$$

4.1.1 A useful example

It is relatively easy to get some intuition for the Laplace transform formula (4.9). For example, for any scalar real number a, we have

$$\mathcal{L}[\mathrm{e}^{-at}\,\mathbf{1}(t)] = \mathcal{L}[\mathrm{e}^{-at}] = \frac{1}{s+a} \tag{4.12}$$

To prove this formula (4.12), we compute

$$\mathcal{L}[\mathrm{e}^{-at}] = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{e}^{-at}\;dt = \int_0^\infty \mathrm{e}^{-(a+s)t}\;dt \tag{4.13}$$

$$= \left[\frac{-1}{a+s}\,\mathrm{e}^{-(a+s)t}\right]_0^{+\infty} \tag{4.14}$$

$$= \lim_{t\to+\infty}\left(\frac{-1}{a+s}\,\mathrm{e}^{-(a+s)t}\right) \;-\; \frac{-1}{a+s}\,\mathrm{e}^{-(a+s)t}\Big|_{t=0} \tag{4.15}$$

$$\overset{(*)}{=} 0 + \frac{1}{s+a}. \tag{4.16}$$

Here, the step (*) is true when the real part of s is greater than $-a$, but the formula is true even without that assumption. Similar derivations are possible for many other functions.

4.1.2 Some nomenclature

The function $\mathcal{L}[\mathrm{e}^{-at}] = \frac{1}{s+a}$ is a fraction of polynomials.

- A function from $\mathbb{C}$ to $\mathbb{C}$ is *rational* if it is the quotient of two polynomial functions. In other words, $F(s)$ is rational if there exist two polynomial function $\mathrm{num}(s)$ and $\mathrm{den}(s)$ such that

$$F(s) = \frac{\mathrm{num}(s)}{\mathrm{den}(s)}$$

- The domain of a rational function includes all complex numbers except for the values of s such that $\mathrm{den}(s) = 0$.
- The points where the denominator equals zero are called the *poles* of the rational function.

4.1.3 Properties of Laplace transforms

The good news is that we never need to compute Laplace transforms and their inverses using the definition (4.9) in the definition. Instead of using the definition, we will use the properties of Laplace transform to quickly derive the result we want.

Properties of the Laplace transform: In what follows, assume $F(s) = \mathcal{L}[f(t)]$ and $G(s) = \mathcal{L}[g(t)]$.

(P1) Linearity: $$\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$$

(P2) Derivative with respect to time: $$\mathcal{L}\Big[\frac{d}{dt}f(t)\Big] = sF(s) - f(0)$$

(P3) Integral with respect to time: $$\mathcal{L}\Big[\int_0^t f(u)\,du\Big] = \frac{1}{s}F(s)$$

(P4) Complex translation: $$\mathcal{L}[\mathrm{e}^{-at}\,f(t)] = F(s+a).$$

The inverse Laplace transform inherits many properties too, e.g., it is linear.

4.1.4 Using the properties to derive other examples of Laplace transforms

We start from the Laplace transform of the *exponential function*:

$$\mathcal{L}[\mathrm{e}^{-at}\,\mathbf{1}(t)] = \mathcal{L}[\mathrm{e}^{-at}] = \frac{1}{s+a} \tag{4.17}$$

Note that, when $a = 0$, we obtain the Laplace transform of the *unit step function*:

$$\mathcal{L}[\mathbf{1}(t)] = \frac{1}{s} \tag{4.18}$$

We can now use the integral property (P3) to compute

$$\mathcal{L}[t] = \mathcal{L}[t\mathbf{1}(t)] = \mathcal{L}\Big[\int_0^t 1(\tau)d\tau\Big] = \frac{1}{s}\mathcal{L}[\mathbf{1}(t)] = \frac{1}{s}\cdot\frac{1}{s},$$

so that the Laplace transform of the *unit ramp function* is:

$$\mathcal{L}[t] = \frac{1}{s^2} \tag{4.19}$$

Next, we use the inverse Euler formula for the sinusoidal function and the linearity property (P1) to compute

$$\begin{aligned}\mathcal{L}[\sin \omega t] &= \mathcal{L}\Big[\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\Big] = \frac{1}{2i}\Big(\mathcal{L}[e^{i\omega t}] - \mathcal{L}[e^{-i\omega t}]\Big) \\ &= \frac{1}{2i}\Big(\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\Big) = \frac{1}{2i}\frac{(s + i\omega) - (s - i\omega)}{(s - i\omega)(s + i\omega)}\end{aligned}$$

so that the Laplace transform of the *sine wave* is:

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} \tag{4.20}$$

If we use the complex translation property (P4), we can compute the Laplace transform of the *damped sine wave*:

$$\mathcal{L}[e^{-at} \sin \omega t] = \frac{\omega}{(s + a)^2 + \omega^2} \tag{4.21}$$

The Laplace transform of the cosine wave and of the damped cosine wave are computed in a similar way from the inverse Euler formula for the cosine. Alternatively, one can use the derivative-with-respect-to-time property (P2) (recalling that $\frac{d}{dt} \sin(\omega t) = \omega \cos(\omega t)$) and check that both methods lead to the same formula.

4.1.5 Unit pulse and impulse functions

A ball bouncing off the floor, a hammer hitting a nail, a bullet hitting a wall, an explosion hitting a flexible structure, or a automobile crash are situation where a dynamical system is subject to a large force for a short period of time.

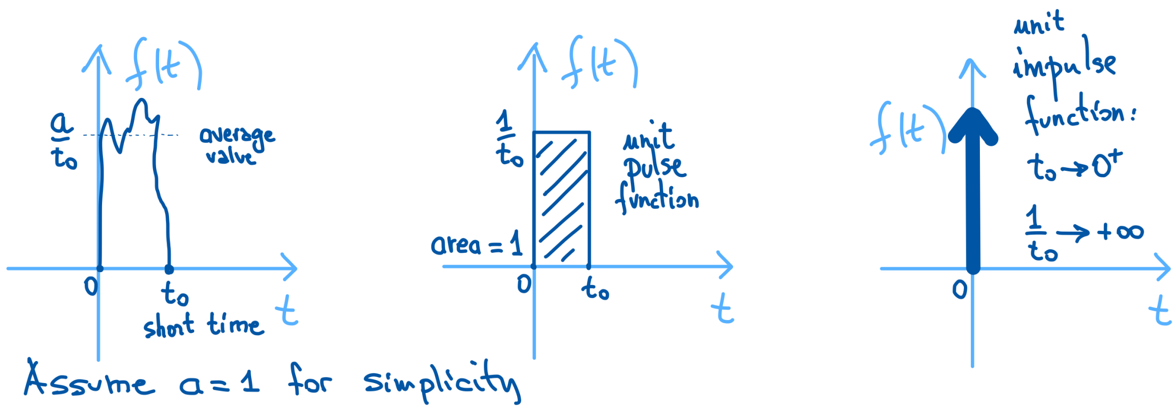


Figure 4.3: Left panel: a realistic pulse with a quick rise, possibly oscillations, and then a drop to zero – over a short interval of time t_0. The area of the pulse is a positive amount a; we assume $a = 1$ to define the unit pulse and unit impulse functions.
Center panel: a *unit pulse function* with duration t_0 and amplitude $1/t_0$.
Right panel: in the limit as $t_0 \to 0+$, we define the *unit impulse function*.

The *unit pulse function* with duration t_0 is

$$\text{pulse}(t) = \begin{cases} \dfrac{1}{t_0} & \text{if } 0 < t < t_0 \\ 0 & \text{if } t < 0 \text{ or } t > t_0 \end{cases} \tag{4.22}$$

Note: The exact shape of a realistic pulse function of time is not important. Only the area is important. If the pulse has area $a \neq 1$, then the correct function to use is $a \cdot \text{pulse}(t)$.

Next, we define an idealized and simplified version of the unit pulse function, which will allow for easier calculations.

