UC Santa Barbara, Department of Mechanical Engineering

ME103 Dynamical Systems. Slides by Chapter.

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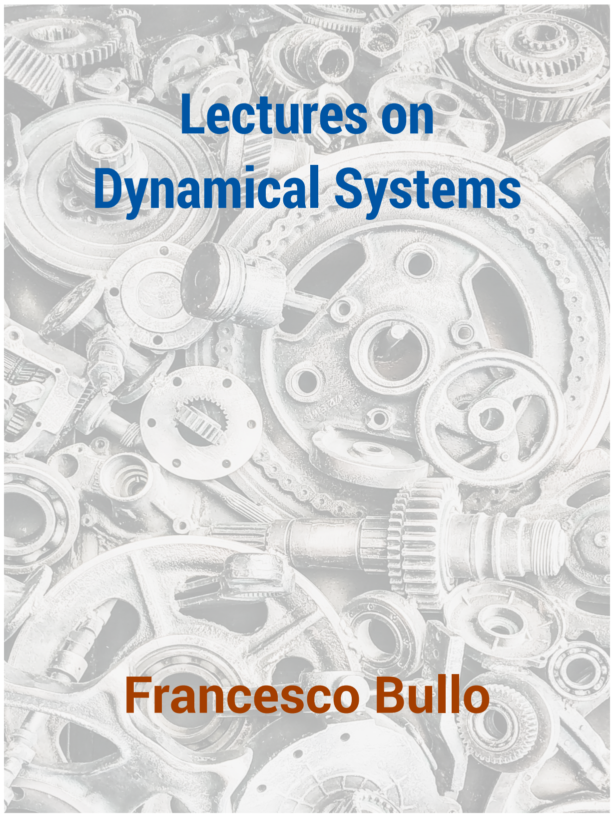
http://motion.me.ucsb.edu/ME103-Fall2024/syllabus.html



Contents

Chapter 2

Mechanical and Electromechanical Dynamical Systems

2.1 Mechanical systems

Newton's law is the starting point for any analysis of mechanical systems. For a *body* composed of a single particle or rigidly interconnected particles moving in a single direction,

$$F = ma \tag{2.1}$$

where:

- F is the resultant force (resultant = algebraic sum of) applied to the body, measured in Newtons (N),
- m is the mass of the body, measured in Kg, and
- $a = a(t) = \ddot{x}(t)$ is the acceleration of the body, that is, the second time derivative of the body position $x(t)$, measured in m/sec^2.

As in Chapter 1, the independent variable t is time and the position $x(t)$ is the dependent variable. The mass m is most often treated as a constant parameter.[1] The force F is independent of Newton's law, it may well be an external force generated by some unspecified means.

Equation (2.1) is referred to as the *equation of motion* as it describes the evolution of the position $x(t)$.

When no force is applied to the particle, then the solution is a translation at constant velocity.

[1]The mass of a rocket burning fuel is a variable of interest and it is not constant.

2.1.1 First-order systems

A *damper* is a mechanical element that dissipates energy. The classic example of a translation damper is a piston connected to a rod and an oil-filled cylinder. Oil resists any relative motion between the piston and the cylinder. Typically, one approximates the force generated by the damper linearly:

$$F_{\text{damper}} = -b\dot{x}(t) \tag{2.2}$$

where $b > 0$ is the *damping coefficient* (aka the viscous friction coefficient and the mechanical resistance).

In a moving car, energy is dissipated by the interaction between air and moving car. Assuming dissipation linearly proportional to car speed (again with damping coefficient b) and assuming the motor produces a constant force f, the equations of motion are

$$m\ddot{x}(t) = -b\dot{x}(t) + f. \tag{2.3}$$

If we concern ourselves only with velocity $v(t) = \frac{d}{dt}x(t)$, the *car velocity system* as

$$m\dot{v}(t) = -bv(t) + f. \tag{2.4}$$

This is a *first-order system*, i.e., a linear decay system with a forcing term.

In class assignment

Is there any difference with the linear growth/decay model in Chapter 1?

Numerical simulation of car velocity system with switching force

```python
# Python libraries
import numpy as np; import matplotlib.pyplot as plt
from scipy.integrate import odeint

# Differential equation model of the dynamical system
def model(v, t, b, m, f):
    dvdt = - (b/m) * v + f(t)/m
    return dvdt

# Parameters and time array
b = 4; m = 3; t = np.linspace(0, 10, 500)
# Force with a step change at time 5
def f(t):
    if t < 5:
        return 20
    else:
        return 30

# Initial conditions
v0_values = [2, 3, 4, 5, 6, 7, 8];
colors = ['#471b00','#752d00', '#a43e00', '#d35000', '#ff6100', '#ff7f1a', '#ff9b56']

# Numerically integrate and plot solutions for each initial condition
plt.figure(figsize=(10, 6))
for idx, v0 in enumerate(v0_values):
    v = odeint(model, v0, t, args=(b, m, f))
    plt.plot(t, v, label=f'v0={v0}', color=colors[idx])

# Annotate and save the plot
plt.title('Solution to $m \dot{v} = - b v + f$, $m=3$, $b=4$, $f=20$ for $t<5$ and ...
    $f=30$ for $t>5$')
plt.xlabel('Time, t'); plt.ylabel('State, v(t)')
plt.legend(); plt.grid(True); plt.xlim(0, 10)
plt.savefig("first-order-ode.pdf", bbox_inches='tight')

# Second figure: Illustrate time constant
tau = m / b
def fzero(t):
    return 0
# Initial condition and time array
v0 = 1

# Numerically integrate and plot solution for the given initial condition
plt.figure(figsize=(10, 3));
v = odeint(model, v0, t, args=(b, m, fzero))
plt.plot(t, v, label=f'v0={v0}', color='#0085ff')

# Highlighting the time constants
time_constants = [tau, 2*tau, 3*tau, 4*tau, 5*tau]
for tc in time_constants:
    plt.axvline(x=tc, color=colors[4], linestyle='--', linewidth=0.75)
plt.xticks(time_constants, [r'${} \tau$'.format(int(tc/tau)) for tc in time_constants])

# Drawing the 1% horizontal dashed line
one_percent_value = 0.01; exp_value = np.exp(-1);
plt.axhline(y=one_percent_value, color=colors[3], linestyle='--', linewidth=1.5)
plt.annotate(f'$1\\% > e^{{-5}} \\approx 0.67\\%$', (6*tau, one_percent_value), ...
    textcoords="offset points", xytext=(0,10), ha='left', color=colors[3])
plt.axhline(y=exp_value, color=colors[3], linestyle='--', linewidth=1.5)
plt.annotate(f'$e^{{-1}}\\approx 36.8\\%$', (6*tau, exp_value), textcoords="offset ...
    points", xytext=(0,10), ha='left', color=colors[3])


# Annotate and save the plot
plt.title(r'Solution to unforced $\tau \dot{x} = -x$')
plt.xlabel('Time, t'); plt.ylabel('state, x(t)');
plt.grid(True); plt.xlim(0, 6); plt.ylim(-0, 1)
plt.savefig("first-order-ode-timeconstant.pdf", bbox_inches='tight')
```

Listing 2.1: Python script generating Figure 2.1. Available at first-order-ode.py

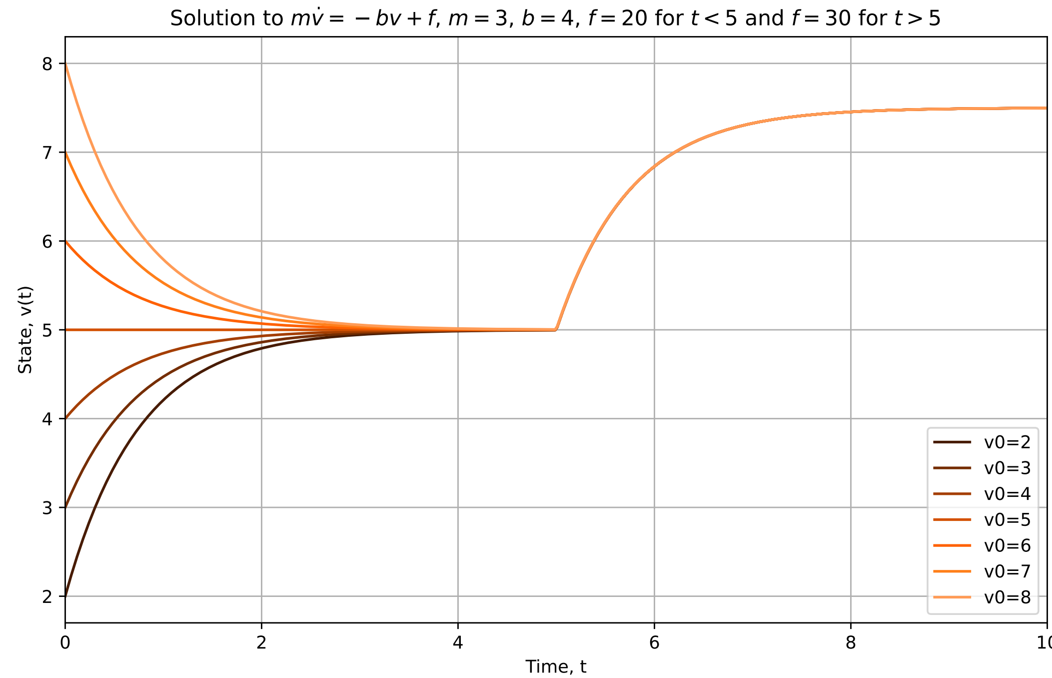


Figure 2.1: Solutions to the first-order equation (2.4): $m\dot{v}(t) = -bv(t) + f$.
When the force is $f = 20$, the final value is $v_{\text{final}} = f/b = 20/4 = 5$.
When the force changes for $f = 30$, the final value is $v_{\text{final}} = f/b = 30/4 = 7.5$.
Loosely speaking, the speed at which the solution starting at $x(0) = 8$ drops to the value 5 is the same with which the solution starting at $x(0) = 2$ rises to the value 5.

Mathematical analysis: Change of coordinates into error system (i.e., an unforced first-order system)

Consider the affine[2] first-order system

$$m\dot{v}(t) = -bv(t) + f \qquad \Longleftrightarrow \qquad \ddot{x}(t) = -\frac{b}{m}\dot{x}(t) + \frac{f}{m}, \tag{2.5}$$

with constant coefficients m, b, and f.

We saw in the numerical simulation that, for each constant force f, the solution converges to a constant *final value*. In other words, the system has a stable equilibrium point, which is easily computed to be:

$$v_{\text{final}} = f/b. \tag{2.6}$$

Next, we consider a *change of coordinates* into a set of *relative velocity* (velocity relative to the final value)

$$v_{\text{relative}}(t) = v(t) - v_{\text{final}} = v(t) - f/b. \tag{2.7}$$

The relative velocity play the role of an *error variable*, measuring the error from current position to final position. We can now perform a simple calculation:

$$\frac{d}{dt}v_{\text{relative}} = \dot{v} - 0 = \frac{1}{m}(-bv + f) = -\frac{b}{m}(v_{\text{relative}} + v_{\text{final}}) + \frac{f}{m} = -\frac{b}{m}v_{\text{relative}}.$$

In summary, the *error system* is an *unforced first-order system*

$$\dot{v}_{\text{relative}}(t) = -\frac{b}{m}v_{\text{relative}}(t). \tag{2.8}$$

[2] A function is affine if it is the sum of a linear function and a constant.

Mathematical analysis: Time constant of unforced first-order systems

We rewrite the first order system with a useful new parameter:

$$\dot{x} = -rx \qquad \Longleftrightarrow \qquad \tau\dot{x} = -x \qquad \Longrightarrow \qquad x(t) = \mathrm{e}^{-t/\tau}\, x(0), \tag{2.9}$$

where we define the *time constant*

$$\tau = 1/r. \tag{2.10}$$

Note:

(i) for an unforced system from a nonzero initial condition,
τ is the time required for the system's response $x(t)$ to decay to $\mathrm{e}^{-1} \approx 36.8\%$ of the initial value $x(0)$, and

(ii) at time $t = 5\tau$, the distance to the final value reaches a value $\mathrm{e}^{-5} \approx 0.67\% < 1\%$ of the initial value $x(0)$. The rule of thumb is to state that, *after five time constants, the error has practically vanished*.

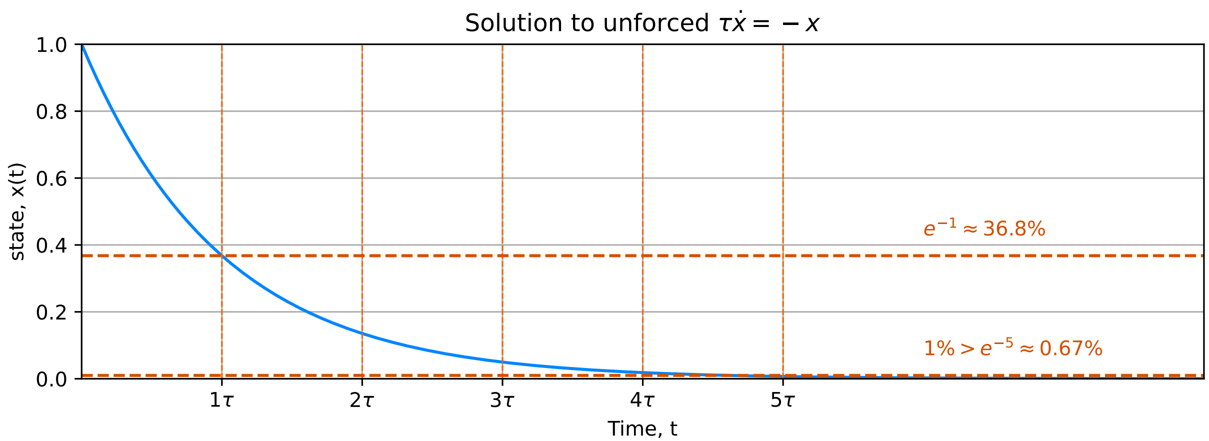


Figure 2.2: Illustrating the time constant of an unforced first-order system $\tau\dot{x} = -x$, $x(0) = 1$.
Note $x(\tau) = \mathrm{e}^{-1}\, x(0)$ and $x(5\tau) = \mathrm{e}^{-5}\, x(0)$. Hence, the state is equal to $\mathrm{e}^{-1} \approx 36.8\%$ at $t = \tau$ and is below 1% at and after $t = 5\tau$.
For the forced system $\tau\dot{x} = -x + 1$ and $x(0) = 0$, the constant τ is the time required for $x(t)$ to reach approximately $1 - \mathrm{e}^{-1} \approx 63.2\%$ of its final value.

2.1.2 Second-order systems: harmonic oscillators

A *spring* is a mechanical element that stores energy. For now, we focus on translational springs. Typically, a spring has a natural *rest length* with the property that, at rest length, the spring produces no force. When the spring is stretched or compressed, it produces a restoring force which is proportional to the displacement.

Assume that the first end of the spring is fixed and the second end is at position 0. When the second end is attached to a body at position x, then the spring force on the body is

$$F_{\text{spring}} = -kx \tag{2.11}$$

where $k > 0$ is a *stiffness* or *spring constant*.

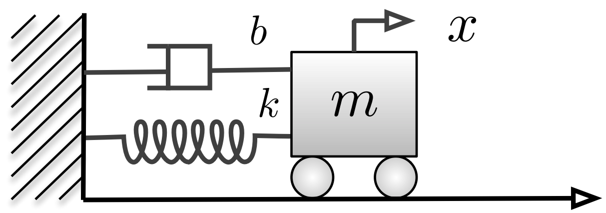


Figure 2.3: A mass-spring-damper system, described by equation (2.12).

When a body (translating along a single axis) is connected to both a spring and damper, the resulting dynamics are called the *damped harmonic oscillator*:

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0. \tag{2.12}$$

It is possible and useful to rewrite this second-order differential equation as a first order equation in two variables. As before, we define the velocity variable $v(t) = \dot{x}(t)$ and write

$$\dot{x}(t) = v(t) \tag{2.13a}$$

$$\dot{v}(t) = -(b/m)v(t) - (k/m)x(t) \tag{2.13b}$$

Since we need two variables, this system is said to have *dimension* 2.

Numerical analysis of the damped harmonic oscillator: *Underdamped oscillator*

```
# Python libraries
import numpy as np; import matplotlib.pyplot as plt
from scipy.integrate import odeint

# Constants
m = 1.0   # Mass
b = 0.5   # Damper
k = 2.0   # Stiffness

# Differential equations for damped harmonic oscillator
def damped_oscillator(y, t, b, k, m):
    x, v = y
    dxdt = v
    dvdt = -(b/m) * v - (k/m) * x
    return [dxdt, dvdt]

# Time vector
t = np.linspace(0, 14, 1000)

# Six different initial conditions [x0, v0]
initial_conditions = [ [2, 0],  [1, 0],  [.5, 0],  [.1, 0], [-1, 0]]
colors = ['#471b00','#752d00', '#a43e00', '#d35000', '#ff6100', '#ff7f1a']

# Plotting solutions as a function of time
plt.figure(figsize=(10, 4)); plt.xlim(0, 14);
for idx, init_cond in enumerate(initial_conditions):
    sol = odeint(damped_oscillator, init_cond, t, args=(b, k, m))
    plt.plot(t, sol[:, 0], label=f'x0={init_cond[0]}, ...
        v0={init_cond[1]}', color=colors[idx])

plt.title('Damped harmonic oscillator: solutions (from zero initial ...
    velocity) and phase portrait');
plt.ylabel('Displacement (x)'); plt.legend(); plt.grid(True); ...
    plt.xlabel('Time')
plt.savefig('damped-harmonic.pdf', bbox_inches='tight')

# Phase portrait
X, Y = np.meshgrid(np.linspace(-2, 2, 20), np.linspace(-3, 3, 20))
U = Y; V = -(b/m) * Y - (k/m) * X; magnitude = np.sqrt(U**2 + V**2)

plt.figure(figsize=(10,8)); plt.grid(True)
plt.xlim(-2, 2); plt.ylim(-3, 3); plt.scatter(0, 0, color='black', ...
    s=50, zorder=5)
plt.streamplot(X, Y, U, V, density=0.75, color='#0085ff', ...
    arrowsize=1.5, linewidth=magnitude)
plt.xlabel('Displacement x'); plt.ylabel('Velocity v')
plt.savefig('damped-harmonic-phase.pdf', bbox_inches='tight')
```

Listing 2.2: Python script generating Figure 2.4. Available at damped-harmonic.py

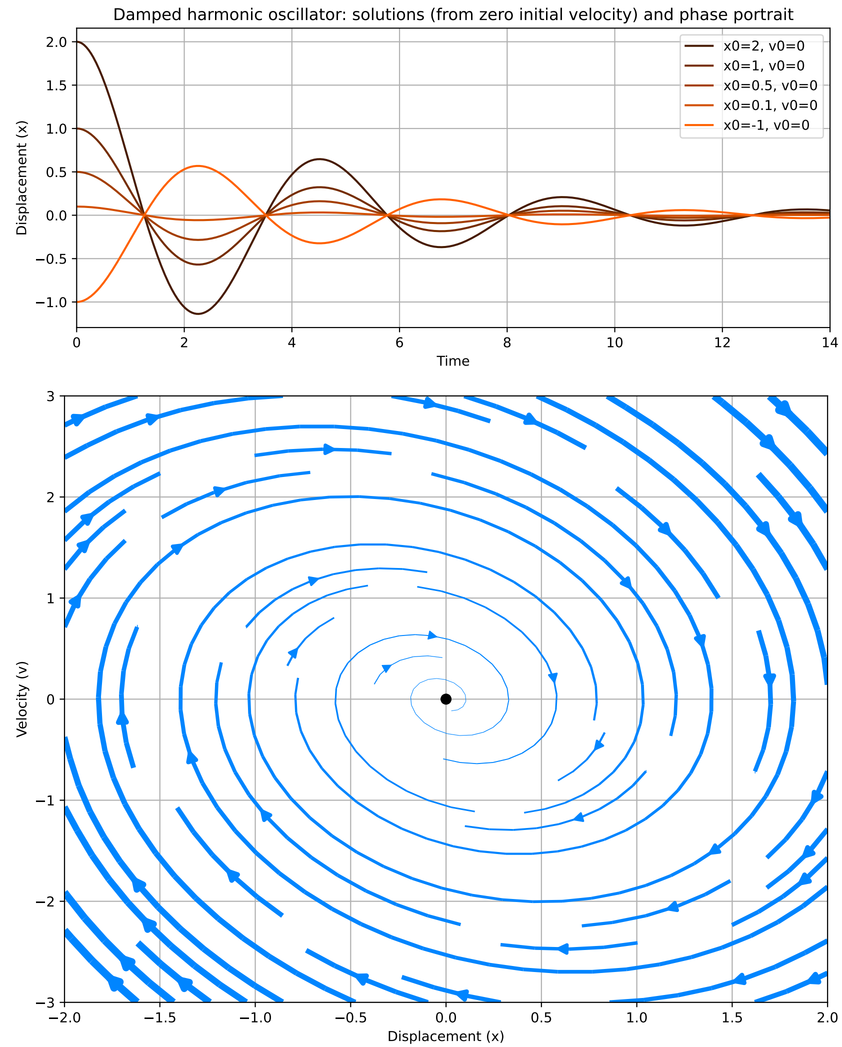


Figure 2.4: Solutions and phase portrait for the damped harmonic oscillator (2.13), with a low value of the damping coefficient. When there are oscillations, the system is said to be *underdamped*.

Numerical analysis of the damped harmonic oscillator: *Overdamped oscillator*

```python
# Python libraries
import numpy as np; import matplotlib.pyplot as plt
from scipy.integrate import odeint

# Constants
m = 1.0   # Mass
b = 3.0   # Damper
k = 2.0   # Stiffness

# Differential equations for overdamped harmonic oscillator
def overdamped_oscillator(y, t, b, k, m):
    x, v = y
    dxdt = v
    dvdt = -(b/m) * v - (k/m) * x
    return [dxdt, dvdt]

# Time vector
t = np.linspace(0, 14, 1000)

# Six different initial conditions [x0, v0]
initial_conditions = [ [2, 0],  [1, 0],  [.5, 0],  [.1, 0], [-1, 0] ]
colors = ['#471b00','#752d00', '#a43e00', '#d35000', '#ff6100', '#ff7f1a']

# Plotting solutions as a function of time
plt.figure(figsize=(10, 4)); plt.xlim(0, 14);
for idx, init_cond in enumerate(initial_conditions):
    sol = odeint(overdamped_oscillator, init_cond, t, args=(b, k, m))
    plt.plot(t, sol[:, 0], label=f'x0={init_cond[0]}, ...
        v0={init_cond[1]}', color=colors[idx])

plt.title('Overdamped harmonic oscillator: solutions (from zero initial ...
    velocity) and phase portrait');
plt.ylabel('Displacement (x)'); plt.legend(); plt.grid(True); ...
    plt.xlabel('Time')
plt.savefig('overdamped-harmonic.pdf', bbox_inches='tight')

# Phase portrait
X, Y = np.meshgrid(np.linspace(-2, 2, 20), np.linspace(-3, 3, 20))
U = Y; V = -(b/m) * Y - (k/m) * X; magnitude = np.sqrt(U**2 + V**2)

plt.figure(figsize=(10,8)); plt.grid(True)
plt.xlim(-2, 2); plt.ylim(-3, 3); plt.scatter(0, 0, color='black', ...
    s=50, zorder=5)
plt.streamplot(X, Y, U, V, density=0.75, color='#0085ff', ...
    arrowsize=1.5, linewidth=magnitude)
plt.xlabel('Displacement (x)'); plt.ylabel('Velocity (v)')
plt.savefig('overdamped-harmonic-phase.pdf', bbox_inches='tight')
```

Listing 2.3: Python script generating Figure 2.5. Available at `overdamped-harmonic.py`

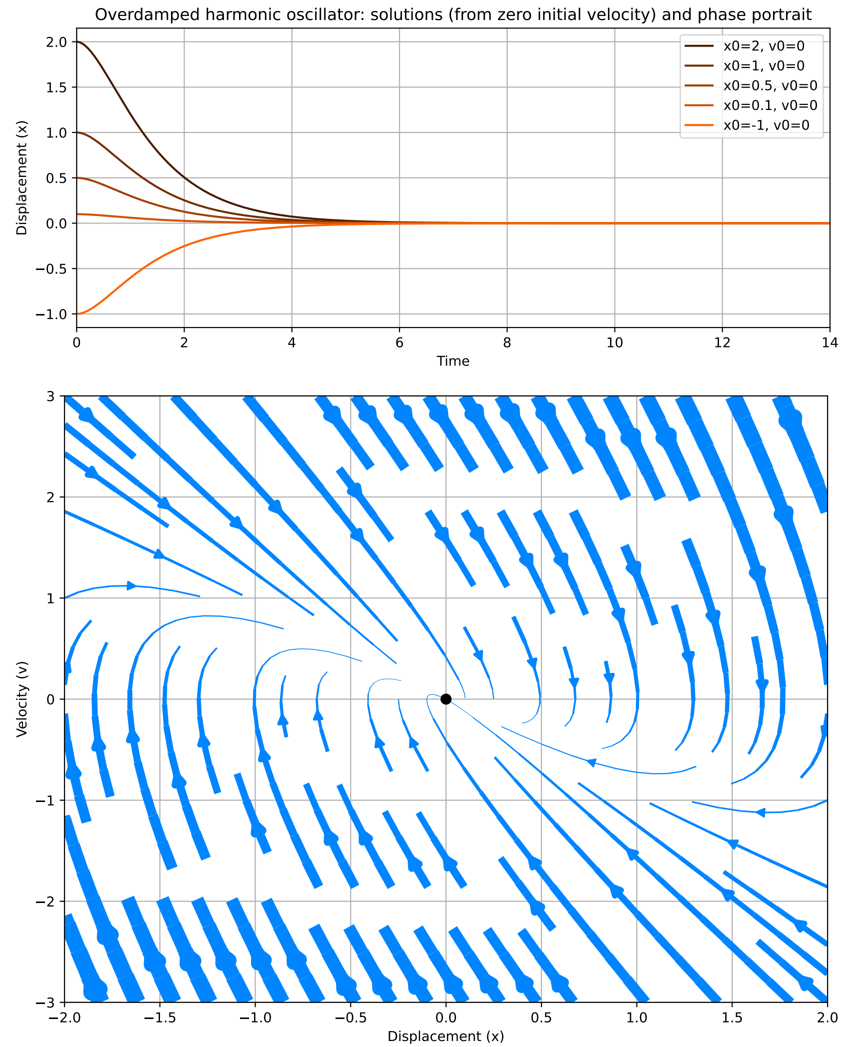


Figure 2.5: Solutions and phase portrait for the damped harmonic oscillator (2.13) with a high value of the damping coefficient. When there are no oscillations, the system is said to be *overdamped*.

It is also possible to set the damping coefficient to zero ($b = 0$) and consider the *undamped harmonic oscillator*:

$$m\ddot{x}(t) + kx(t) = 0. \tag{2.14}$$

Writing this second order differential equation as first-order in two variables, we get

$$\dot{x}(t) = v(t) \tag{2.15a}$$

$$\dot{v}(t) = -(k/m)x(t) \tag{2.15b}$$

Numerical analysis of the damped harmonic oscillator: *Undamped oscillator*

```
# Python libraries
import numpy as np; import matplotlib.pyplot as plt
from scipy.integrate import odeint

# Constants
m = 1.0   # Mass
k = 2.0   # Stiffness

# Differential equations for undamped harmonic oscillator
def undamped_oscillator(y, t, k, m):
    x, v = y
    dxdt = v
    dvdt = - (k/m) * x
    return [dxdt, dvdt]

# Time vector
t = np.linspace(0, 14, 1000)

# Six different initial conditions [x0, v0]
initial_conditions = [ [2, 0],  [1, 0],  [.5, 0],  [.1, 0], [-1, 0]]
colors = ['#471b00','#752d00', '#a43e00', '#d35000', '#ff6100', '#ff7f1a']

# Plotting solutions as a function of time
plt.figure(figsize=(10, 4)); plt.xlim(0, 14);
for idx, init_cond in enumerate(initial_conditions):
    sol = odeint(undamped_oscillator, init_cond, t, args=(k, m))
    plt.plot(t, sol[:, 0], label=f'x0={init_cond[0]}, ...
        v0={init_cond[1]}', color=colors[idx])

plt.title('Undamped harmonic oscillator: solutions (from zero initial ...
    velocity) and phase portrait');
plt.ylabel('Displacement (x)'); plt.legend(); plt.grid(True); ...
    plt.xlabel('Time')
plt.savefig('undamped-harmonic.pdf', bbox_inches='tight')

# Phase portrait
X, Y = np.meshgrid(np.linspace(-2, 2, 20), np.linspace(-3, 3, 20))
U = Y; V = - (k/m) * X; magnitude = np.sqrt(U**2 + V**2)

plt.figure(figsize=(10,8)); plt.grid(True)
plt.xlim(-2, 2); plt.ylim(-3, 3); plt.scatter(0, 0, color='black', ...
    s=50, zorder=5)
plt.streamplot(X, Y, U, V, density=0.75, color='#0085ff', ...
    arrowsize=1.5, linewidth=magnitude)
plt.xlabel('Displacement (x)'); plt.ylabel('Velocity (v)')
plt.savefig('undamped-harmonic-phase.pdf', bbox_inches='tight')
```

Listing 2.4: Python script generating Figure 2.6. Available at `undamped-harmonic.py`