The *unit impulse function* is defined to have area equal to 1. The impulse function is a mathematical construction to simplify calculations when the duration of a pulse t_0 is much smaller than the time constant of the system.

$$\delta(t) = \begin{cases} +\infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0. \end{cases} \quad \text{such that} \quad \int_{-\infty}^{+\infty} \delta(\tau) d\tau = 1 \tag{4.23}$$

Even though we do not provide the proof here, it is useful to note:

$$\mathcal{L}[\delta(t)] = 1 \tag{4.24}$$

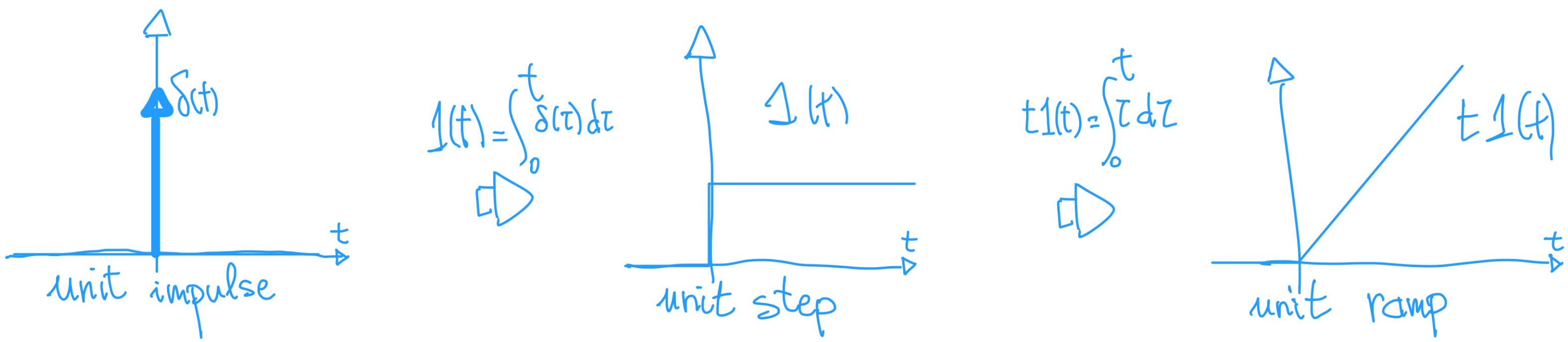


Figure 4.4: From unit impulse to unit step to unit ramp function, via integration with respect to time.

4.1.6 Table of Laplace transforms

Using the example of the exponential function $\mathcal{L}[\mathrm{e}^{-at}] = \frac{1}{s+a}$ and the properties of the Laplace transform, we can derive this table of examples. Here a is positive or negative, ω is positive, and n is a natural number.

	Function of time $f(t)$		*Laplace transform $F(s)$ and its poles*	
	In this table we consider only *exponential-like functions*.		Laplace transforms of *exponential-like functions* are rational functions.	
(1)	Unit impulse	$\delta(t)$	1	none
(2)	Unit step	$\mathbf{1}(t)$	$\frac{1}{s}$	$s = 0$
(3)	Unit ramp	t	$\frac{1}{s^2}$	$s = 0$ repeated
(4)	Exponential function	e^{-at}	$\frac{1}{s+a}$	$s = -a$
(5)	Sine wave	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s = \pm \mathrm{i}\omega$
(6)	Cosine wave	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s = \pm \mathrm{i}\omega$
(7)	Damped sine wave	$\mathrm{e}^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$s = -a \pm \mathrm{i}\omega$
(8)	Damped cosine wave	$\mathrm{e}^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	$s = -a \pm \mathrm{i}\omega$

Table 4.1: Table of Laplace transforms. Rows (7) and (8) are more general than (5) and (6), in the sense that they are equal to (5) and (6) when $a = 0$.

In summary, from the Table of Laplace transforms, we have learned that each exponential-like function of t transforms into a rational function of s. In turn, each rational function of s has zero, one, or multiple poles. In Figure 4.5, for each point s^* in the complex plane (drawn with an $\times$ symbol), we illustrate the function of time whose Laplace transform has a pole at s^*.

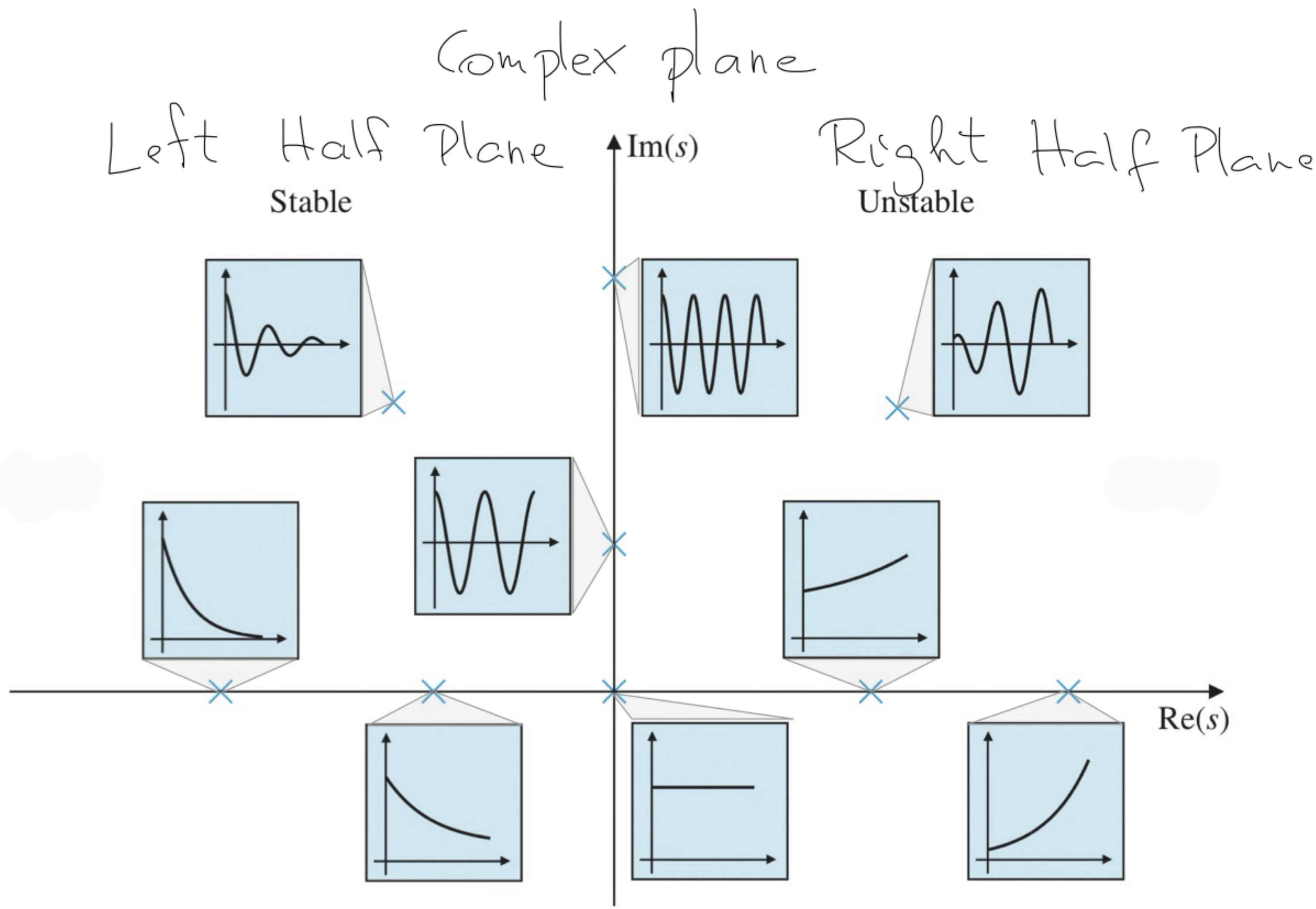


Figure 4.5: Time functions associated with poles in the complex plane. For each point s^* in the complex plane (drawn with an $\times$ symbol), we illustrate the function of time whose Laplace transform has a pole at s^*.

4.2 The inverse Laplace transform

We now discuss how to compute the inverse Laplace transform of rational functions. Specifically, given a rational function $F(s)$, we discuss how to find the function of time $f(t)$ such that $\mathcal{L}[f(t)] = F(s)$. There are three methods:

(i) the use of extensive *lookup tables*, with example pairs $(f(t), F(s))$,

(ii) the method of *partial fraction expansion*

(iii) the use of a symbolic manipulation software, such as the SymPy symbolic computing library in Python.

Regarding lookup tables, we have a more expansive table of Laplace transforms in Appendix 4.5. But, in general, there are too many cases to consider and it is not practical to generate huge Laplace transform tables. Therefore, in practice a combination of lookup tables and partial fraction expansions is used.

4.2.1 Partial fraction expansions

Typically, we will consider functions $F(s)$ that can be rewritten as the sum of simpler functions:

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s) \tag{4.25}$$

so that

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[F_1(s)] + \mathcal{L}^{-1}[F_2(s)] + \cdots + \mathcal{L}^{-1}[F_n(s)] \tag{4.26}$$

$$= f_{\mathbf{1}}(t) + f_2(t) + \cdots + f_n(t) \tag{4.27}$$

where $f_{\mathbf{1}}(t)$, $f_2(t)$, ..., $f_n(t)$ are the inverse Laplace transforms of $F_1(s)$, $F_2(s)$, ..., $F_n(s)$, respectively. Therefore, it is useful to have a lookup table with a lot of different pairs $(f(t), F(s))$.

Problem setup: We often consider a rational function

$$F(s) = \frac{\mathrm{num}(s)}{\mathrm{den}(s)} \tag{4.28}$$

where the degree of the numerator is equal or lower than the degree of the denominator. We aim to compute the inverse Laplace transform of $F(s)$.

Step 1: The first step is to identify the poles of F and factorize the denominator:

$$F(s) = \frac{\mathrm{num}(s)}{(s+p_1)(s+p_2)\dots(s+p_n)} \tag{4.29}$$

Step 2: The second step is to expand F in a so-called *partial fraction expansion*

$$F(s) = \frac{r_1}{s+p_1} + \frac{r_2}{s+p_2} + \dots + \frac{r_n}{s+p_n}.$$

where each coefficient r_i is called a *residue* at the pole $-p_i$. Sometime the expansion contains more complex terms (e.g., when there are complex conjugate poles, as in the fourth example below, or repeated poles, as in the fifth example below).

Step 3: Compute the residues $r_1, \dots, r_n$, using one of the following two methods:
Matching the numerators: always applicable, sometimes lengthy
Single-pole residue formula: quick shortcut, when applicable
We will illustrate the methods in the examples below.

Step 4: The final easy step is to use linearity and obtain

$$\mathcal{L}^{-1}[F(s)] = r_1\,\mathcal{L}^{-1}\Big[\frac{1}{s+p_1}\Big] + r_2\,\mathcal{L}^{-1}\Big[\frac{1}{s+p_2}\Big] + \dots + r_n\,\mathcal{L}^{-1}\Big[\frac{1}{s+p_n}\Big] = r_1\,\mathrm{e}^{-p_1 t} + r_2\,\mathrm{e}^{-p_2 t} + \dots + r_n\,\mathrm{e}^{-p_n t} \tag{4.30}$$

First example (rational function already written as sum of simple terms)

We now consider various examples of partial fraction expansions and corresponding inverse Laplace transform. As first example, we consider a function with real poles only and already written in partial fraction expansion:

$$F_1(s) = 2 + \frac{3}{s} + \frac{4}{s+5} \tag{4.31}$$

Using rows (1), (2) and (4) of Table 4.1 and the linearity property, we compute

$$f_{\mathbf{1}}(t) \;=\; \mathcal{L}^{-1}[F_1(s)] \;=\; 2\delta(t) + 3 \cdot \mathbf{1}(t) + 4\,\mathrm{e}^{-5t} \tag{4.32}$$

Second example (two isolated real poles)

As second example, we consider

$$F_2(s) = \frac{s+2}{s^2+7s+12} \tag{4.33}$$

We compute

$$s^2+7s+12=0 \iff s=-3,-4 \iff s^2+7s+12=(s+3)(s+4). \tag{4.34}$$

Therefore, we write the partial fraction expansion

$$F_2(s) = \frac{s+2}{(s+3)(s+4)} = r_1\frac{1}{s+3} + r_2\frac{1}{s+4} \tag{4.35}$$

We now describe first the *matching the numerators* method and then the *single-pole residue formula*.

Matching the numerators We start by writing equation (4.35) as

$$\frac{s+2}{s^2+7s+12} = \frac{r_1(s+4)+r_2(s+3)}{(s+3)(s+4)} \tag{4.36}$$

The denominators are identical because of the computation of the partial fraction expansion. We therefore focus on the numerators:

$$s+2 = r_1(s+4)+r_2(s+3) = (r_1+r_2)s + (4r_1+3r_2) \tag{4.37}$$

We now equate each power of s:

$$\begin{cases} 1 = r_1+r_2 \\ 2 = 4r_1+3r_2 \end{cases} \tag{4.38}$$

This is a *linear set of 2 equation in 2 variables*. After some tedious calculations, we obtain

$$r_1=-1 \quad \text{and} \quad r_2=2. \tag{4.39}$$

Single-pole residue formula We now describe the *single-pole residue formula* method. This method applies only to *single real poles*. The residue r_i associated to a pole $-p_i$ is given by

$$r_i = (s + p_i)F(s)\Big|_{s=-p_i} \tag{4.40}$$

Note: this formula is correct only when the pole p is not repeated; if the root is complex, it's typically easier to use the matching method.

For $F_2(s) = \dfrac{s+2}{(s+3)(s+4)} = r_1\dfrac{1}{s+3} + r_2\dfrac{1}{s+4}$, the single-pole residue formula (4.40) gives:

$$r_1 = (s+3)F_2(s)\Big|_{s=-3} = \frac{s+2}{s+4}\Big|_{s=-3} = \frac{-1}{+1} = -1 \tag{4.41}$$

$$r_2 = (s+4)F_2(s)\Big|_{s=-4} = \frac{s+2}{s+3}\Big|_{s=-4} = \frac{-2}{-1} = +2 \tag{4.42}$$

In summary, from either methods, we obtain

$$f_2(t) \;=\; \mathcal{L}^{-1}[F_2(s)] = \mathcal{L}^{-1}\left[(-1)\frac{1}{s+3} + (+2)\frac{1}{s+4}\right] \;=\; -\,\mathrm{e}^{-3t} + 2\,\mathrm{e}^{-4t} \tag{4.43}$$

Third example (three isolated real poles)

As third example, we consider the case of multiple isolated poles by analyzing the following rational function:

$$F_3(s) = \frac{2}{s^3 + 6s^2 + 11s + 6} \tag{4.44}$$

In general, computing the roots of a high-order polynomial can be challenging. However, the rational root theorem simplifies this task for polynomials with integer coefficients: the only possible integer roots are the divisors of the constant term.

For our example function $F_3(s)$, the denominator has integer coefficients and the constant term is 6. Therefore, the possible integer roots are ± 1, ± 2, ± 3, and ± 6. By plugging these possible values into the denominator we find that $s = -1$, $s = -2$ and $s = -3$ are indeed roots. We can therefore factor the denominator as

$$s^3 + 6s^2 + 11s + 6 = (s+1)(s+2)(s+3) \tag{4.45}$$

and write the partial fraction expansion as

$$F_3(s) = \frac{2}{(s+1)(s+2)(s+3)} = r_1 \frac{1}{s+1} + r_2 \frac{1}{s+2} + r_3 \frac{1}{s+3} \tag{4.46}$$

Another approach is to use mathematical software.

```
# Python code to compute the roots of a polynomial
from sympy import symbols, solve
s = symbols('s')
roots = solve(s**3 + 6*s**2 + 11*s + 6, s)
print(roots)
```

Since $F_3(s) = \frac{2}{(s+1)(s+2)(s+3)} = r_1\frac{1}{s+1} + r_2\frac{1}{s+2} + r_3\frac{1}{s+3}$, the single-pole residue formula (4.40) implies

$$r_1 = (s+1)F_3(s)\Big|_{s=-1} = \frac{2}{(s+2)(s+3)}\Big|_{s=-1} = +1 \tag{4.47}$$

$$r_2 = (s+2)F_3(s)\Big|_{s=-2} = \frac{2}{(s+1)(s+3)}\Big|_{s=-2} = -2 \tag{4.48}$$

$$r_3 = (s+3)F_3(s)\Big|_{s=-3} = \frac{2}{(s+1)(s+2)}\Big|_{s=-3} = +1 \tag{4.49}$$

In summary,

$$f_3(t) \;=\; \mathcal{L}^{-1}\Big[(+1)\frac{1}{s+1} + (-2)\frac{1}{s+2} + (+1)\frac{1}{s+3}\Big] \;=\; \mathrm{e}^{-t} - 2\,\mathrm{e}^{-2t} + \mathrm{e}^{-3t} \tag{4.50}$$

Fourth example (one pair of complex conjugate poles)

As fourth example, we consider a rational function with a complex conjugate pair of roots:

$$F_4(s) = \frac{8s+12}{s^2+6s+25} \tag{4.51}$$

We compute[3]

$$s^2+6s+25=0 \iff s=-3\pm 4\mathrm{i} \iff s^2+6s+25=(s+3)^2+4^2 \tag{4.52}$$

Therefore, the denominator is of the form $(s+a)^2+\omega^2$ where $a=3$ and $\omega=4$. In this case, the single-pole residue formula (4.40) is inconvenient to apply. Therefore we proceed by *matching the numerators*.

We now recall rows (7) and (8) for damped sine and cosine waves in the Table 4.1 of Laplace transforms, and look for coefficients α, β such that:

$$\frac{8s+12}{s^2+6s+25} = \left(\alpha\frac{\omega}{(s+a)^2+\omega^2}+\beta\frac{s+a}{(s+a)^2+\omega^2}\right)_{a=3,\omega=4} = \alpha\frac{4}{s^2+6s+25}+\beta\frac{s+3}{s^2+6s+25} \tag{4.53}$$

By matching the numerators and each power of s, we obtain

$$8s+12=4\alpha+\beta(s+3) \implies \begin{cases} 8=\beta \\ 12=4\alpha+3\beta \end{cases} \tag{4.54}$$

so that $\beta=+8$ and $12=4\alpha+24$, that is, $\alpha=-3$. In summary, we know

$$f_4(t) \;=\; \mathcal{L}^{-1}\left[(-3)\frac{4}{s^2+6s+25}+(+8)\frac{s+3}{s^2+6s+25}\right] \;=\; -3\,\mathrm{e}^{-3t}\sin(4t)+8\,\mathrm{e}^{-3t}\cos(4t) \tag{4.55}$$

[3]Given the second order algebraic equation $az^2+bz+c=0$, recall the classic formula for the roots $z_{1,2}=(-b\pm\sqrt{b^2-4ac})/(2a)$.