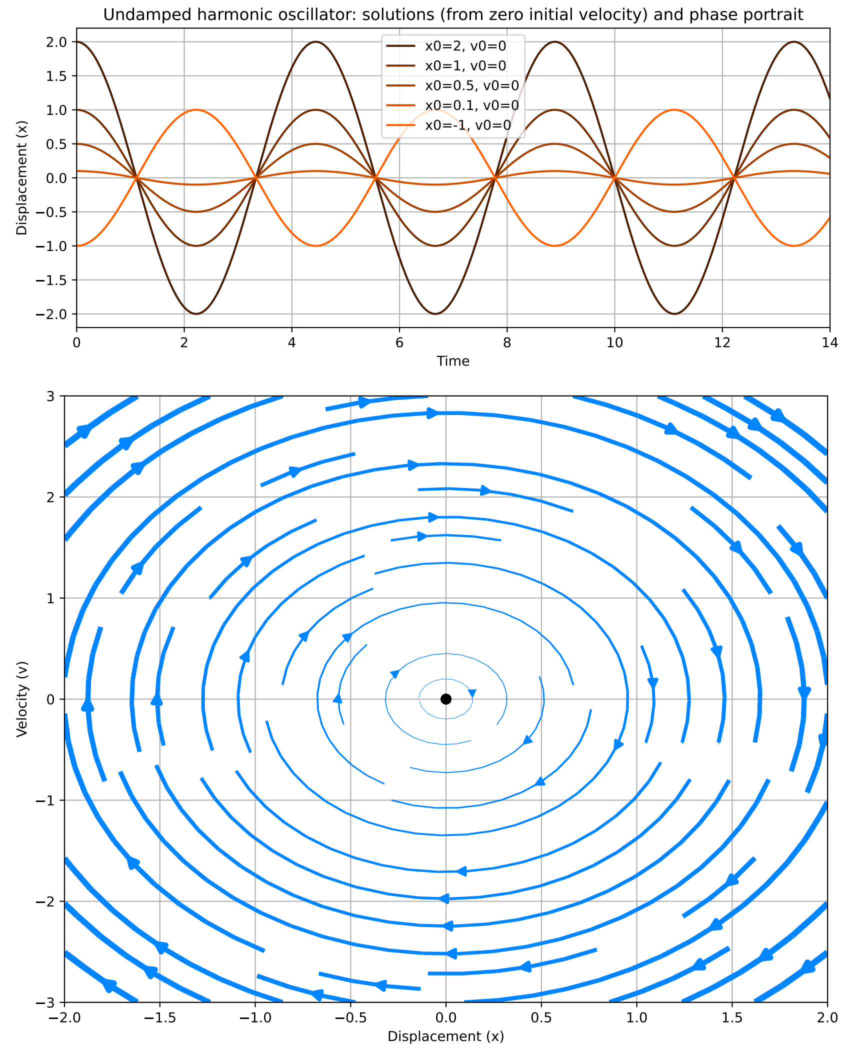
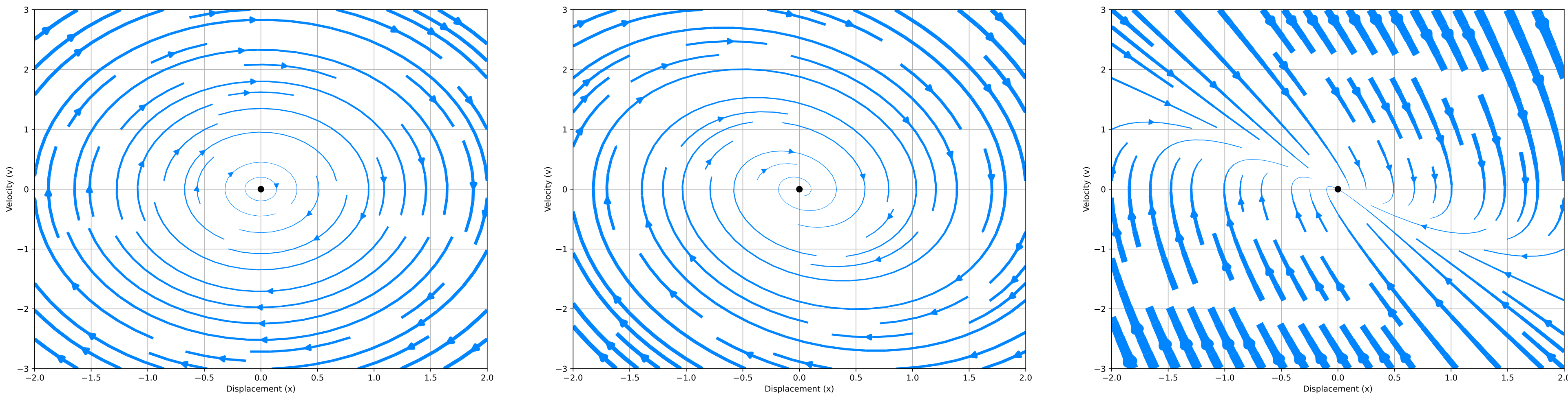


Figure 2.6: Solutions and phase portrait for the undamped harmonic oscillator (2.15).

Transition from zero to high damping values



(a) Undamped system: $b = 0$

(b) Underdamped system: small $b > 0$

(c) Underdamped system: large $b > 0$

2.1.3 Mathematical analysis: Harmonic solutions

Each solution to the undamped harmonic oscillator

$$m\ddot{x} + kx = 0 \tag{2.16}$$

is of the form

$$x(t) = a\sin(\omega_{\mathsf{n}} t) + b\cos(\omega_{\mathsf{n}} t) \tag{2.17}$$

where

- $\omega_{\mathsf{n}} = \sqrt{\frac{k}{m}}$ is called the *natural frequency*, measured in radians per second,
- the parameters a and b in equation (2.17) are uniquely determined by the initial condition (and vice versa) $(x(0), \dot{x}(0))$,
- it is possible to define the *period of oscillation*

$$T = \frac{2\pi}{\omega_{\mathsf{n}}}. \tag{2.18}$$

  Natural frequency and period of oscillation are intrinsic to the system and independent of initial conditions,
- the sinusoidal function $a\sin(\omega t) + b\cos(\omega t)$ is called a *harmonic motion*. Each harmonic motion is determined by a frequency, magnitude, and phase. For[3] $\phi = \operatorname{arctan}_2(b, a)$, one can show that

$$a\sin(\omega t) + b\cos(\omega t) = \sqrt{a^2 + b^2}\sin(\omega t + \phi) \tag{2.19}$$

We will study harmonic, overdamped and underdamped oscillators carefully in a later chapter.

[3] The function $\operatorname{arctan}_2(y, x)$ computes the angle of the point (x, y) in the Cartesian coordinate system, measured counterclockwise from the positive x-axis. It returns the angle in the range $(-\pi, \pi]$, taking into account the signs of both x and y to correctly determine the quadrant in which the angle lies. When both x and y are positive, $\operatorname{arctan}_2(y, x) = \arctan(y/x)$.

2.2 Two degrees of freedom systems: The suspension

In automotive engineering, a suspension system is a set of components including springs, shock absorbers, and linkages that connect a vehicle to its wheels. Its primary purpose is to absorb and dampen shocks from the irregularities in the road surface, while also maintaining contact between the tires and the road surface. A well-designed suspension system enhances ride comfort, vehicle handling, and the overall safety of the vehicle.

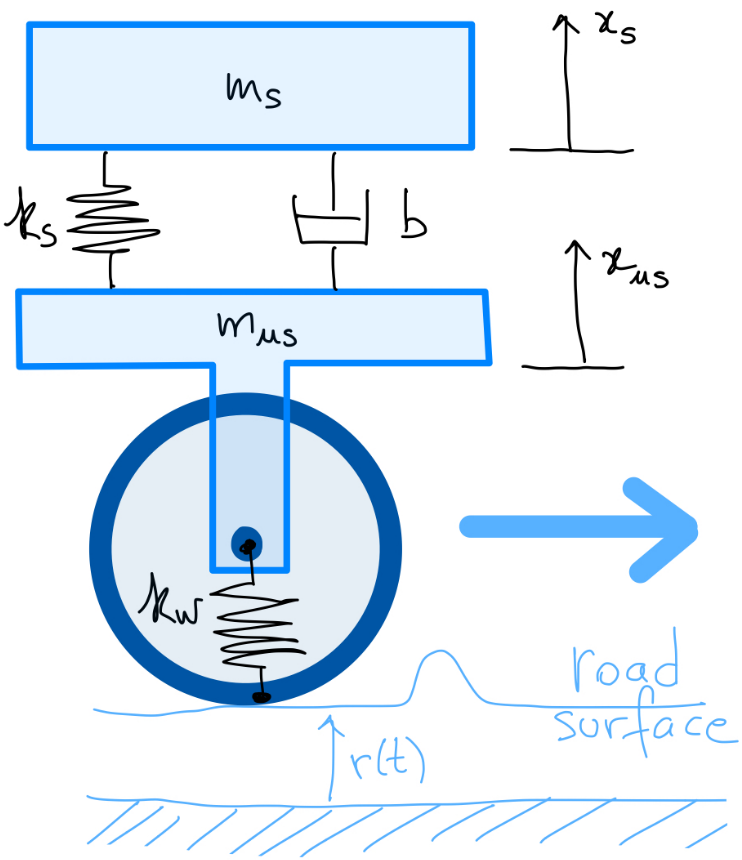


Figure 2.7: The suspension in an automobile.

The body of the vehicle, passengers and cargo is called the *sprung mass* and denoted by m_s; the vertical position of the sprung mass is denoted by x_s.

The wheels, axles, and other parts that are directly connected to the road surface are called the *unsprung mass*, denoted by m_{us}; the vertical position is x_{us}.

The interaction wheel/road is usually described as a spring with stiffness k_w.

The suspension includes a shock absorber, that is a damper with coefficient b, and a spring with stiffness k_s.

We assume the automobile is moving forward at constant speed over a possibly-uneven terrain.

From a full to a quarter vehicle suspension system

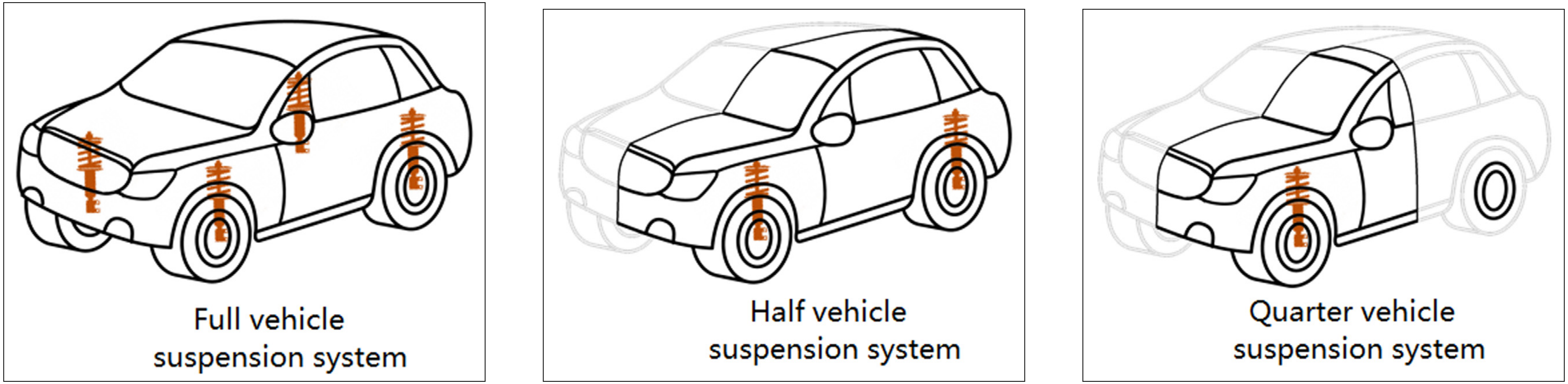


Figure 2.8: From a full to a quarter vehicle suspension system. Image sources from (Zhang et al., 2020) without permission.

As in Figure 2.7, let x_{s} and x_{us} denote the vertical displacements of two masses from their equilibrium position. (The equilibrium position accounts for gravity and a counterbalancing compression of the two springs; therefore we do not consider gravity). It is important to understand that the sprung mass moves relative to the unsprung mass.

From Newton's law, we know that the equations will be of the form

$$\begin{aligned} m_{\mathrm{s}}\ddot{x}_{\mathrm{s}} &= \text{resultant force on sprung mass, and} \\ m_{\mathrm{us}}\ddot{x}_{\mathrm{us}} &= \text{resultant force on unsprung mass.} \end{aligned}$$

Paying special attention to the signs of the terms in the spring and damper terms (see remark below, check twice, and read Exercise E2.1), one can compute:

$$m_{\mathrm{s}}\ddot{x}_{\mathrm{s}} = k_{\mathrm{s}}(x_{\mathrm{us}} - x_{\mathrm{s}}) + b(\dot{x}_{\mathrm{us}} - \dot{x}_{\mathrm{s}}), \tag{2.20a}$$

$$m_{\mathrm{us}}\ddot{x}_{\mathrm{us}} = -k_{\mathrm{s}}(x_{\mathrm{us}} - x_{\mathrm{s}}) - b(\dot{x}_{\mathrm{us}} - \dot{x}_{\mathrm{s}}) - k_{\mathrm{w}}\big(x_{\mathrm{us}} - r(t)\big), \tag{2.20b}$$

where $r(t)$ is the height of the road surface as function of time.

In summary, after rearranging these equations, the *suspension dynamics* are:

$$m_{\mathrm{s}}\ddot{x}_{\mathrm{s}} + b(\dot{x}_{\mathrm{s}} - \dot{x}_{\mathrm{us}}) + k_{\mathrm{s}}(x_{\mathrm{s}} - x_{\mathrm{us}}) = 0, \tag{2.21a}$$

$$m_{\mathrm{us}}\ddot{x}_{\mathrm{us}} + b(\dot{x}_{\mathrm{us}} - \dot{x}_{\mathrm{s}}) + k_{\mathrm{s}}(x_{\mathrm{us}} - x_{\mathrm{s}}) + k_{\mathrm{w}}x_{\mathrm{us}} = k_{\mathrm{w}}r(t), \tag{2.21b}$$

Note: Just like the force in the simulation of the car velocity system, the road surface $r(t)$ is now an *input* into the the dynamical system. An input signal is an exogenous, i.e., an external input affects the system's behavior but is not influenced by the system's state.