Fifth example (repeated real poles)

As fifth and last example, we consider the case of a repeated pole. We consider

$$F_5(s) = \frac{s^2+3s+3}{(s+2)^3} \tag{4.56}$$

where the pole $s=-2$ is repeated three times. In this case, the correct partial fraction expansion contains three terms (the same number as the multiplicity of the pole):

$$F_5(s) = \frac{s^2+3s+3}{(s+2)^3} = \alpha\frac{1}{s+2} + \beta\frac{1}{(s+2)^2} + \gamma\frac{1}{(s+2)^3} \tag{4.57}$$

Since the single-pole residue formula (4.40) does not apply, we proceed by *matching the numerators*:

$$\frac{s^2+3s+3}{(s+2)^3} = \alpha\frac{1}{s+2} + \beta\frac{1}{(s+2)^2} + \gamma\frac{1}{(s+2)^3} = \frac{\alpha(s+2)^2+\beta(s+2)+\gamma}{(s+2)^3} \tag{4.58}$$

$$\implies \quad s^2+3s+3 = \alpha(s^2+4s+4)+\beta(s+2)+\gamma \tag{4.59}$$

$$\implies \quad \begin{cases} 1 = \alpha \\ 3 = \alpha 4 + \beta \\ 3 = 4\alpha + 2\beta + \gamma \end{cases} \quad \implies \quad \begin{cases} \alpha = +1 \\ \beta = 3-4 = -1 \\ \gamma = 3-4-2\cdot(-1) = 1. \end{cases} \tag{4.60}$$

In summary, we know

$$f_5(t) = \mathcal{L}^{-1}\Big[\frac{s^2+3s+3}{(s+2)^3}\Big] = \mathcal{L}^{-1}\Big[(+1)\frac{1}{s+2} + (-1)\frac{1}{(s+2)^2} + (+1)\frac{1}{(s+2)^3}\Big] \tag{4.61}$$

$$= \mathrm{e}^{-2t} - t\,\mathrm{e}^{-2t} + \frac{1}{2!}t^2\,\mathrm{e}^{-2t} = \big(1 - t + \tfrac{1}{2}t^2\big)\,\mathrm{e}^{-2t} \tag{4.62}$$

where we have used row (11) from the Table 4.2 of additional Laplace transforms to compute

$$\mathcal{L}^{-1}\Big[\frac{1}{(s+2)^2}\Big] = t\,\mathrm{e}^{-2t} \qquad \text{and} \qquad \mathcal{L}^{-1}\Big[\frac{1}{(s+2)^3}\Big] = \frac{1}{2}t^2\,\mathrm{e}^{-2t}. \tag{4.63}$$

4.2.2 Symbolic mathematics software: the Python SymPy library

Programming notes

It is useful to briefly compare leading software libraries for symbolic mathematics, also known as computer algebra systems.

- Leading commercial software systems are Maple, Mathematica, and Maxima. These software systems include advanced tools for calculus, linear algebra, number theory, and differential equations.
- SymPy is an open-source Python library for symbolic mathematics. It implements basic functionality in calculus, algebra, discrete math, and geometry and it can compute Laplace transforms and inverse Laplace transforms, see (Meurer et al, 2017). SymPy is well-suited for those who prioritize open-source flexibility and Python integration (while commercial systems possibly offer more extensive features and optimizations).

```python
# Import SymPy for symbolic math operations
from sympy import symbols, diff, integrate, Eq, solve, sin, cos, series

# Define a symbol x
x = symbols('x')

print("Example calculations performed by SymPy:\n")

# 1. Differentiate a symbolic expression
expr = sin(x) * cos(x)
derivative = diff(expr, x)
print(f"Derivative of sin(x) * cos(x): {derivative}")

# 2. Integrate a symbolic expression
integral = integrate(expr, x)
print(f"Integral of sin(x) * cos(x): {integral}")

# 3. Solve a symbolic equation
equation = Eq(x**2 + 2*x - 8, 0)
solutions = solve(equation, x)
print(f"Solutions to x^2 + 2x - 8 = 0: {solutions}")

# 4. Perform a Taylor series expansion
taylor_series = series(sin(x), x, 0, 6)
print(f"Taylor series of sin(x) (5th degree): {taylor_series}")
```

Listing 4.1: Python script illustrating SymPy's abilities, see Figure 4.6. Available at sympy-demo.py

```
Example calculations performed by SymPy:

Derivative of sin(x) * cos(x): -sin(x)**2 + cos(x)**2
Integral of sin(x) * cos(x): sin(x)**2/2
Solutions to x^2 + 2x - 8 = 0: [-4, 2]
Taylor series of sin(x) (5th degree): x - x**3/6 + x**5/120 + O(x**6)
```

Figure 4.6: Output of the sympy-demo.py program.

Symbolic computation of inverse Laplace transforms

```python
from sympy import symbols, inverse_laplace_transform, latex
from sympy.abc import s, t

# Define symbols
a, b, c, omega = symbols('a b c omega', real=True)

# Define the rational functions for which we want the inverse Laplace transform
functions = [
    2 + 3/s + 4/(s + 5),             # ex1: a single pole
    (s + 2) / (s**2 + 7*s + 12),    # ex2: two real poles
    2 / (s**3 + 6*s**2 + 11*s + 6),# ex3: multiple isolated poles
    (8*s + 12) / (s**2 + 6*s + 25),# ex4: complex conjugate poles
    (s**2 + 3*s + 3) / (s + 2)**3, # ex5: a repeated pole
    # Symbolic examples
    (a*s + b) / (s**2 + 7*s + 12), # ex6: two poles
    1 / ((s + a) * (s + b) * (s + c)), # ex7: three poles
    1 / ((s + a) * (s**2 + omega**2))  # ex8: a real, two conjugate poles
]

# Prepare the LaTeX content
latex_content = "Examples of inverse Laplace transforms:\n\\begin{align}\n"

# Loop through the functions
for i, F in enumerate(functions):
    # Compute the inverse Laplace transform
    f = inverse_laplace_transform(F, s, t)

    # Generate the LaTeX line for the current function
    latex_line = f"\\mathcal{{L}}^{{-1}} \\left[ {latex(F)} \\right] &= {latex(f)} \\label{{eq:example{i+1}}}"

    # Add a line break after each equation, except the last one
    if i < len(functions) - 1:
        latex_line += " \\\\ \n"

    # Add vertical space after the first five examples
    if i == 4:
        latex_line += "\\nonumber\\\\ \n"

    # Add the current line to the LaTeX content
    latex_content += latex_line

# Close the LaTeX content with align
latex_content += "\\end{align}\n"

# Write the LaTeX to a file
with open("inverseLaplace.tex", "w") as file:
    file.write(latex_content)
```

Listing 4.2: Python script generating the LaTeX output in Figure 4.7. Available at inverseLaplace.py

Examples of inverse Laplace transforms:

$$\mathcal{L}^{-1}\left[2+\frac{4}{s+5}+\frac{3}{s}\right]=2\delta(t)+3\theta(t)+4e^{-5t}\theta(t) \tag{4.64}$$

$$\mathcal{L}^{-1}\left[\frac{s+2}{s^2+7s+12}\right]=\left(2-e^{t}\right)e^{-4t}\theta(t) \tag{4.65}$$

$$\mathcal{L}^{-1}\left[\frac{2}{s^3+6s^2+11s+6}\right]=\left(e^{2t}-2e^{t}+1\right)e^{-3t}\theta(t) \tag{4.66}$$

$$\mathcal{L}^{-1}\left[\frac{8s+12}{s^2+6s+25}\right]=-\left(3\sin(4t)-8\cos(4t)\right)e^{-3t}\theta(t) \tag{4.67}$$

$$\mathcal{L}^{-1}\left[\frac{s^2+3s+3}{(s+2)^3}\right]=\frac{\left(t^2-2t+2\right)e^{-2t}\theta(t)}{2} \tag{4.68}$$

$$\mathcal{L}^{-1}\left[\frac{as+b}{s^2+7s+12}\right]=\left(4a-b-(3a-b)e^{t}\right)e^{-4t}\theta(t) \tag{4.69}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(a+s)(b+s)(c+s)}\right]=\frac{\left((a-b)e^{t(a+b)}-(a-c)e^{t(a+c)}+(b-c)e^{t(b+c)}\right)e^{-t(a+b+c)}\theta(t)}{(a-b)(a-c)(b-c)} \tag{4.70}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(a+s)(\omega^2+s^2)}\right]=\frac{\left(\omega+\left(a\sin(\omega t)-\omega\cos(\omega t)\right)e^{at}\right)e^{-at}\theta(t)}{\omega\left(a^2+\omega^2\right)} \tag{4.71}$$

Figure 4.7: Examples of inverse Laplace transforms of rational functions, via the SymPy symbolic computing library (Meurer et al, 2017).
In SymPy, the function $\theta(t)$ is the unit step function $\mathbf{1}(t)$.
The first five examples are the same inverse Laplace transforms that we computed in the previous section.

4.3 Solving linear differential equations

As before we let $F(s) = \mathcal{L}[f(t)]$. We now recall the derivative-with-respect-to-time property (P2) and extend it to higher order derivatives (simply by applying it repeatedly). We recall

$$\mathcal{L}\Big[\frac{d}{dt}f(t)\Big] = sF(s) - f(0) \tag{4.72}$$

so that

$$\mathcal{L}\Big[\frac{d^2}{dt^2}f(t)\Big] = s\,\mathcal{L}\Big[\frac{d}{dt}f(t)\Big] - \frac{df}{dt}(0) = s\big(sF(s) - f(0)\big) - \frac{df}{dt}(0) \tag{4.73}$$

$$= s^2F(s) - sf(0) - \frac{df}{dt}(0) \tag{4.74}$$

Applying the property repeatedly, we can compute higher-order time derivatives:

$$\mathcal{L}\Big[\frac{d^nf}{dt^n}(t)\Big] = s^nF(s) - s^{n-1}f(0) - s^{n-2}\frac{df}{df}(0) - \ldots - \frac{d^{n-1}f}{dt^{n-1}}(0) \tag{4.75}$$

$$= s^nF(s) - \sum_{k=0}^{n-1} s^{n-1-k}\frac{d^kf}{dt^k}(0) \tag{4.76}$$

We use these properties to *transform differential equations into algebraic equations*. In what follows we let

$$X(s) = \mathcal{L}[x(t)], \qquad Y(s) = \mathcal{L}[y(t)], \qquad \text{and} \qquad U(s) = \mathcal{L}[u(t)].$$

4.3.1 A differential equation with non-zero initial conditions and zero input

Consider the differential equation

$$\ddot{y} + 7\dot{y} + 12y = 0, \qquad y(0) = y_0, \quad \dot{y}(0) = v_0 \tag{4.77}$$

We aim to compute the solution $y(t)$ as a function of the unknown initial conditions y_0 and v_0.

To do so, we take Laplace transform of both the left and the right hand side to obtain

$$\big(s^2 Y(s) - sy_0 - v_0\big) + 7\big(sY(s) - y_0\big) + 12Y(s) = 0$$

where we used $Y(s) = \mathcal{L}[y(t)]$ and the derivative properties (4.72) and (4.73). We can now collect the terms multiplying $Y(s)$:

$$\begin{aligned} &\big(s^2 + 7s + 12\big)Y(s) - sy_0 - v_0 - 7y_0 = 0 \\ \iff \qquad &Y(s) = \frac{sy_0 + (v_0 + 7y_0)}{s^2 + 7s + 12} = \frac{sy_0 + (v_0 + 7y_0)}{(s+3)(s+4)}. \end{aligned} \tag{4.78}$$

Equation (4.69) in the previous section on partial fraction expansions states:

$$\mathcal{L}^{-1}\left[\frac{sa+b}{s^2+7s+12}\right] = (4a - b)\,\mathrm{e}^{-4t} - (3a-b)\,\mathrm{e}^{-3t}.$$

Since here $a = y_0$ and $b = v_0 + 7y_0$, the solution to the differential equation (4.77) is

$$y(t) = (4y_0 - v_0 - 7y_0)\,\mathrm{e}^{-4t} - (3y_0 - v_0 - 7y_0)\,\mathrm{e}^{-3t} = -(3y_0 + v_0)\,\mathrm{e}^{-4t} + (4y_0 + v_0)\,\mathrm{e}^{-3t} \tag{4.79}$$

In summary, using to the Laplace transform, we have transformed the differential equation (4.77) into the algebraic equation (4.78). We have then solved the algebraic equation and computed the solution to the differential equation via a partial fraction expansion.

4.3.2 A differential equation with zero initial conditions and non-zero input

Given a constant scalar f, consider the differential equation

$$\ddot{y} + 7\dot{y} + 12y = f, \qquad y(0) = 0,\ \dot{y}(0) = 0 \tag{4.80}$$

We now take Laplace transform of both the left and the right hand side to obtain

$$s^2 Y(s) + 7sY(s) + 12Y(s) = \frac{f}{s} \tag{4.81}$$

where we used $Y(s) = \mathcal{L}[y(t)]$, the derivative properties (4.72) and (4.73), and the equality $\mathcal{L}[f] = \mathcal{L}[f \cdot \mathbf{1}(t)] = f/s$. Hence,

$$Y(s) = \frac{f}{s(s^2 + 7s + 12)} = \frac{f}{s(s+3)(s+4)} \tag{4.82}$$

Equation (4.70) in the previous section on partial fraction expansions states:

$$\mathcal{L}^{-1}\left[\frac{1}{(s+a)\,(s+b)\,(s+c)}\right] = \frac{1}{(a-b)(a-c)(b-c)}\Big((b-c)\,\mathrm{e}^{-at} + (a-b)\,\mathrm{e}^{-ct} + (c-a)\,\mathrm{e}^{-bt}\Big) \tag{4.83}$$

Since here we have $a = 0$, $b = 3$ and $c = 4$, the solution to the differential equation (4.80) is

$$y(t) = \frac{f}{(-3)(-4)(3-4)}\Big((3-4)\mathbf{1}(t) + (-3)\,\mathrm{e}^{-4t} + (4)\,\mathrm{e}^{-3t}\Big) \tag{4.84}$$

$$= \frac{f}{12}\Big(\mathbf{1}(t) + 3\,\mathrm{e}^{-4t} - 4\,\mathrm{e}^{-3t}\Big) \tag{4.85}$$

4.3.3 Combining the previous two examples

We now consider the differential equation

$$\ddot{y} + 7\dot{y} + 12y = f, \qquad y(0) = y_0,\ \dot{y}(0) = v_0 \tag{4.86}$$

Note: this problem is the same differential equation as in the previous two Sections 4.3.1 and 4.3.2 but here there are both: non-zero initial conditions y_0 and v_0 and a constant scalar input f.

We claim that the solution is the sum of the solutions in the two previous case. Summing the solution in equation (4.79) to the solution in equation (4.84) we obtain

$$y(t) = -(3y_0 + v_0)\,\mathrm{e}^{-4t} + (4y_0 + v_0)\,\mathrm{e}^{-3t} + \frac{f}{12}\Big(\mathbf{1}(t) + 3\,\mathrm{e}^{-4t} - 4\,\mathrm{e}^{-3t}\Big) \tag{4.87}$$

The reason why this formula is correct is given in the next section.

4.3.4 General case

Suppose we are given a controlled dynamical system in the form:

$$\sum_{i=0}^{n} a_i \frac{d^i y}{dt^i}(t) = u(t) \tag{4.88}$$

where

- y is the output and u is the control input
- $a_0, \ldots, a_n$ are constant coefficients, and
- we use the abbreviation $\sum_{i=0}^{n} a_i \frac{d^i y}{dt^i}(t) = a_0 y(t) + a_1 \frac{dy}{dt}(t) + \cdots + a_n \frac{d^n y}{dt^n}(t)$

We assume we are also given the initial conditions:

$$\frac{d^i y}{dt^i}(0) = y_0^i \qquad \text{for } i = 0, 1, \ldots, n-1 \tag{4.89}$$

The Laplace transform of both left and right hand side of the differential equation (4.88) gives:

$$\sum_{i=0}^{n} a_i \Big(s^i Y(s) - \sum_{k=0}^{i-1} s^{i-1-k} y_0^k \Big) = U(s) \tag{4.90}$$

Therefore, after some reorganizing and book keeping

$$Y(s) = \frac{U(s)}{\sum_{i=0}^{n} a_i s^i} + \frac{\sum_{i=0}^{n} \sum_{k=0}^{i-1} a_i s^{i-1-k} y_0^k}{\sum_{i=0}^{n} a_i s^i} =: Y_{\text{forced}}(s) + Y_{\text{free}}(s)$$

which implies

$$y(t) = \mathcal{L}^{-1}\left[Y_{\text{forced}}(s)\right] + \mathcal{L}^{-1}\left[Y_{\text{free}}(s)\right] = y_{\text{forced}}(t) + y_{\text{free}}(t)$$

We learn a few useful lessons:

(i) the *forced response* $y_{\text{forced}}(t)$ is due to a non-zero input $u(t)$, with zero initial conditions,

(ii) the *free response* $y_{\text{free}}(t)$ is due to non-zero initial conditions, with zero input $u(t) = 0$,

(iii) *the response $y(t)$ is the sum of forced and free response*. In other words, in *a linear control system, the response is driven by two causes: the initial conditions and the external input. The overall response is the sum of the effects produced by each of the two causes*;

(iv) the *characteristic polynomial* is the denominator in the Laplace transform of both forced and free responses:

$$\sum_{i=0}^{n} a_i s^i = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \tag{4.91}$$

(v) the roots of the characteristic polynomials determine the exponential-like functions in the free response.