Remark 2.1 (Absolute versus relative effects). *Consider a body with position x_1 and velocity v_1. Note the difference between a "absolute force" like*

$$-kx_1 \quad \text{or} \quad -bv_1 \qquad \text{(absolute spring and absolute damper forces)}$$

and a "relative force" (due to the interconnection with a second body with position x_2 and velocity v_2):

$$-k(x_1 - x_2) \quad \text{or} \quad -b(v_1 - v_2) \qquad \text{(relative spring and relative damper forces)}$$

But, for clarity, truly all springs and dampers generate always only relative forces, i.e., forces proportional to relative position and relative velocity. The reason why (absolute spring and absolute damper forces) *appear is because the second body is assumed to be at zero position and zero velocity ($x_2 = v_2 = 0$).*

Remark 2.2 (How to get the correct signs). *Recall the damped harmonic oscillator in equation* (2.12)*: $m\ddot{x} + b\dot{x} + kx = 0$. Similarly, to ensure that the signs are correct in the first equation* (2.21a)*, note that the acceleration, velocity and position terms in x_s and its derivative need to be multiplied by positive coefficients. The same is true in equation* (2.21b) *for the coefficients of x_us and its derivatives.*•

Remark 2.3. *A choice of realistic automobile parameters taken from (Franklin et al., 2015, Section 2.2) is:*

sprung mass	m_s	*1500 kg*
unsprung mass	m_s	*80 kg*
wheel stiffness	k_w	*1,000,000 N/m*
suspension stiffness	k_s	*130,000 N/m*
suspension damping	b	*9800 N sec / m*

Remark 2.4. *It is usually preferable to have low unsprung weight (and high sprung to unsprung weight ratio) in order to allow the suspension to respond more effectively to road imperfections, improving ride quality and handling.*

Numerical analysis of the syspension system

```python
import numpy as np; from scipy.integrate import odeint;
import matplotlib.pyplot as plt

# Define the system of ODEs with state = [xs, xs_dot, xu, xu_dot]
def system_of_eqns(state, t, mu, ms, ks, b, kw, road):
    xs, xs_dot, xu, xu_dot = state
    xs_ddot = (-ks*(xs-xu) - b*(xs_dot-xu_dot)) / ms
    xu_ddot = (ks*(xs-xu) + b*(xs_dot-xu_dot) - kw*xu + kw*road(t)) / mu
    return [xs_dot, xs_ddot, xu_dot, xu_ddot]

# Parameters for a "quarter automobile" and time array
ms = 375       # Sprung mass (for a quarter of a car)
mu = 20        # Unsprung mass
kw = 1000000   # Wheel stiffness
ks = 130000    # Suspension stiffness
b  = 9800      # Suspension damping coefficient

# Initial conditions: [xs, xs_dot, xu, xu_dot]. positions in meters.
t = np.linspace(0, 1.4, 300)
initial_conditions = [-0.1, 0.0, 0.00, 0.0]
sol = odeint(system_of_eqns, initial_conditions, t, args=(mu, ms, ks, b, ...
    kw, lambda t: 0))

# Plotting the unforced solution
plt.figure(figsize=(10, 5)); plt.plot(t, sol[:, 0], label='x sprung')
plt.plot(t, sol[:, 2], label='x unsprung'); plt.grid(True)
plt.xlabel('Time'); plt.ylabel('Position (meters)'); plt.xlim(0, 1.4);
plt.title('Unforced suspension system'); plt.legend();
plt.savefig("suspension-unforced.pdf", bbox_inches='tight')

# Road surface: zero for first .5 seconds, then a sinusoidal bump
bumpheight = .116   # typical bump height = 4 inches = .116 meters
duration = .46 / 4.4  # typical bump width = 18 inches = 0.46 meters. 10 ...
    miles/hour = 4.4 meter/sec
def bump_road(t):
    if 0.5 <= t < 0.5 + duration:
        return bumpheight * np.sin((t - .5) * np.pi / duration)
    else:
        return 0

# Solving for forced case from equilibrium initial condition
initial_conditions_forced = [-0.1, 0.0, 0.0, 0.0]
sol_forced = odeint(system_of_eqns, initial_conditions_forced, t, ...
    args=(mu, ms, ks, b, kw, bump_road))
road_data = np.array([bump_road(time) for time in t])

# Plotting the forced solution
plt.figure(figsize=(10, 5)); plt.plot(t, sol_forced[:, 0], label='s ...
    (Sprung mass)')
plt.plot(t, sol_forced[:, 2], label='u (Unsprung mass)'); plt.grid(True)
plt.plot(t, road_data, label='Road surface', linestyle='--'); plt.xlim(0, 1.4);
plt.xlabel('Time'); plt.ylabel('Position (meters)'); plt.title('Forced ...
    suspension system')
plt.legend(); plt.savefig("suspension-forced.pdf", bbox_inches='tight')
```

Listing 2.5: Python script generating Figure 2.9. Available at suspension.py

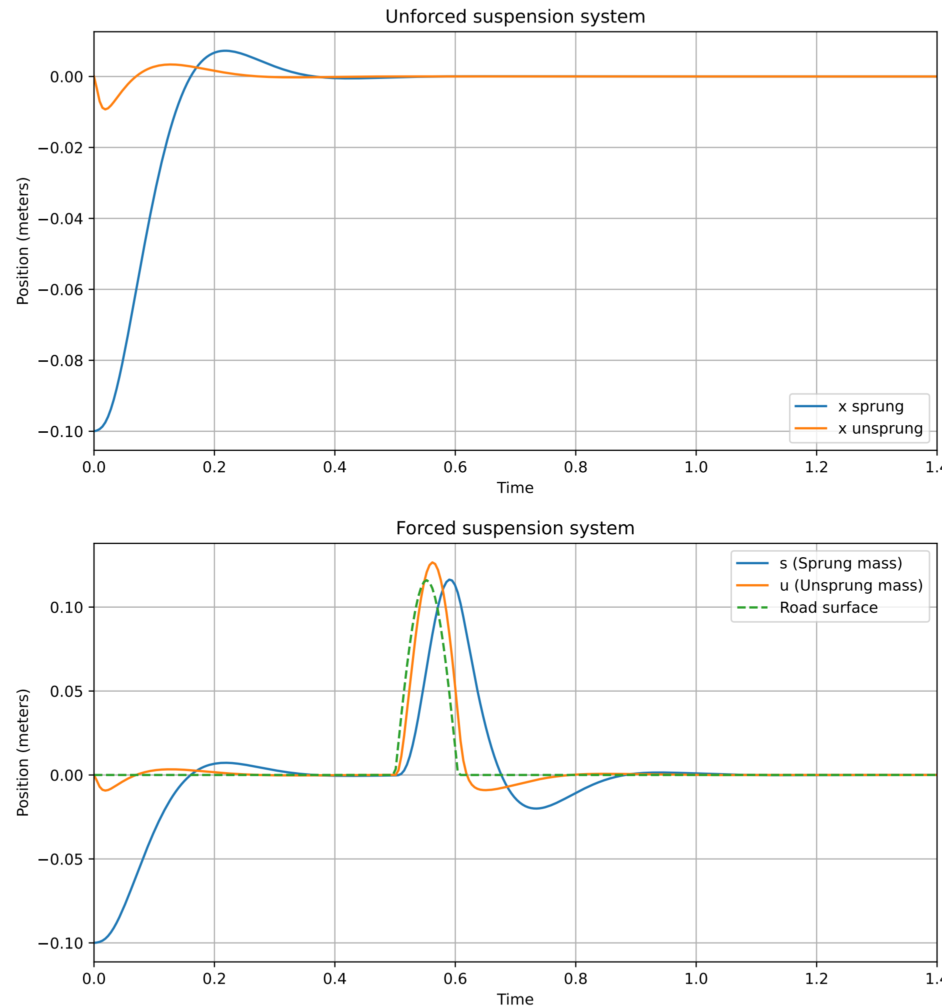
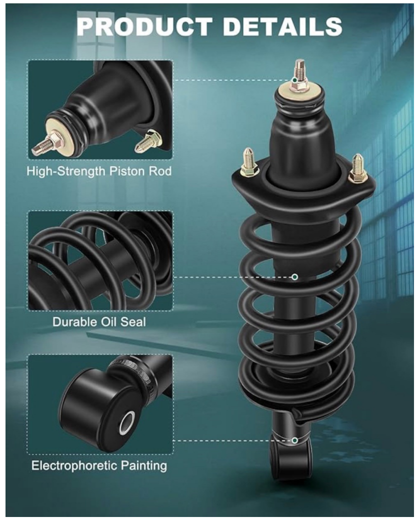


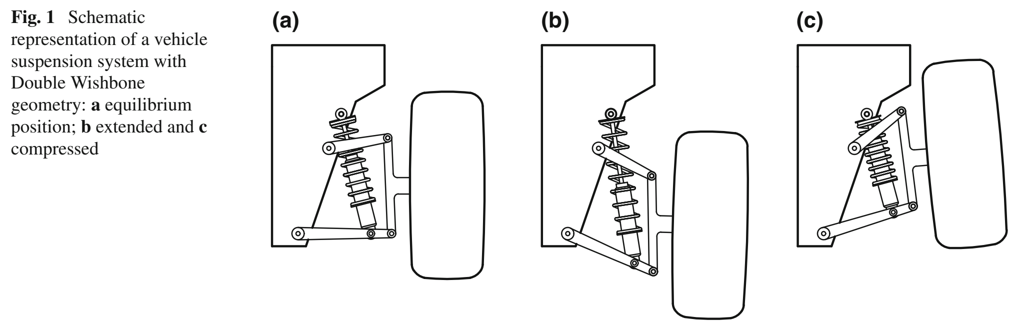
Figure 2.9: Solutions of the suspension system (2.21): unforced solution (road height = 0 for all time) and forced solution due to a speed bump at time 0.5.

Comments on vehicle suspension systems

(a) A *MacPherson suspension system* for a front wheel, featuring:
(i) a *strut* (with coil spring and damper), where the upper part of the strut is connected to the chassis via a pivoting bearing mount that allows rotational movement for steering, and the lower part of the strut connects to the wheel hub via a ball joint; and
(ii) a *lower control arm*, also known as a *wishbone arm*, (i.e., an A or V-shaped arm connected to revolute joints on the chassis and to a ball joint on the wheel), which provides lateral support and manages the vertical movement of the wheel.



(b) A strut combining a shock absorber and a coil spring, sourced from `https://amazon.com` without permission.



(c) From (Fernandes et al., 2019) without permission. Illustration of the automobile *double wishbone suspension system* based upon a four-bar linkage.

2.3 Rotational motion

Newton's law applies also to rotational mechanical systems such as pendula, pulleys, and any mechanical system with an axis of rotation. The law is simply

$$\tau = I\ddot{\theta} \tag{2.22}$$

where

- τ is the resultant torque (resultant = algebraic sum of) applied to the body, measured in $\text{N} \cdot \text{m}$,
- I is the moment of inertia of the body, measured in Kg m^2, and
- $\ddot{\theta}$ is the angular acceleration of the body, i.e., the second time derivative of the angular position $\theta(t)$, measured in rad/sec^2.

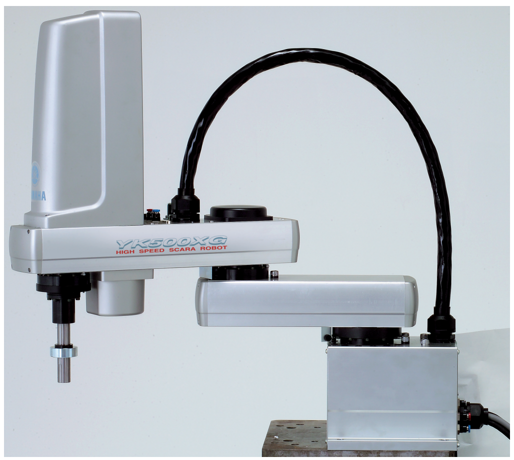


Figure 2.10: The Yamaha© YK500XG is a high-speed SCARA robot with two revolute joints and a vertical prismatic joint. Image courtesy of Yamaha Motor Co., Ltd, http://global.yamaha-motor.com/business/robot.

Figure 2.11: Illustration of NASA's Transiting Exoplanet Survey Satellite: TESS. Credit: NASA's Goddard Space Flight Center.

In class assignment

How many distinct bits of information are required to unambiguously specify the meaning of an angle θ?
(e.g., one bit of information is the direction of angle measurement: clockwise vs counterclockwise)

How to fully describe an angle

In order to unambiguously specify the meaning of an angle θ, one needs to specify:

(i) reference angle: where is the zero angle,

(ii) direction: counterclockwise (or clockwise, but in this text, all angles are measured counterclockwise),

(iii) unit: radians (not degrees), and

(iv) range: $(-\pi, \pi]$.

Rotary dampers and torsion springs

Just like we saw for translational motion, there exist dampers and springs for rotational motion, see Figure 2.12. Therefore, even for rotational mechanics it is possible and common to encounter damped harmonic oscillators:

$$I\ddot{\theta}(t) + b\dot{\theta}(t) + k\theta(t) = 0 \tag{2.23}$$

As for the translational system depicted in Figure 2.3, equation (2.23) is based on the assumption that the rotatory damper and torsional springs are connected at one end to a fixed body and at the other end to the rotating angle θ.

(a) A small rotary damper. Exploiting fluid viscosity, rotary dampers slow down the motion of rotating parts. For example, in automotive applications, rotary dampers are used for glove compartments, cup holders, and grad handles.

(b) A small torsional spring. Torsional springs store and release energy. For example, in automotive applications, torsional springs help balance the weight of the trunk or tailgate, making it easier to open and close.

Figure 2.12: Rotary dampers and torsional springs play the same role for rotational motion as linear dampers and springs.

2.3.1 The pendulum

As illustrated in Figure 2.13, consider a pendulum of length ℓ with mass m concentrated at its end, subject to gravity with constant g and to linear friction (due to air or due to the mechanical rotation at the pivot point) described by a damping coefficient b. We note:

- the moment of inertia is $m\ell^2$, and
- the gravity force tangent to the circular motion of the pendulum is $mg\sin(\theta)$, hence the torque on the pendulum is $m\ell g\sin(\theta)$.

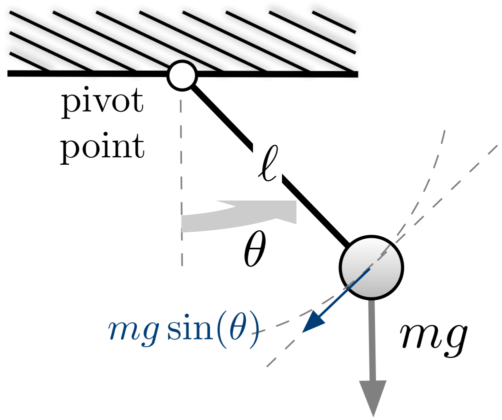


Figure 2.13: A pendulum subject to gravity, connected to a pivot point.
The variable is the angle θ, measured counterclockwise from the zero value when the pendulum is in its vertical rest state.
The moment of inertia of the pendulum about the pivot point is $I = m\ell^2$.
The pendulum is subject to the gravity force of magnitude mg, which translates into a restoring torque of magnitude $m\ell g\sin(\theta)$.

Equations of motion

To compute the equations of motion of the pendulum we adopt Newton's law (2.22) for rotational motion (see also (2.23)).

In summary, the *pendulum dynamics* are

$$m\ell^2\ddot{\theta} + b\dot{\theta} + m\ell g\sin(\theta) = 0. \tag{2.24}$$

If there is no friction, the equations of motion simplify to:

$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0. \tag{2.25}$$

We can write the equation in first order form

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\frac{b}{m\ell^2}\omega - \frac{g}{\ell}\sin(\theta) \end{aligned} \tag{2.26}$$

or in vector form

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{b}{m\ell^2}\omega - \frac{g}{\ell}\sin(\theta) \end{bmatrix} \tag{2.27}$$

Equilibrium points: pendulum down and up

We look for equilibria by setting both right-hand sides to zero and obtain

$$\omega = 0 \quad \text{and} \quad \sin\theta = 0 \quad \Longleftrightarrow \quad \theta = \pm n\pi \tag{2.28}$$

for arbitrary integers $n = 0, +1, -1, +2, -2, \dots$. In what follows, we restrict our attention to the range $-\pi < \theta \leq \pi$ so that we have only two equilibria:

- the equilibrium point $\begin{bmatrix}\theta^*_{\text{down}} \\ \omega^*\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$, corresponding to the pendulum in its *down position*,
- the equilibrium point $\begin{bmatrix}\theta^*_{\text{up}} \\ \omega^*\end{bmatrix} = \begin{bmatrix}\pi \\ 0\end{bmatrix}$, corresponding to the pendulum in its *up position*.

We know from intuition that the “pendulum down” equilibrium is stable and the “pendulum up” equilibrium is unstable.

Numerical simulation of the pendulum without friction

```
# Python libraries
import numpy as np; import matplotlib.pyplot as plt
from scipy.integrate import odeint

# Pendulum dynamics
def pendulum(Y, t, g, ell):
    theta, omega = Y
    dtheta = omega
    domega = -g/ell * np.sin(theta)
    return [dtheta, domega]

# Parameters and time array
g   = 9.81     # gravity
ell = 1.0      # length of the pendulum
m   = 0.5      # mass (not directly used in the equations, but provided for ...
    completeness)
t = np.linspace(0, 10, 1000)

# Initial conditions: [theta0, omega0] and plot the solution
initial_conditions = [[.1*np.pi, 0], [.4*np.pi, 0], [.7*np.pi, 0], [.99*np.pi, 0]]
colors = ['#752d00', '#a43e00', '#d35000', '#ff6100']
plt.figure(figsize=(10, 4))

for idx, ic in enumerate(initial_conditions):
    Y = odeint(pendulum, ic, t, args=(g, ell))
    theta, omega = Y.T
    plt.plot(t, theta, label=f'theta0={ic[0]:.2f}, omega0={ic[1]:.2f}', ...
        color=colors[idx])

# Set y-ticks to be fractions of pi
plt.yticks( [-np.pi, -np.pi/2, 0, np.pi/2, np.pi],
            ['$-\pi$', '$-\pi/2$', '0', '$\pi/2$', '$\pi$'] )
plt.title('Undamped pendulum dynamics (theta(t) vs time) and phase portrait')
plt.xlabel('Time, t'); plt.ylabel('Theta(t)'); plt.xlim(0, 10);
plt.grid(True); plt.savefig("pendulum.pdf", bbox_inches='tight')

# Phase portrait
theta_range, omega_range = np.meshgrid(np.linspace(-2*np.pi, 2*np.pi, 20), ...
    np.linspace(-7, 7, 20))
dtheta, domega = pendulum([theta_range, omega_range], 0, g, ell)
magnitude = np.sqrt(dtheta**2 + domega**2)/2; plt.figure(figsize=(12,8));
plt.streamplot(theta_range, omega_range, dtheta, domega, density=.5, ...
    linewidth=magnitude, color='#0085ff' , broken_streamlines=False, arrowsize=3)

# Plotting the trajectories in the phase portrait
for idx, ic in enumerate(initial_conditions):
    Y = odeint(pendulum, ic, t, args=(g, ell))
    theta, omega = Y.T
    plt.plot(theta, omega, color=colors[idx], label=f'theta0={ic[0]:.2f}, ...
        omega0={ic[1]:.2f}')

# Plotting the scatter points at theta = -2pi, -pi, 0, pi, 2pi
for scatter_theta in [-2*np.pi, -np.pi, 0, np.pi, 2*np.pi]:
    plt.scatter(scatter_theta, 0, color='black', s=50, zorder=5)
plt.xticks([-2*np.pi, -3*np.pi/2, -np.pi, -np.pi/2, 0, np.pi/2, np.pi, 3*np.pi/2, ...
    2*np.pi],
    ['$-2\pi$', '$-3\pi/2$', '$-\pi$', '$-\pi/2$', '0', '$\pi/2$', '$\pi$', ...
        '$3\pi/2$', '$2\pi$'])
plt.xlabel('Theta'); plt.ylabel('Omega'); plt.xlim([-2*np.pi, 2*np.pi]); ...
    plt.ylim([-7, 7])
plt.grid(True); plt.savefig("pendulum-phase.pdf", bbox_inches='tight')
```

Listing 2.6: Python script generating Figure 2.14. Available at pendulum.py

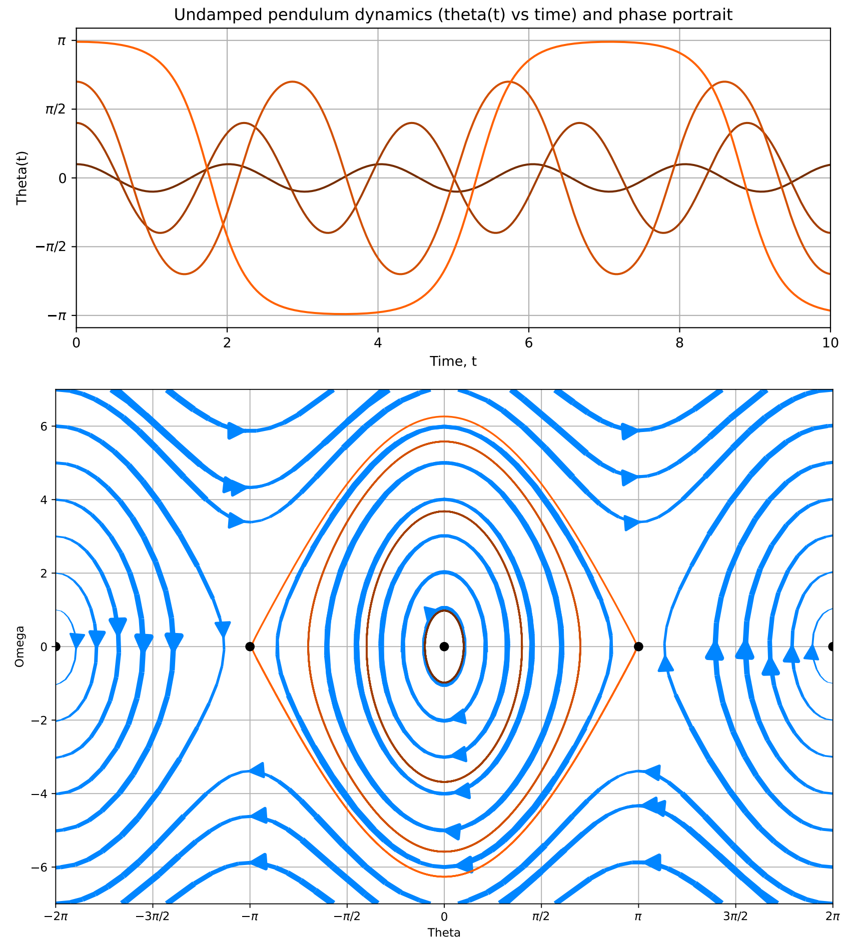


Figure 2.14: Solutions and phase portrait for the undamped pendulum dynamics (2.25) (i.e., the pendulum without friction).
Top image: Four initial conditions are randomly selected within the angular range $[-\pi/2, \pi/2]$ and with low initial angular velocity. Bottom image: phase portrait with the four trajectories superimposed.

Numerical simulation of the pendulum with friction

```python
# Python libraries
import numpy as np; import matplotlib.pyplot as plt
from scipy.integrate import odeint

# Pendulum dynamics with damping
def pendulum(Y, t, g, ell, b):
    theta, omega = Y
    dtheta = omega
    domega = -g/ell * np.sin(theta) - b/(m*ell**2) * omega
    return [dtheta, domega]

# Parameters and time array
g   = 9.81     # gravity
ell = 1.0      # length of the pendulum
m   = 0.5      # mass
b   = 0.2      # damping coefficient (adjust this value as desired)
t = np.linspace(0, 10, 1000)

# Initial conditions: [theta0, omega0] and plot the solution
initial_conditions = [[.1*np.pi, 0], [.4*np.pi, 0], [.7*np.pi, 0], [.99*np.pi, 0]]
colors = ['#752d00', '#a43e00', '#d35000', '#ff6100']

plt.figure(figsize=(10, 4))
for idx, ic in enumerate(initial_conditions):
    Y = odeint(pendulum, ic, t, args=(g, ell, b))
    theta, omega = Y.T
    plt.plot(t, theta, label=f'theta0={ic[0]:.2f}, omega0={ic[1]:.2f}', ...
        color=colors[idx])

plt.yticks([-np.pi, -np.pi/2, 0, np.pi/2, np.pi],
           ['$-\pi$', '$-\pi/2$', '0', '$\pi/2$', '$\pi$'])
plt.title('Pendulum dynamics (theta(t) vs time) with damping')
plt.xlabel('Time, t'); plt.ylabel('Theta(t)'); plt.xlim(0, 10); plt.grid(True)
plt.savefig("pendulum-damped.pdf", bbox_inches='tight')

# Phase portrait
theta_range, omega_range = np.meshgrid(np.linspace(-2*np.pi, 2*np.pi, 20), ...
    np.linspace(-7, 7, 20))
dtheta, domega = pendulum([theta_range, omega_range], 0, g, ell, b)
magnitude = np.sqrt(dtheta**2 + domega**2)/2; plt.figure(figsize=(12,8));
plt.streamplot(theta_range, omega_range, dtheta, domega, density=.5, ...
    linewidth=magnitude, color='#0085ff', broken_streamlines=False, arrowsize=3)

for idx, ic in enumerate(initial_conditions):
    Y = odeint(pendulum, ic, t, args=(g, ell, b))
    theta, omega = Y.T
    plt.plot(theta, omega, color=colors[idx], label=f'theta0={ic[0]:.2f}, ...
        omega0={ic[1]:.2f}')

# Plotting the scatter points at theta = -2pi, -pi, 0, pi, 2pi
for scatter_theta in [-2*np.pi, -np.pi, 0, np.pi, 2*np.pi]:
    plt.scatter(scatter_theta, 0, color='black', s=50, zorder=5)

plt.xticks([-2*np.pi, -3*np.pi/2, -np.pi, -np.pi/2, 0, np.pi/2, np.pi, 3*np.pi/2, ...
    2*np.pi],
           ['$-2\pi$', '$-3\pi/2$', '$-\pi$', '$-\pi/2$', '0', '$\pi/2$', ...
               '$\pi$', '$3\pi/2$', '$2\pi$'])
plt.xlabel('Theta'); plt.ylabel('Omega'); plt.xlim([-2*np.pi, 2*np.pi])
plt.ylim([-7, 7]); plt.grid(True); plt.savefig("pendulum-damped-phase.pdf", ...
    bbox_inches='tight')
```

Listing 2.7: Python script generating Figure 2.15. Available at pendulum-damped.py

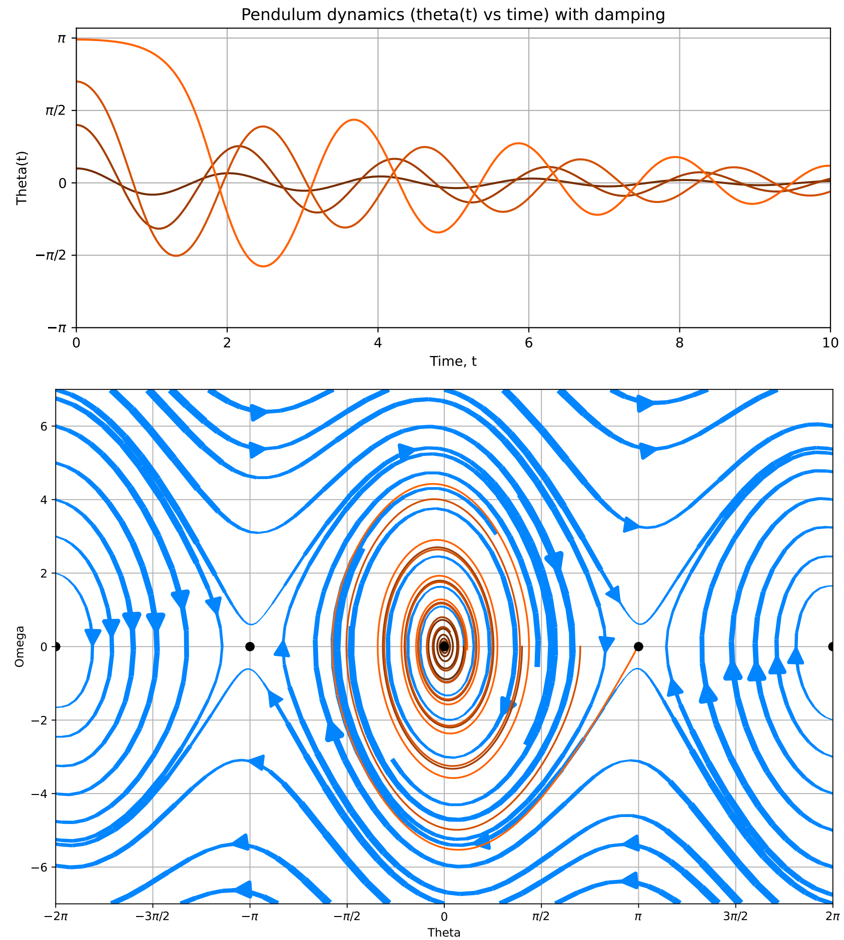
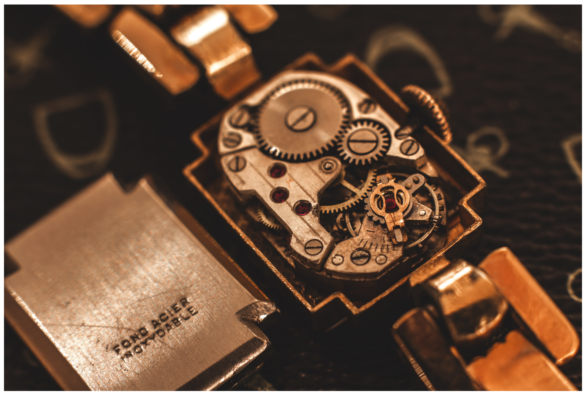


Figure 2.15: Solutions and phase portrait for the damped pendulum dynamics (2.24) (i.e., the pendulum with friction).
Top image: Four initial conditions are randomly selected within the angular range $[-\pi/2, \pi/2]$ and with low initial angular velocity. Bottom image: phase portrait with the four trajectories superimposed.

2.3.2 Mechanical gears

- *Gears* are toothed elements that transfer motion and power between mechanical subsystems. Gears operate in pairs, with their teeth meshing to prevent slippage (there might be backlash). Each gear is attached to a shaft.
- The *input gear* (also known as driver gear) affects the movement of the *output gear* (also known as driven gear).
- Unequal gear sizes lead to a *mechanical advantage*, changing output speed and output torque.
- Example applications include clocks, windmills, bicycles, and automobile transmissions.



(a) Vintage internal clockwork (spring and toothed gearwheels inside a mechanical clock), sourced from https://unsplash.com.
Traditional mechanical clocks rely on gear trains to transfer energy from a wound spring or suspended weight to the clock hands, ensuring precise movement. The intricate design allows for accurate timekeeping by compensating for variations in power delivery, maintaining the clock's consistency over time.

(b) Bicycle drivetrain
The crankset and rear wheel of a bicycle are connected by a chain that engages with sprockets, commonly known as "chainrings" at the front and the "cassette" at the rear. The gear ratio determines how many times the rear wheel rotates for each full revolution of the crank.
On a single-speed bicycle, the gear ratio is fixed.
On a multi-speed bicycle, shifting the chain between larger or smaller sprockets alters the gear ratio, adjusting the bicycle's resistance. Depending on the terrain, the rider selects an optimal gear for slowly climbing hills or quickly riding on flat surfaces.