4.4 Appendix: Additional Laplace transform properties

The Laplace transforms satisfies numerous properties beyond (P1)–(P4) listed in Section 4.1.3. Here are some additional properties that are sometimes useful to analyze dynamical systems.

(P5) Final Value Theorem: $\lim_{t\to+\infty} f(t) = \lim_{s\to 0} sF(s)$, if the limit of f exists finite[4]

(P6) Initial Value Theorem: $\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$

(P7) Time delay: $\mathcal{L}[f(t-T)] = \mathrm{e}^{-sT} F(s)$ (recall $f(t-T) = 0$ for all $t < T$)

(P8) Convolution: $\mathcal{L}\Big[\int_0^t f(\tau)g(t-\tau)d\tau\Big] = F(s)G(s)$, where $\int_0^t f(\tau)g(t-\tau)d\tau$ is the *convolution integral*.

(P9) Time scaling: $\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

(P10) Complex derivative: $\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$

Higher-order complex derivatives: $\mathcal{L}[(-1)^n t^n f(t)] = (-1)^s \frac{d}{ds}F(s)$

[4]The limit of f exists finite if all the poles of the rational functino $sF(s)$ are on the left half plane, as we will study in the next chapter.

4.5 Appendix: Additional Laplace transform pairs, the 2nd table

In this appendix we present Laplace transform pairs that complement those presented in the first Table 4.1 of Laplace transform pairs in Section 4.1.6.

	Function of time $f(t)$	*Laplace transform $F(s)$ and its poles*	
	In this table we consider only exponential-like functions.	Laplace transforms of exponential-like functions are rational functions.	
(9)	t^n (for any $n = 1, 2, 3, \dots$)	$\frac{n!}{s^{n+1}}$	$s = 0$, repeated $n+1$ times
(10)	$t\,\mathrm{e}^{-at}$	$\frac{1}{(s+a)^2}$	$s = -a$, repeated
(11)	$t^n\,\mathrm{e}^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$s = -a$, repeated $n+1$ times
(7)	$\mathrm{e}^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$s = -a \pm \mathrm{i}\omega$
(8)	$\mathrm{e}^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	$s = -a \pm \mathrm{i}\omega$
(12)	$\frac{1}{b-a}(\mathrm{e}^{-at} - \mathrm{e}^{-bt})$	$\frac{1}{(s+a)(s+b)}$	$s = -a, -b$, $a \neq b$
(13)	$\frac{1}{b-a}(b\,\mathrm{e}^{-bt} - a\,\mathrm{e}^{-at})$	$\frac{s}{(s+a)(s+b)}$	$s = -a, -b$, $a \neq b$
(14)	$\frac{1}{(a-b)(a-c)(b-c)}\Big((c-a)\,\mathrm{e}^{-bt} + (a-b)\,\mathrm{e}^{-ct} + (b-c)\,\mathrm{e}^{-at}\Big)$	$\frac{1}{(s+a)(s+b)(s+c)}$	$s = -a, -b, -c$, $a \neq b \neq c$
(14)	$\frac{1}{\omega(a^2+\omega^2)}\Big(\omega\,\mathrm{e}^{-at} + a\sin(\omega t) - \omega\cos(\omega t)\Big)$	$\frac{1}{(s+a)(s^2+\omega^2)}$	$s = -a, \pm\mathrm{i}\omega$

Table 4.2: Row (11) is more general than rows (9) and (10) (as well as rows (1), (2) and (4) in Table 4.1) and focuses on the case of a single real pole, possibly repeated. Recall $n!$ is the factorial of n: $0! = 1, 1! = 1, 2! = 2, 3! = 6, \dots$.
Rows (7), (8), (12) and (13) capture all possible cases of two poles, not repeated. Either both poles are real or the two poles are complex conjugate. Rows (7), (8) are repeated here for convenience.
Rows (14) and (15) are only two examples of a rational function with three distinct poles.

4.6 Exercises

E4.1 **Laplace transforms (example #1).** Given a signal $x(t)$, let $X(s)$ denote its Laplace transform. Using the properties and tables of Laplace transforms, compute:

(i) $\mathcal{L}\left[\dot{x}(t) + 4\,\mathrm{e}^{-2t} - 3\right]$,

(ii) $\mathcal{L}\left[\int_0^t x(\tau)\,d\tau + \cos(5t)\right]$, and

(iii) $\mathcal{L}\left[\ddot{x}(t) - t^2\,\mathrm{e}^{-t} + \mathrm{e}^t \sin(7t)\right]$.
Hint: Use row (11) in Table 4.2.

Answer: The solutions are given as follows:

(i) The Laplace transform of the derivative (P2) is given by $\mathcal{L}[\dot{x}(t)] = sX(s) - x(0)$. The exponential term $4\,\mathrm{e}^{-2t}$ transforms to $\frac{4}{s+2}$, and the constant -3 transforms to $\frac{-3}{s}$. Using the linearity property, we combine these terms:

$$\mathcal{L}\left[\dot{x}(t) + 4\,\mathrm{e}^{-2t} - 3\right] = sX(s) - x(0) + \frac{4}{s+2} - \frac{3}{s}$$

(ii) For the integral term, we apply the integral property (P3): $\mathcal{L}\left[\int_0^t x(\tau)\,d\tau\right] = \frac{1}{s}X(s)$. Additionally, the Laplace transform of $\cos(5t)$ is $\frac{s}{s^2+25}$. Using linearity to combine the two transforms, we get:

$$\mathcal{L}\left[\int_0^t x(\tau)\,d\tau + \cos(5t)\right] = \frac{1}{s}X(s) + \frac{s}{s^2+25}$$

(iii) The Laplace transform of the second derivative is given by $\mathcal{L}[\ddot{x}(t)] = s^2X(s) - sx(0) - \dot{x}(0)$, as we saw in equation (4.73). For the second term, we use $\mathcal{L}[t^2\,\mathrm{e}^{-t}] = \frac{2}{(s+1)^3}$ (see row (11) in Table 4.2), and for the third term, $\mathcal{L}[\mathrm{e}^t \sin(7t)] = \frac{7}{(s-1)^2+49}$. Combining everything:

$$\mathcal{L}\left[\ddot{x}(t) - t^2\,\mathrm{e}^{-t} + \mathrm{e}^t \sin(7t)\right] = s^2X(s) - sx(0) - \dot{x}(0) - \frac{2}{(s+1)^3} + \frac{7}{(s-1)^2+49}$$

■

E4.2 **Using the Laplace transform to solve a differential equation (example #1).** Consider the differential equation (without inputs)

$$\ddot{x} + 4\dot{x} + 5x = 0, \qquad x(0) = \dot{x}(0) = 1. \tag{E4.1}$$

(i) Compute the solution in the Laplace domains $X(s) = \mathcal{L}[x(t)]$.
(ii) Expand the solution in a partial fraction.
(iii) Compute the inverse Laplace transform to obtain $x(t)$.

Answer:

(i) We take the Laplace transform of left and right hand side:

$$s^2X(s) - sx(0) - \dot{x}(0) + 4sX(s) - 4x(0) + 5X(s) = 0 \tag{E4.2}$$

$$\iff \quad s^2X(s) - s - 1 + 4sX(s) - 4 + 5X(s) = 0 \tag{E4.3}$$

We then compute $X(s)$ and write it in partial fraction expansion

$$X(s) = \frac{s+5}{s^2+4s+5} \tag{E4.4}$$

(ii) We then write $X(s)$ in partial fraction expansion. First, we note that $s^2 + 4s + 5 = (s+2)^2 + 1$, that is, we have a pair of complex conjugate roots. Therefore, as in the fourth example in Section 4.2.1, we know that damped sine and cosine waves will appear. After solving the linear equations to match the numerators we obtain:

$$X(s) = \frac{s+5}{s^2+4s+5} = \frac{s+2}{(s+2)^2+1} + \frac{3}{(s+2)^2+1} \tag{E4.5}$$

where, in the last equality, we wrote the denominator as the sum of two positive numbers.

(iii) Finally, using the inverse Laplace transform of damped sine and cosine from the Table 4.1 of Laplace transforms:

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathrm{e}^{-2t}\cos(t) + 3\,\mathrm{e}^{-2t}\sin(t) \tag{E4.6}$$

■

E4.3 **Using the Laplace transform to solve a differential equation (example #2).** Consider the differential equation with an input

$$\ddot{y} - y = t, \qquad y(0) = \dot{y}(0) = 1. \tag{E4.7}$$

(i) Compute the solution in the Laplace domain $Y(s) = \mathcal{L}[y(t)]$.
(ii) Compute the inverse Laplace transform of $Y(s)$ to obtain $y(t)$.