Sprockets and chains function similarly to gears in transferring rotational motion and power. Gear design can change the *direction of rotation or movement* (e.g., read up on bevel gears). *Rack and pinion systems* convert between rotational motion and linear motion. Here are some instructive videos:

- quick review of spur gear, helical gear (quiter), double helical gear (even quiter and stable), worm gear (high reduction gear, self-locking), screw gear, rack & pinion gear (conversion between rotational and linear motion), straight bevel gear (transfers motion between intersecting axes), helical bevel gear, and internal/external gear (read up also on planetary gears): basic gear types (short);
- advanced gear types (short);
- explanations of gear trains and composite gear trains (8m 47s);
- the automobile differential is a gear system designed to allow two wheels to rotate at different speed: differential steering (3m 45s), and (short 49s); and
- a GOOGOL:1 reduction with lego gears (9m 58s)

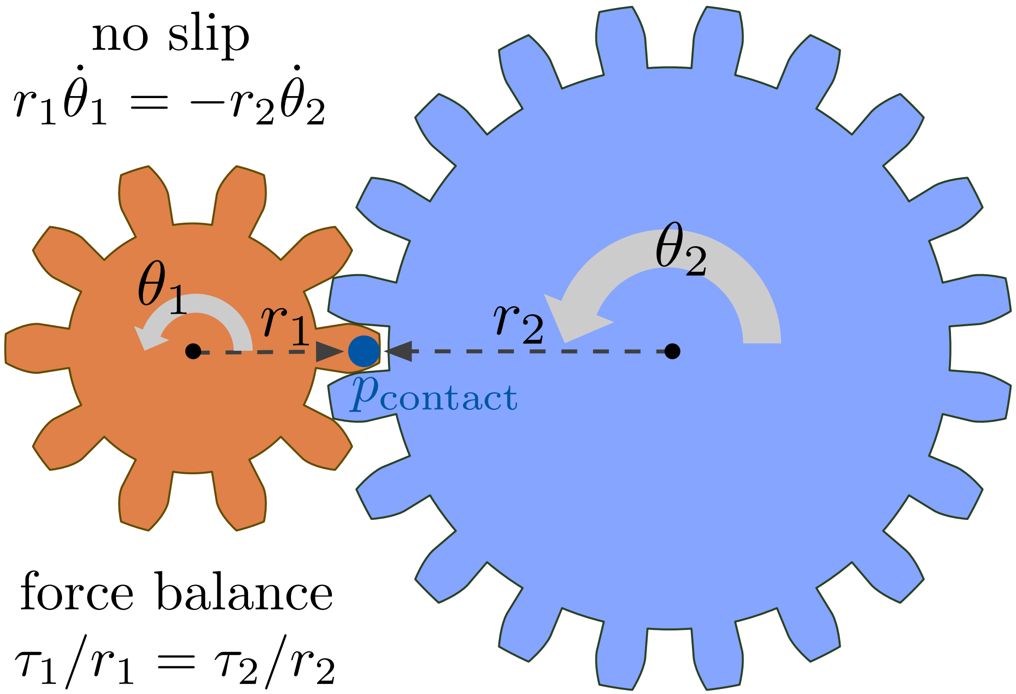


Figure 2.16: Two gears interconnecting two parallel shafts (not drawn).

The angles θ_1 and θ_2 are measured counterclockwise (this is our convention for all angles in these notes). The two shaft move in opposite direction so that $\dot{\theta}_1 > 0$ if and only if $\dot{\theta}_2 < 0$.

At the contact point p_{contact}:
(i) the no-slip condition is: $\text{velocity}_{\text{contact}} = r_1\dot{\theta}_1 = -r_2\dot{\theta}_2$.
(ii) the force-balance condition is: $\text{force}_{\text{contact}} = \dfrac{\tau_1}{r_1} = \dfrac{\tau_2}{r_2}$.

Note: It appears that the radius of the two gears is proportional to their number of teeth ($n_1 = 10$, $n_2 = 20$, and it appears $r_2 = 2r_1$). Why would that be true?

Mathematical analysis: transmission of velocities

We adopt the following notation:

- the input gear has angle θ_{input}, radius r_{input} and n_{input} teeth, and
- the output gear has angle θ_{output}, radius r_{output} and n_{output} teeth.

(i) The *equal tooth pitch* property is that the distance between the same point on two consecutive teeth. Under this property, the radius of a gear is proportional to its number of teeth:

$$\frac{r_{\text{input}}}{n_{\text{input}}} = \frac{r_{\text{output}}}{n_{\text{output}}} \qquad \iff \qquad \frac{r_{\text{output}}}{r_{\text{input}}} = \frac{n_{\text{output}}}{n_{\text{input}}}. \tag{2.29}$$

It is customary to define the *gear ratio* to be

$$\boxed{\text{gear ratio} = \frac{n_{\text{output}}}{n_{\text{input}}}} \tag{2.30}$$

(ii) We assume the gear interconnection satisfies the *no slip condition*, namely the linear velocities of the two gears at the point of contact are equal. Since linear velocity = radius $\times$ angular velocity, we obtain:

$$r_{\text{input}}\,\dot{\theta}_{\text{input}} \;=\; -r_{\text{output}}\,\dot{\theta}_{\text{output}}. \tag{2.31}$$

Therefore, we can write

$$\boxed{\frac{\dot{\theta}_{\text{output}}}{\dot{\theta}_{\text{input}}} = -\frac{r_{\text{input}}}{r_{\text{output}}} = -\frac{n_{\text{input}}}{n_{\text{output}}} = -\frac{1}{\text{gear ratio}}} \tag{2.32}$$

Mathematical analysis: transmission of torques

When the two gears are in contact, a contact force is applied on both gears. This force is equal in magnitude and opposite in direction. Let torque τ_{input} is applied on the input gear, a torque τ_{output} is felt at the output gear through the contact point. Because both angles are measured in the same direction (counterclockwise), as illustrated in Figure 2.16, we obtain

$$\frac{\tau_{\text{input}}}{r_{\text{input}}} = \frac{\tau_{\text{output}}}{r_{\text{output}}}. \tag{2.33}$$

In turn, the equal tooth pitch property implies

$$\frac{\tau_{\text{output}}}{\tau_{\text{input}}} = \frac{n_{\text{output}}}{n_{\text{input}}} = \text{gear ratio} \tag{2.34}$$

In our example in Figure 2.16, treating the gear $\#1$ (with $n_1 = 10$) as the input and the gear $\#2$ (with $n_2 = 20$) as the output, we calculate the gear ratio to be $20/10 = 2$. Therefore,

(i) the angular velocity of gear #1 is halved at gear #2 (with opposite direction), and

(ii) a torque at gear #1 is perceived twice as large at gear #2 (with same direction).

This is the meaning of *mechanical advantage:* Engaging a smaller gear (in terms of radius or number of teeth) with a larger gear increases the torque in the system, but reduces the speed. Such a gear pair provides a mechanical advantage, making it easier to do tasks like lifting heavy objects or climbing steep inclines in vehicles.

2.3.3 Dynamics of interconnected gears

Here we study a natural dynamical system associated to the interconnected gears in Figure 2.16. Specifically, we suppose a torque T is applied to the first gear and we obtain the dynamics for θ_2. (Other combinations are possible, e.g., see E2.5.)

When the shafts are not interconnected, assuming I_1 and I_2 are the two moments of inertia, we have

$$I_1\ddot{\theta}_1 = T \tag{2.35}$$

$$I_2\ddot{\theta}_2 = 0 \tag{2.36}$$

When the gear interconnection is included, two torques τ_1 and τ_2 appear:

$$I_1\ddot{\theta}_1 = T + \tau_1 \tag{2.37}$$

$$I_2\ddot{\theta}_2 = \tau_2 \tag{2.38}$$

Since $n_1\dot{\theta}_1 = -n_2\dot{\theta}_2$ and $n_2\tau_1 = n_1\tau_2$, we know

$$\dot{\theta}_1 = -\frac{n_2}{n_1}\dot{\theta}_2, \quad \ddot{\theta}_1 = -\frac{n_2}{n_1}\ddot{\theta}_2 \quad \text{and} \quad \tau_1 = \frac{n_1}{n_2}\tau_2. \tag{2.39}$$

Plugging these expressions into the dynamics, we obtain:

$$I_1\Big(-\frac{n_2}{n_1}\ddot{\theta}_2\Big) = T + \frac{n_1}{n_2}\tau_2, \tag{2.40}$$

$$I_2\ddot{\theta}_2 = \tau_2. \tag{2.41}$$

We now wish to eliminate τ_2; to do so, we multiply the first equation by $-\frac{n_2}{n_1}$ and sum the two resulting equations:

$$\boxed{\Big(I_2 + \frac{n_2^2}{n_1^2}I_1\Big)\ddot{\theta}_2 = (-\frac{n_2}{n_1})T} \tag{2.42}$$

The moment of inertia $I_2 + \frac{n_2^2}{n_1^2}I_1$ in equation (2.42) is called the *equivalent moment of inertia* of the interconnected shafts.

2.4 Electrical circuits

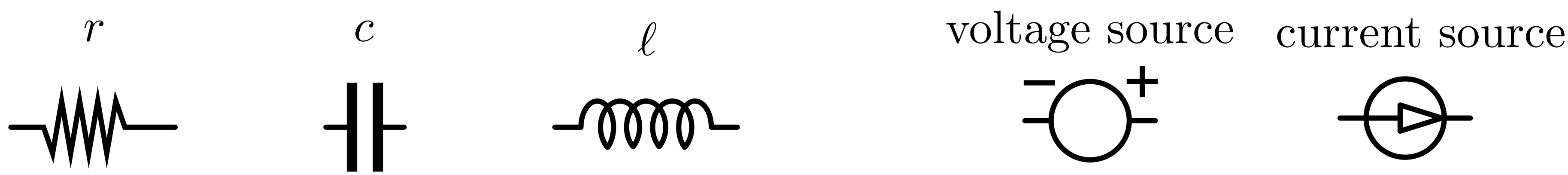


Figure 2.17: Passive elements (resistor, capacitor, and inductor) and active elements (voltage source and current source)

Components and their constitutive relations

Resistor: The constitutive relation of a *resistor* is $v = ri$ (*Ohm's law*), where the *resistance* r is measured in Ohms (Ω).

Here, v is the voltage across the resistor and i is the current through the resistor.

Capacitor: The constitutive relation of a *capacitor* is $i = c\frac{dv}{dt}$, where the *capacitance* c is measured in Farads (F).

Note: $i = c\frac{dv}{dt}$ is equivalent to $v(t) = v(0) + \frac{1}{c}\int_0^t i(\tau)d\tau$

Note: we assume "pure capacitors" which are capable of storing energy and releasing all of it; there are no energy losses.

Inductor: The constitutive relation of an *inductor* is $v = \ell\frac{di}{dt}$, where the *inductance* ℓ is measured in Henrys (H).

Note: $v = \ell\frac{di}{dt}$ is equivalent to $i(t) = i(0) + \frac{1}{\ell}\int_0^t v(\tau)d\tau$

Voltage source: A *voltage source* provides a fixed or varying voltage irrespective of the current drawn from it, measured in Volts (V).

For example, a battery is often modelled as a constant voltage source.

Current source: A *current source* supplies a fixed or varying current irrespective of the voltage across it, measured in Amperes (A).

Kirchhoff's voltage law (KVL) KVL states that the algebraic sum of all voltages around a closed loop or closed path in a circuit is zero. Specifically, for voltages v_k across components around a given loop, measured with consistent reference direction (either all clockwise or all counterclockwise):

$$\sum_k v_k = 0.$$

Kirchhoff's current law (KCL) KCL states that the algebraic sum of all currents entering and leaving a node (or junction) in a circuit is zero. Specifically, for currents i_k associated with a given node k, currents entering the node are considered positive, and those leaving are considered negative (or vice versa, based on convention):

$$\sum_k i_k = 0.$$

In other words, the total current flowing into the node equals the total current flowing out.

RC circuit with a voltage generator

We start by consider a circuit with a voltage source, a resistor and a capacitor in series, as illustrated in Figure 2.18. Let $v_{\text{input}}(t)$ be the voltage at the input, $r > 0$ be a resistance, $c > 0$ be a capacitance, and $v_{\text{output}}(t)$ be the voltage at the output.

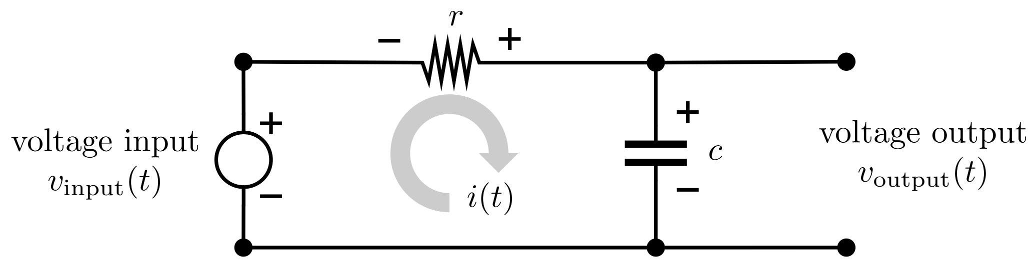


Figure 2.18: Series RC circuit. Based upon the convention for the voltages: $v_{\text{resistor}} = ri$ and $i = -c\frac{d}{dt}v_{\text{output}}$.

From Kirchhoff's voltage law (KVL), the sum of the voltage drops across the resistor and the capacitor equals the supplied voltage v_{input}:

$$v_{\text{input}} + v_{\text{resistor}} - v_{\text{capacitor}} = 0.$$

Substituting in the constitutive relation for the resistor and noting $v_{\text{capacitor}} = v_{\text{output}}$, we obtain:

$$v_{\text{input}} + ri(t) - v_{\text{output}} = 0.$$

From the constitutive relation of the capacitor, we know $i = -c\frac{d}{dt}v_{\text{output}}$, so that the overall differential equation governing the RC circuit is

$$v_{\text{input}} - cr\dot{v}_{\text{output}} - v_{\text{output}} = 0 \qquad \iff \qquad \boxed{\dot{v}_{\text{output}}(t) + \frac{1}{rc}v_{\text{output}}(t) = \frac{1}{rc}v_{\text{input}}(t)} \tag{2.43}$$

This is a first-order model in v_{output} with input v_{input}.

RLC circuit with a voltage generator

We now consider a circuit with a voltage source, a resistor, an inductor, and a capacitor in series, as illustrated in Figure 2.19. Let $v_{\text{input}}(t)$ be the voltage at the input, $r > 0$ be a resistance, $c > 0$ be a capacitance, ℓ be an inductance, and $v_{\text{output}}(t)$ be the voltage at the output.

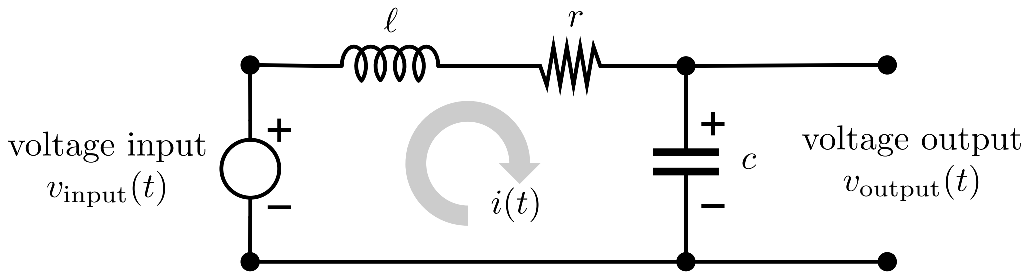


Figure 2.19: Series RLC circuit

As in Figure, we consider a *series RLC circuit with a voltage source* $v_{\text{input}}(t)$. The governing differential equation can be obtained from the KVL and from the constitutive relations for each element. In short:

$$v_{\text{input}}(t) = ri(t) + \ell\frac{di(t)}{dt} + \frac{1}{c}\int_0^t i(\tau)d\tau$$

Taking the derivative with respect to time[4] of both left and right had side, and rearranging terms, we obtain:

$$\ell\frac{d^2i(t)}{dt^2} + r\frac{di(t)}{dt} + \frac{1}{c}i(t) = \dot{v}_{\text{input}}(t) \tag{2.44}$$

This is a second-order model in the current i with input v_{input}. Following similar reasoning, we can obtain a second-order model for the output voltage:

$$\ddot{v}_{\text{output}} + \frac{r}{\ell}\dot{v}_{\text{output}} + \frac{1}{\ell c}\dot{v}_{\text{output}} = \frac{1}{\ell c}v_{\text{input}}(t). \tag{2.45}$$

[4]Recall that the fundamental theorem of calculus states $\frac{d}{dt}\int_0^t i(\tau)d\tau = i(t)$.

The equation (2.44) is analogous to the forced damped harmonic oscillator, described by the equation of motion

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

for a spring-mass-damper system subject to a force $f(t)$.

mass-spring-damper mechanical system		RLC electrical circuit	
m	mass	ℓ	inductance
b	damping coefficient	r	resistance
k	stiffness	$1/c$	inverse capacitance,
$f(t)$	external force	$\frac{d}{dt}v_{\text{input}}(t)$	forcing term

Table 2.1: Analogies between mechanical and electrical systems

2.5 Electromechanical systems: the DC motor

An *electromechanical system* is an engineering device composed of both electrical and mechanical components. Specifically, a *direct-current motor (DC motor)* converts electrical energy into mechanical energy or, more precisely, direct current into a torque. A DC motor is illustrated in Figure 2.20 and its functioning is illustrated in this wikipedia animation. Here are some highlights on how the DC motor functions:

Physical principle: A current-carrying conductor experiences a mechanical force when placed in a magnetic field. This force is called the Lorentz force.

From physical principle to engineering design: The Lorentz force cause the conductor to rotate, thus turning the motor's shaft and producing mechanical work.

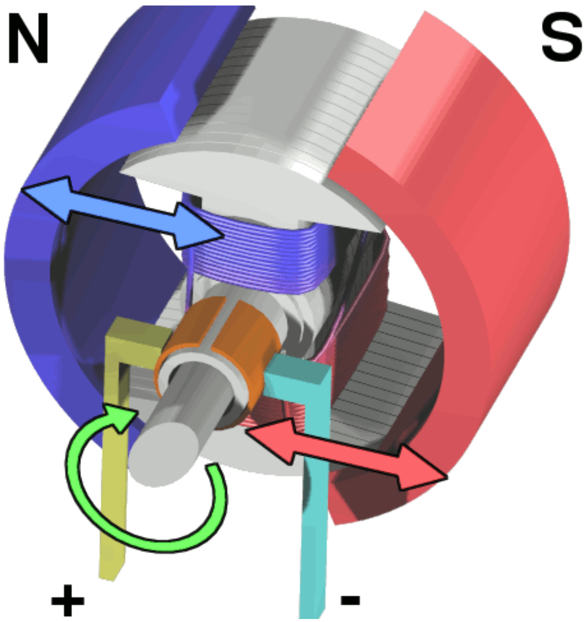


Figure 2.20: In a typical DC motor, the conductor is a coil, that is, a series of loops made from conductive wires wound around a core.

In a brushed DC motor, brushes are used to ensure that the current in the conductor is in the correct direction to produce maximum torque. In this image, a brushed DC electric motor generates torque from a supplied DC power, by using internal commutation (via brushes) and stationary permanent magnets.

Brushless DC motors (which do not use brushes) rely on electronic controllers to switch the current in the motor's windings.

Public domain image from Wikipedia.

A complete derivation of the governing equations for a DC motor is outside the scope of these notes. Based upon the formulas for the Lorentz force and upon the geometry and design of the motor circuit, it suffices to say that

(i) *the current through the conductor i_{cond} generates a torque on the shaft*, with magnitude equal to $K_{\text{torque}} i_{\text{cond}}$ where $K_{\text{torque}} > 0$ is constant, and

(ii) *the shaft's angular velocity $\dot{\theta}_{\text{m}}$ generates a "back emf" voltage*[5] *on the conductor circuit*, with magnitude equal to $K_{\text{velocity}} \dot{\theta}_{\text{m}}$ with $K_{\text{velocity}} > 0$ and opposed to the voltage applied to the motor.

We assume some rotational damping with coefficient b and a moment of inertia I_m for the rotor. We also let ℓ denote the inductance and r denote the resistance of the conductor circuit.

In summary, the equations of motion for the DC motor are:

$$I_{\text{m}} \ddot{\theta}_{\text{m}}(t) + b \dot{\theta}_{\text{m}}(t) = K_{\text{torque}} i_{\text{cond}}(t) \tag{2.46a}$$

$$\ell \frac{d}{dt} i_{\text{cond}}(t) + r i_{\text{cond}}(t) = v_{\text{source}}(t) - K_{\text{velocity}} \dot{\theta}_{\text{m}}(t) \tag{2.46b}$$

where $v_{\text{source}}(t)$ is the externally applied voltage to the conductor circuit.

[5] "Back emf" stands for "back electromotive force."

These equations are *electromechanical* since they involve both mechanical and electrical quantities:

- equation (2.46a) with state θ_{m} is a rotational mechanical system with damping and with a forcing torque $K_{\mathrm{torque}} i_{\mathrm{cond}}$, and
- equation (2.46a) with state i_{cond} is an RL circuit with a forcing voltage source $v_{\mathrm{source}}(t) - K_{\mathrm{velocity}}\dot{\theta}_{\mathrm{m}}$

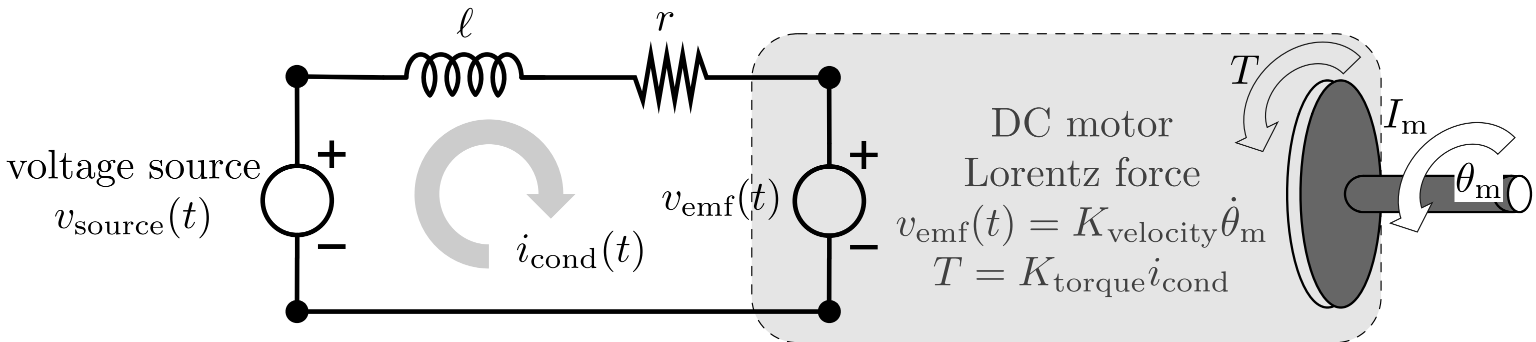


Figure 2.21: A DC motor relies upon the physical principle of the Lorentz force the transduce a voltage into a torque:
(i) the current through the conductor i_{cond} generates a torque on the shaft, with magnitude equal to $K_{\mathrm{torque}} i_{\mathrm{cond}}$, and
(ii) the shaft's angular velocity $\dot{\theta}_{\mathrm{m}}$ generates a "back emf" voltage , with magnitude equal to $K_{\mathrm{velocity}}\dot{\theta}_{\mathrm{m}}$ and opposed to the voltage applied to the motor.

Numerical analysis of the DC motor

```python
import numpy as np; from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

def motor_dynamics(t, state, I_m, b, K_torque, L, R, K_velocity, V_input):
    theta_m, theta_m_dot, ic = state
    theta_m_ddot = (K_torque * ic - b * theta_m_dot) / I_m
    ic_dot = (V_input(t) - K_velocity * theta_m_dot - R * ic) / L
    return [theta_m_dot, theta_m_ddot, ic_dot]

# Parameters for the DC motor
I_m = 0.01          # Moment of inertia of the motor
b = 0.1             # Damping coefficient
K_torque = 0.01     # Torque constant
L = 0.5             # Motor inductance
R = 1               # Motor resistance
K_velocity = 0.01   # Back EMF constant

# Time array
t = np.linspace(0, 6, 1000)

# Voltage input: step function at 1V
V_input = lambda t: 1.0 if t > 1 else 0.0

# Initial conditions: [theta_m, theta_m_dot, ic]
initial_conditions = [0.0, 0.0, 0.0]
sol = solve_ivp(motor_dynamics, [t[0], t[-1]], initial_conditions, ...
     t_eval=t, args=(I_m, b, K_torque, L, R, K_velocity, V_input))

# Plotting
plt.figure(figsize=(12, 10)); plt.subplot(3, 1, 1); plt.xlim(0, 6)
plt.plot(sol.t, sol.y[0], label='Motor Position (rad)')
plt.grid(True); plt.ylabel('Position (rad)'); plt.legend()

plt.subplot(3, 1, 2)
plt.plot(sol.t, sol.y[1], label='Motor Speed (rad/s)'); plt.xlim(0, 6)
plt.grid(True); plt.ylabel('Speed (rad/s)'); plt.legend()

plt.subplot(3, 1, 3)
plt.plot(sol.t, sol.y[2], label='Current (A)', color='red'); ...
     plt.xlim(0, 6)
plt.grid(True); plt.xlabel('Time'); plt.ylabel('Current (A)'); plt.legend()

plt.tight_layout()
plt.savefig("dcmotor.pdf", bbox_inches='tight')
```

Listing 2.8: Python script generating Figure 2.22. Available at dcmotor.py

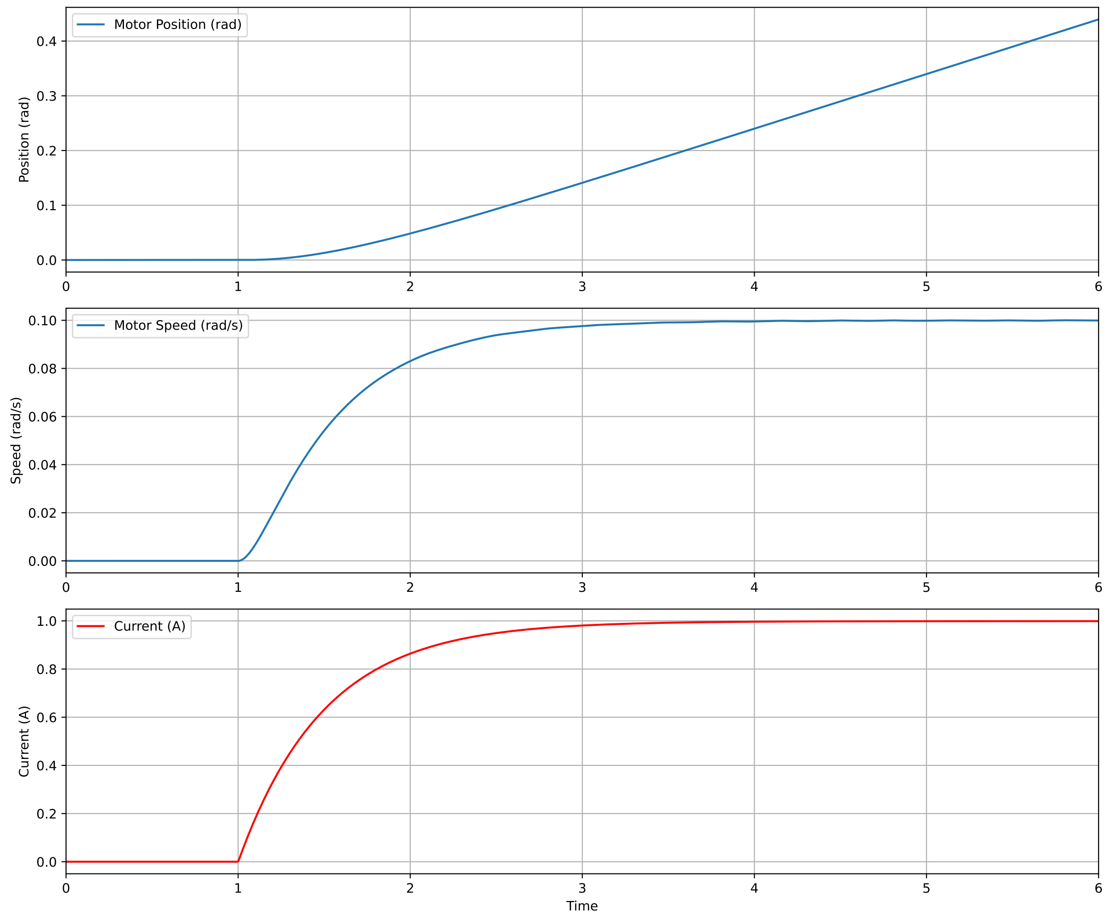


Figure 2.22: Solutions of the DC motor system (2.46): response to a 1V step input voltage at time $t = 1$s, that is, the input voltage satisfies $v_{\text{source}}(t) = 0$V from time $0 \leq t \leq 1$ and then $v_{\text{source}}(t) = 1$V for $t > 1$.

2.6 Historical notes, further reading, and online resources

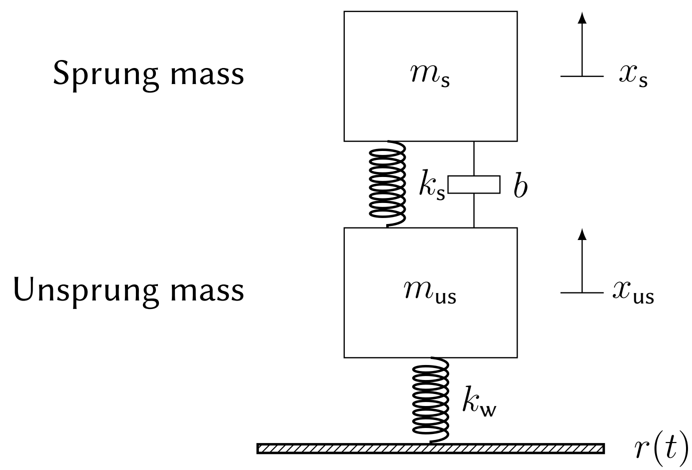
Instructive videos:

- how do car suspensions work (20m 23s) and (short) (2m 49s) with animations and explanation of different types of automobile suspensions;
- applications of rotary dampers and torsion springs: how to install rotary dampers (2m 26s);
- the only mechanical gears known to occur in nature are reviewed in this article and video interview (3m 41s).