Hint: There are at least two distinct ways to solve this problem. One possible way is to use these additional Laplace transform pairs: $\mathcal{L}[\cosh(t)] = \frac{s}{s^2-1}$ and $\mathcal{L}[\sinh(t)] = \frac{1}{s^2-1}$.

Answer: We take the Laplace transform of left and right hand side:

$$s^2Y(s) - sy(0) - \dot{y}(0) - Y(s) = \frac{1}{s^2} \tag{E4.8}$$

$$\iff \quad (s^2-1)Y(s) = s + 1 + \frac{1}{s^2} = \frac{s^3+s^2+1}{s^2} \tag{E4.9}$$

We then compute $Y(s)$ and write it in an appropriate partial fraction expansion:

$$Y(s) = \frac{s^3+s^2+1}{s^2(s^2-1)} = -\frac{1}{s^2} + \frac{s+2}{s^2-1} \tag{E4.10}$$

Finally, using the inverse Laplace transform of hyperbolic sine and cosine:

$$y(t) = \mathcal{L}^{-1}[Y(s)] = -t + \cosh(t) + 2\sinh(t) \tag{E4.11}$$

Alternative approach: The partial fraction expansion of $Y(s)$ is of the form:

$$Y(s) = \frac{s^3+s^2+1}{s^2(s^2-1)} = \frac{\alpha}{s} + \frac{\beta}{s^2} + \frac{\gamma}{s-1} + \frac{\delta}{s+1} \tag{E4.12}$$

Using the inverse Laplace transform we know

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \alpha + \beta t + \gamma\,\mathrm{e}^{t} + \delta\,\mathrm{e}^{-t} \tag{E4.13}$$

To compute the values of the coefficients, we use the method of "matching the numerators of left and right hand side" and we get

$$s^3 + s^2 + 1 = (\alpha+\gamma+\delta)s^3 + (\gamma-\delta+\beta-\alpha)s^2 - (\alpha+\beta)s - \beta. \tag{E4.14}$$

and we obtain $\alpha = 0$, $\beta = -1$, $\gamma = 3/2$, and $\delta = -1/2$.

■

E4.4 **Laplace transform of suspension system.** Consider the suspension system described in Section 2.2 and Figure 2.7. Recall that the equations of motion for the system were found to be:

$$\begin{aligned} m_\mathrm{s}\ddot{x}_\mathrm{s} + b\,(\dot{x}_\mathrm{s} - \dot{x}_\mathrm{us}) + k_\mathrm{s}\,(x_\mathrm{s} - x_\mathrm{us}) &= 0 \\ m_\mathrm{us}\ddot{x}_\mathrm{us} + b\,(\dot{x}_\mathrm{us} - \dot{x}_\mathrm{s}) + k_\mathrm{s}\,(x_\mathrm{us} - x_\mathrm{s}) + k_\mathrm{w}(x_\mathrm{us} - r(t)) &= 0. \end{aligned}$$

where $x_\mathrm{s}(t)$ is the vertical position of the sprung mass, $x_\mathrm{us}(t)$ the vertical position of the sprung mass, and $r(t)$ is the height of the road surface. Assume that the initial positions and velocities of both masses are equal to zero: $x_\mathrm{s}(0) = x_\mathrm{us}(0) = \dot{x}_\mathrm{s}(0) = \dot{x}_\mathrm{us}(0) = 0$

Define the Laplace transforms: $X_\mathrm{us}(s) = \mathcal{L}[x_\mathrm{us}(t)]$, $X_\mathrm{s}(s) = \mathcal{L}[x_\mathrm{s}(t)]$ and $R(s) = \mathcal{L}[r(t)]$.

(i) Using the properties of Laplace transforms, find the Laplace transforms of the two equations.

(ii) Use the two equations to eliminate the intermediate variable $X_\mathrm{us}(s)$ to obtain an expression for $X_\mathrm{s}(s)$ in terms of $R(s)$.

Answer:

(i) Applying the Laplace transform yields

$$\begin{aligned} m_s s^2 X_\mathrm{s}(s) + bsX_\mathrm{s}(s) - bsX_\mathrm{us}(s) + k_s X_\mathrm{s}(s) - k_s X_\mathrm{us}(s) &= 0 \\ m_{us} s^2 X_\mathrm{us}(s) + bsX_\mathrm{us}(s) - bsX_\mathrm{s}(s) + k_s X_\mathrm{us}(s) - k_s X_\mathrm{s}(s) + k_w X_\mathrm{us}(s) - k_w R(s) &= 0. \end{aligned}$$

Grouping like terms, we get

$$\begin{aligned} (m_s s^2 + bs + k_s)X_\mathrm{s}(s) - (bs + k_s)X_\mathrm{us}(s) &= 0 \\ (m_{us} s^2 + bs + k_s + k_w)X_\mathrm{us}(s) - (bs + k_s)X_\mathrm{s}(s) - k_w R(s) &= 0 \end{aligned}$$

(ii) We solve for $X_\mathrm{us}(s)$ in the first equation, ans substitute this into the second equation to obtain

$$(m_{us} s^2 + bs + k_s + k_w)\frac{(m_s s^2 + bs + k_s)}{(bs + k_s)} X_\mathrm{s}(s) - (bs + k_s)X_\mathrm{s}(s) - k_w R(s) = 0.$$

Solving for $X_\mathrm{s}(s)$ in terms of $R(s)$ yields

$$X_\mathrm{s}(s) = \frac{k_w(bs + k_s)}{(m_{us} s^2 + bs + k_s + k_w)(m_s s^2 + bs + k_s) - (bs + k_s)^2} R(s)$$

■

E4.5 **Optional: Programming exercise.** Verify the solutions to ordinary differential equations in Section 4.3 by modifying the following Python SymPy code.

```python
# Python code to compute the roots of a polynomial
from sympy import Function, dsolve, Eq, Derivative, symbols, init_printing
from sympy.abc import t

# Define the symbols
y0, v0 = symbols('y0 v0')

# Define the function which represents y(t)
y = Function('y')

# Define the differential equation as in Section 4.3
diffeq = Eq(y(t).diff(t, t) + 7*y(t).diff(t) + 12*y(t), 0)

# Solve the differential equation with initial conditions
solution = dsolve(diffeq, y(t), ics={y(0): y0, y(t).diff(t).subs(t, 0): v0})

# Display the solution
print("The ode solution with the specified initial conditions is:")
print(solution)
```

Listing 4.3: Python script illustrating SymPy's abilities, see Figure E4.1. Available at sympy-demo-ode.py

```
The ode solution with the specified initial conditions is:
Eq(y(t), (v0 + 4*y0 + (-v0 - 3*y0)*exp(-t))*exp(-3*t))
```

Figure E4.1: Output of the sympy-demo-ode.py program. Simple calculations show that this solution is the same as in equation (4.79).

Answer: No solution is available at this time. ■

E4.6 **Laplace transforms (example #2).** Given a signal $x(t)$, let $X(s)$ denote its Laplace transform. Using the properties and tables of Laplace transforms, compute:

(i) $\mathcal{L}\left[\ddot{x}(t) + 3\,\mathrm{e}^{-2t} - 5\right]$,

(ii) $\mathcal{L}\left[t\,\mathrm{e}^{-2t} - t^2\,\mathrm{e}^{-t} + \mathrm{e}^{3t}\sin(5t)\right]$, and

(iii) $\mathcal{L}\left[\sin(\omega t + \theta)\right]$, where ω and θ are constant parameters.

E4.7 **Inverse Laplace transforms.** Compute the inverse Laplace transforms of:

(i) $\dfrac{s+3}{s(s+1)}$,

(ii) $\dfrac{s+3}{s^2+2s+10}$,

(iii) $\dfrac{s+3}{(s+1)^2(s+2)}$, and

(iv) $\dfrac{s+3}{s(s^2+\omega^2)}$.

E4.8 **Using the Laplace transform to solve a differential equation (example #3).** Consider the differential equation (without inputs):

$$\dddot{x} + 6\ddot{x} + 5\dot{x} - 12x = 0, \qquad x(0) = 1, \quad \dot{x}(0) = 0, \quad \ddot{x}(0) = -1. \tag{E4.15}$$

(i) Compute the solution in the Laplace domain $X(s) = \mathcal{L}[x(t)]$.

(ii) Compute the roots of the denominator of the rational function $X(s)$.

(iii) Expand $X(s)$ in partial fractions and compute the unknown coefficients.

(iv) Compute the inverse Laplace transform to obtain $x(t)$.

E4.9 **Dynamical behavior of a muscle.** In E2.8, we modeled a muscle using a spring damper system and obtained its equations of motion[5] by applying Newton's law. The system variables are $x(t)$, $\dot{x}(t)$, and $x_{\text{interm}}(t)$. The system parameters are the positive constants m, k, and b. The system inputs are $f_{\text{load}}(f)$ and $f_{\text{muscle}}(t)$. Assume that the equations of motion for this system are:

$$m\ddot{x} = -k(x - x_{\text{interm}}) + f_{\text{load}}(t),$$
$$b\dot{x}_{\text{interm}} = -k(x_{\text{interm}} - x) - f_{\text{muscle}}(t).$$

(i) Assuming that $x(0) = \dot{x}(0) = x_{\text{interm}}(0) = 0$, convert these differential equations to the Laplace domain.

(ii) Obtain expressions for $X(s)$ and $X_{\text{interm}}(s)$ in terms of the parameters m, b, k, and the Laplace transform of the input forces $F_{\text{load}}(s) = \mathcal{L}[f_{\text{load}}]$ and $F_{\text{muscle}}(s) = \mathcal{L}[f_{\text{muscle}}]$.

(iii) Assume that the load force $f_{\text{load}}(t)$ and the muscle force $f_{\text{muscle}}(t)$ are equal to a constant value f_0 that begins acting at time $t = 0$. Substitute the Laplace transform of these forces into the expressions for $X(s)$ and $X_{\text{interm}}(s)$ computed in part (ii).

(iv) Use the Final Value Theorem (P5) in Appendix 4.4 (assume the limit exists) in order to find the final positions of $x(t)$ and $x_{\text{interm}}(t)$.

[5] As a reminder, a muscle connected to a fixed point and subject to a load force can be modeled by the equivalent mechanical system shown in the Figure E2.4. The key elements of the system are: (1) The muscle connects the fixed point to a mass m at position x. (2) The muscle is represented by the interconnection of two components, with the intermediate point at coordinate x_{interm}. (3) The muscle exerts a force $f_{\text{muscle}}(t)$ at the intermediate point. (4) A damper with damping coefficient b connects the intermediate point to the stationary point. (5) A spring with stiffness k and zero rest length connects the intermediate point to the mass. (6) The mass m is subject to a load force $f_{\text{load}}(t)$.

E4.10 **Laplace transforms (example #3).** Perform the following calculations using the properties in Appendix 4.4 on the additional Laplace transform properties.

(i) Compute $\mathcal{L}[z(t-2)\mathbf{1}(t-2)]$, using the time-delay property (P7).

(ii) Compute $\mathcal{L}\left[\int_0^t (t-\tau)x(\tau)\,d\tau\right]$, using the convolution integral property (P8).

(iii) Compute the final value of $y(t)$ where $Y(s) = \dfrac{5}{s(s+2)(s+4)}$ using the final value theorem (P5) and assuming the limit exists.

Answer: The solutions are given as follows:

(i) From the time-delay property (P7), the Laplace transform of $z(t-2)\mathbf{1}(t-2)$ is given by $\mathrm{e}^{-2s}Z(s)$, since the signal is delayed by 2 units. Thus, the solution is:

$$\boxed{\mathcal{L}[z(t-2)\mathbf{1}(t-2)] = \mathrm{e}^{-2s}Z(s)}$$

(ii) The convolution theorem states that $\mathcal{L}\left[\int_0^t (t-\tau)x(\tau)\,d\tau\right] = \frac{X(s)}{s^2}$. Thus, the solution is:

$$\boxed{\mathcal{L}\left[\int_0^t (t-\tau)x(\tau)\,d\tau\right] = \frac{X(s)}{s^2}}$$

(iii) The final value theorem states that

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s).$$

We first compute the expression for $sY(s)$:

$$sY(s) = s\cdot\frac{5}{s(s+2)(s+4)} = \frac{5}{(s+2)(s+4)}.$$

Now, we take the limit as $s\to 0$:

$$\lim_{s\to 0}\frac{5}{(s+2)(s-4)} = \frac{5}{(0+2)(0+4)} = \frac{5}{2\cdot 4} = \frac{5}{8}.$$

Thus, the final value of $y(t)$ is:

$$\boxed{\lim_{t\to\infty} y(t) = \frac{5}{8}}$$

■

E4.11 **Padé approximation for systems with time delayed inputs.** Consider a first-order system with state variable x and time constant $\tau > 0$. Assume that the initial condition is zero and that the input to the system is a *delayed unit impulse*, i.e., a unit impulse that occurs at time $T > 0$.

(i) Write down the differential equation that describes this system and the input. Use the unit impulse function δ.

Hint: In other words, the input is zero at each time except $t = T$.

(ii) Compute the solution to the delayed impulse by simply delaying the solution to the unit impulse. In other words,

(a) write down the solution $x(t)$ to the same first-order system with an unit impulse applied at time $t = 0$, and then
(b) modify the solution by delaying by time T.

(iii) Take the Laplace transform of the differential equation from part (i) and compute an expression for $X(s)$. Is $X(s)$ a rational function?

Hint: The time-delay property (P7) in Appendix 4.4 may be useful.

(iv) For analyzing systems with time delays, the so-called Padé approximation is a useful tool. The *first-order Padé approximation* of the exponential function is

$$e^y \approx \frac{2+y}{2-y}. \tag{E4.16}$$

Use the first-order Padé approximation and the inverse Laplace transform to obtain an approximate solution $x_{\text{approx}}(t)$ in the time domain as a function of the parameters τ and T.

(v) Show that, at fixed time constant $\tau > 0$ and time $t > 0$, we have $\lim_{T\to 0}\big(x(t) - x_{\text{approx}}(t)\big) = 0$.

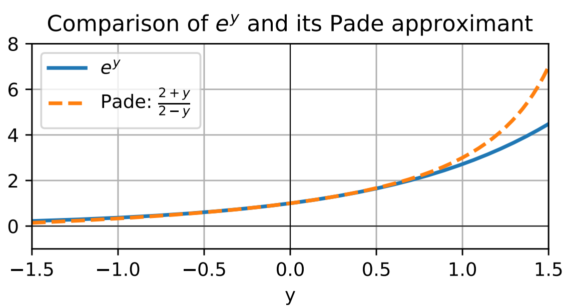


Figure E4.2: Comparison of the exponential function e^y with its Padé approximant $(2+y)/(2-y)$. The Padé approximant provides a close approximation around $y = 0$, but diverges with a vertical asymptote at $y = 2$.

Answer:

(i) The differential equation is

$$\tau\dot{x}(t) + x(t) = \delta(t-T).$$

(ii) The response of a first-order system subject to a unit impulse at time zero is

$$x(t) = \frac{1}{\tau}\,\mathrm{e}^{-t/\tau}, \quad t \geq 0.$$

Therefore, the response to a delayed unit impulse at $t = T$ is

$$x(t) = \begin{cases} 0, & 0 \leq t < T \\ \frac{1}{\tau}\,\mathrm{e}^{-(t-T)/\tau}, & t \geq T. \end{cases}$$

(iii) Taking the Laplace transform using the time-delay property (P7) and rearranging yields

$$X(s) = \frac{\mathrm{e}^{-sT}}{1+\tau s}$$

No, $X(s)$ is not a rational function because it is not the ratio of polynomials. Hence, the usual methods to compute inverse Laplace transform do not apply.

(iv) Using the Padé approximation yields

$$X_{\text{approx}}(s) = \frac{2 - Ts}{(2+Ts)(1+\tau s)}.$$

We can compute the partial fraction expansion as

$$X_{\text{approx}}(s) = \frac{2 - Ts}{(2+Ts)(1+\tau s)} = \frac{\alpha}{2+Ts} + \frac{\beta}{1+\tau s}.$$

The matching method yields the expressions $\alpha = \frac{4T}{T-2\tau}$ and $\beta = -\frac{T+2\tau}{T-2\tau}$. Substitution and the inverse Laplace transform yields

$$x_{\text{approx}}(t) = \frac{4}{T-2\tau}\,\mathrm{e}^{-2t/T} - \frac{T+2\tau}{\tau(T-2\tau)}\,\mathrm{e}^{-t/\tau}$$

(v) We compute

$$\lim_{T\to 0^+} x_{\text{approx}}(t) = \lim_{T\to 0^+} \frac{4}{T-2\tau}\,\mathrm{e}^{-2t/T} - \lim_{T\to 0^+} \frac{T+2\tau}{\tau(T-2\tau)}\,\mathrm{e}^{-t/\tau} = 0 - \frac{2\tau}{\tau(-2\tau)}\,\mathrm{e}^{-t/\tau} = \frac{1}{\tau}\,\mathrm{e}^{-t/\tau} = \lim_{T\to 0^+} x(t)$$

■

Bibliography

A. Meurer et al. SymPy: symbolic computing in Python. *PeerJ Computer Science*, 3:e103, 2017. doi.