The loss of the *Mars Climate Orbiter* in 1999 was a significant engineering failure due to a unit conversion error—specifically, a miscommunication between metric (SI) and imperial (U.S. customary) units. In this disaster, the problem was that NASA's Jet Propulsion Laboratory (JPL) used the metric system, while the spacecraft's contractor, Lockheed Martin, used imperial units. Lockheed Martin provided data for the spacecraft's thrusters in pounds of force, but NASA was expecting the data in Newtons (the SI unit for force). This discrepancy led to the orbiter's trajectory being incorrect, causing it to enter the Martian atmosphere at a much lower altitude than intended, leading to its destruction.

2.7 Exercises

E2.1 **Equations of motion for the suspension system.** Consider the suspension system studied in Section 2.2: m_s and m_{us} are sprung mass and unsprung mass, respectively; k_s and k_w are the spring constants for m_s and m_{us} respectively, b is the damping coefficient, and $r(t)$ is the road surface.



(i) Draw a free body diagram for each mass and accounting for each spring and damper (ignore gravity).

(ii) Write equations for each force acting on the masses m_s and m_{us}.

(iii) Write the equations of motion for the sprung and unsprung masses based on Newton's law.

Answer:

(i) Assume the positive axis is pointing upwards. In this system, we have spring and damping forces.

$$F_{\text{spring}} = \text{(stiffness coefficient)} \times \text{displacement} \qquad \text{and} \qquad F_{\text{damper}} = \text{(damping coefficient)} \times \text{velocity} \tag{E2.1}$$

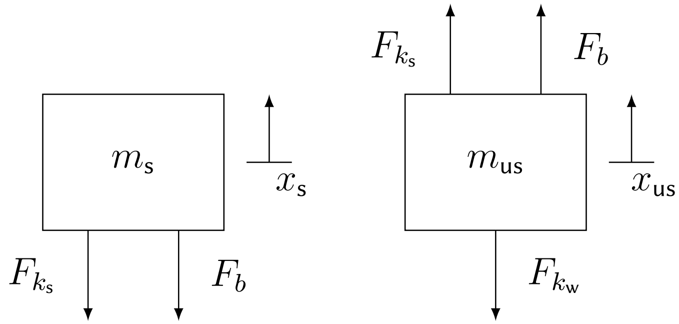


Figure E2.1: Free Body Diagram

(ii) For mass m_{s}: There are two forces acting on the mass

$$F_{k_{\text{s}}} = k_{\text{s}}x_{\text{us}} - k_{\text{s}}x_{\text{s}} = k_{\text{s}}(x_{\text{us}} - x_{\text{s}})$$
$$F_b = b\dot{x}_{\text{us}} - b\dot{x}_{\text{s}} = b(\dot{x}_{\text{us}} - \dot{x}_{\text{s}})$$

so that $m_{\text{s}}\ddot{x}_{\text{s}} = F_{k_{\text{s}}} + F_b = k_{\text{s}}(x_{\text{us}} - x_{\text{s}}) + b(\dot{x}_{\text{us}} - \dot{x}_{\text{s}})$.

For mass m_{us}: There are three forces acting on the mass. Specifically, the directions of the forces $F_{k_{\text{s}}}$ and F_b are opposite

$$F_{k_{\text{s}}} = k_{\text{s}}x_{\text{s}} - k_{\text{s}}x_{\text{us}} = -k_{\text{s}}(x_{\text{us}} - x_{\text{s}}),$$
$$F_b = b\dot{x}_{\text{s}} - b\dot{x}_{\text{us}} = -b(\dot{x}_{\text{us}} - \dot{x}_{\text{s}}),$$
$$F_{k_{\text{w}}} = k_{\text{w}}r(t) - k_{\text{w}}x_{\text{us}} = -k_{\text{w}}(x_{\text{us}} - r(t)),$$

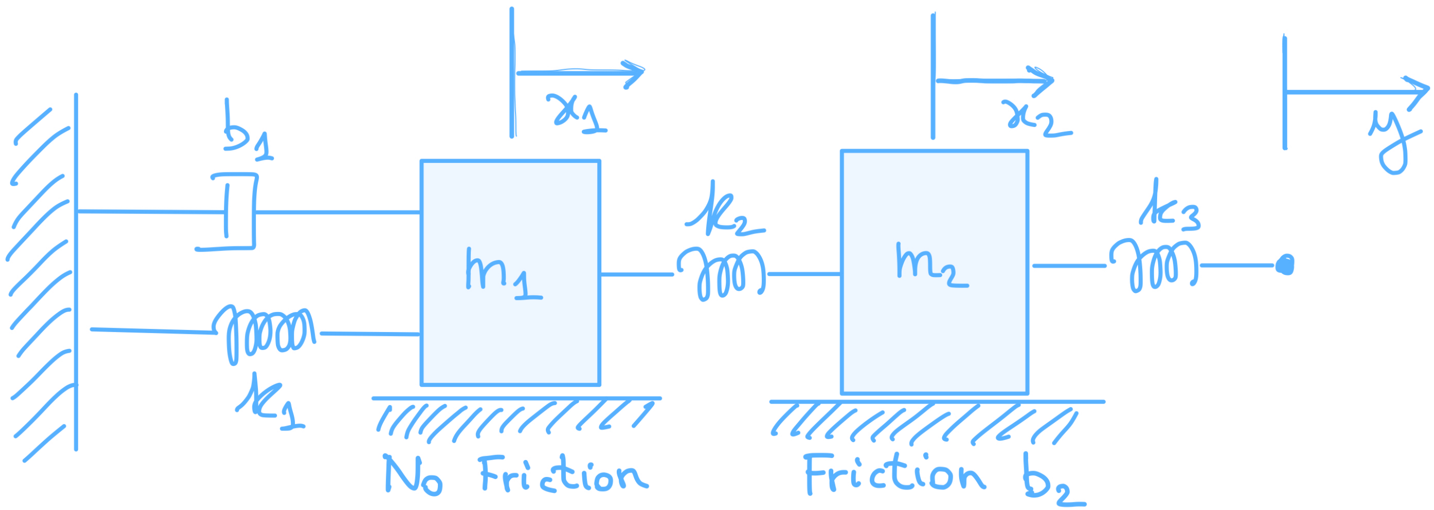
so that $m_{\text{us}}\ddot{x}_{\text{us}} = F_{k_{\text{s}}} + F_b + F_{k_{\text{w}}} = -k_{\text{s}}(x_{\text{us}} - x_{\text{s}}) - b(\dot{x}_{\text{us}} - \dot{x}_{\text{s}}) - k_{\text{w}}(x_{\text{us}} - r(t))$.

(iii) Rearrange the equations:

$$m_{\text{s}}\ddot{x}_{\text{s}} + b\,(\dot{x}_{\text{s}} - \dot{x}_{\text{us}}) + k_{\text{s}}\,(x_{\text{s}} - x_{\text{us}}) = 0$$
$$m_{\text{us}}\ddot{x}_{\text{us}} + b\,(\dot{x}_{\text{us}} - \dot{x}_{\text{s}}) + k_{\text{s}}\,(x_{\text{us}} - x_{\text{s}}) + k_{\text{w}}x_{\text{us}} = k_{\text{w}}r(t)$$

■

E2.2 **Governing equations for two interconnected masses .** Consider the mass-spring-damper system in figure, where the spring have zero rest length and the friction on the second mass acts like a damper with damping coefficient b_2.



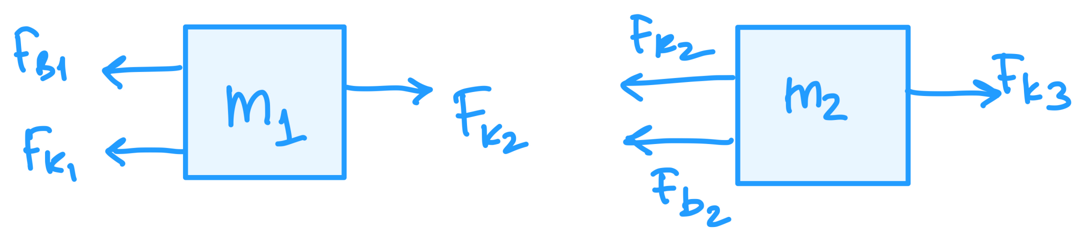
Perform the following steps:

(i) Draw the free body diagram for each mass.

(ii) Use Newton's 2nd Law (i.e., $F = ma$) to write the equations of motion for the system.

Answer:

(i) Here is the free body diagram:



Note that F_{k_2} is in opposite directions for first and second body.

Given the conventions in the drawing, we compute

$$F_{b_1} = b_1\dot{x}_1, \quad F_{k_1} = k_1\dot{x}_1, \quad F_{k_2} = k_2(x_2 - x_1) \tag{E2.2}$$

$$F_{b_2} = b_2\dot{x}_2, \quad F_{k_3} = k_3(y - x_2). \tag{E2.3}$$

(ii) Given the convention in the drawing, Newton's 2nd Law gives us

$$\begin{aligned} F_{k_2} - F_{b_1} - F_{k_1} &= m_1\ddot{x}_1 \\ F_{k_3} - F_{k_2} - F_{b_2} &= m_2\ddot{x}_2. \end{aligned}$$

Substituting expressions for each of the forces gives

$$\begin{aligned} k_2(x_2 - x_1) - b_1\dot{x}_1 - k_1x_1 &= m_1\ddot{x}_1 \\ k_3(y - x_2) - k_2(x_2 - x_1) - b_2\dot{x}_2 &= m_2\ddot{x}_2. \end{aligned}$$

Putting these in standard form, we have

$$\ddot{x}_1 = -\frac{(k_1 + k_2)}{m_1}x_1 - \frac{b_1}{m_1}\dot{x}_1 + \frac{k_2}{m_1}x_2$$

$$\ddot{x}_2 = \frac{k_2}{m_2}x_1 - \frac{(k_2 + k_3)}{m_2}x_2 - \frac{b_2}{m_2}\dot{x}_2 + \frac{k_3}{m_2}y$$

■

E2.3 **Solution to the harmonic oscillator.** Consider a mass-spring system described by the harmonic oscillator (i.e., an undamped harmonic oscillator) as in Figure E2.2.

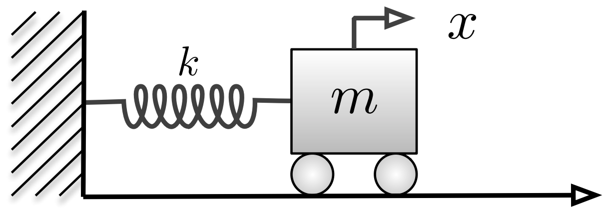


Figure E2.2: A mass-spring system described by the harmonic oscillator system $m\ddot{x} + kx = 0$.

Let m denote the mass of the oscillating object and k the stiffness of the spring. From equation (2.17), each solution to the undamped harmonic oscillator is of the form

$$x(t) = a\sin(\omega t) + b\cos(\omega t) \tag{E2.4}$$

where the natural frequency is $\omega = \sqrt{k/m}$. Define the abbreviations: $x_0 := x(0)$ and $v_0 := \dot{x}(0)$.

(i) Write a formula for (a, b) as function of (x_0, v_0) and viceversa.

(ii) Consider the equality

$$a\sin(\omega t) + b\cos(\omega t) = A\sin(\omega t + \phi) \tag{E2.5}$$

Write a formula for (a, b) as function of (A, ϕ) and viceversa.

Hint: Recall that in class we saw $A = \sqrt{a^2 + b^2}$. You will need to use trigonometric identities.

(iii) *Optional:* Show that the solution can also be written as

$$x(t) = C_1 e^{-j\omega t} + C_2 e^{j\omega t} \tag{E2.6}$$

for appropriate complex numbers C_1 and C_2. Write C_1 and C_2 as a function of (x_0, v_0).

Hint: Recall Euler's formula for complex numbers.

Answer:

(i) From equation (E2.4), the time derivative of $x(t)$ is given by

$$\dot{x}(t) = a\omega\cos(\omega t) - b\omega\sin(\omega t). \tag{E2.7}$$

Evaluating equation (E2.4) and equation (E2.7) at time 0 we obtain:

$$x(0) = a\sin(0) + b\cos(0)$$
$$\dot{x}(0) = a\omega\cos(0) - b\omega\sin(0)$$
$$\implies \boxed{b = x_0} \quad \text{and} \quad \boxed{a = \frac{v_0}{\omega}}$$

Similarly, we can write (x_0, v_0) as a function of (a, b) by rearranging the terms above to find

$$\boxed{x_0 = b} \quad \text{and} \quad \boxed{v_0 = a\omega}$$

Finally, it is useful to substitute the expression for (a, b) into the formula for $x(t)$ to obtain:

$$x(t) = \frac{v_0}{\omega}\sin(\omega t) + x_0\cos(\omega t)$$

(ii) Recall that the angle sum trigonometric identity for the sinusoidal function is $\sin\big(\alpha + \beta\big) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. Using this identity, we expand the right hand side of the equation (E2.5) to find

$$a\sin\big(\omega t\big) + b\cos\big(\omega t\big) = A\sin(\omega t)\cos(\phi) + A\cos(\omega t)\sin(\phi)$$

Therefore, by matching terms, we obtain

$$\boxed{a = A\cos(\phi)} \quad \text{and} \quad \boxed{b = A\sin(\phi)}$$

To write A in terms of (a, b) we make use of the fact that $\sin^2 + \cos^2 = 1$. Observe that

$$a^2 + b^2 = A^2\big(\cos^2(\phi) + \sin^2(\phi)\big) \quad \implies \quad \boxed{A = \sqrt{a^2 + b^2}}$$

Next, rearranging the expressions for (a, b) in terms of (A, ϕ) we see

$$\begin{aligned} a &= A\cos(\phi) \\ b &= A\sin(\phi) \end{aligned} \quad \implies \quad \frac{b}{a} = \tan\phi \quad \implies \quad \boxed{\phi = \arctan(b/a)}$$

(iii) We want to show that, for some complex numbers C_1 and C_2,

$$x(t) = a\sin(\omega t) + b\cos(\omega t) = C_1 e^{-j\omega t} + C_2 e^{j\omega t}$$

From Euler's formulas, we have the relationships

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) \qquad \text{and} \qquad e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t).$$

Substituting these equations into the exponential form (E2.6) of the solution we have

$$\begin{aligned} x(t) &= C_1\big(\cos(\omega t) - j\sin(\omega t)\big) + C_2\big(\cos(\omega t) + j\sin(\omega t)\big) \\ &= (C_1 + C_2)\cos(\omega t) + j(C_2 - C_1)\sin(\omega t) \end{aligned}$$

Observe that the time derivative of this equation is given by

$$\dot{x}(t) = -\omega(C_1 + C_2)\sin(\omega t) + j\omega(C_2 - C_1)\cos(\omega t)$$

Similarly to before, by substituting our initial conditions for $(x(t), \dot{x}(t))$, we can write (C_1, C_2) as a function (x_0, v_0) as follows

$$\begin{aligned} x(0) &= \big(C_1 + C_2\big)\cos(0) + j\big(C_2 - C_1\big)\sin(0) &&\implies C_1 + C_2 = x_0 \\ \dot{x}(0) &= -\omega\big(C_1 + C_2\big)\sin(0) + j\omega(C_2 - C_1)\cos(0) &&\implies C_2 - C_1 = \frac{v_0}{j\omega} \end{aligned}$$

From here, we can find C_1 and C_2 independently to be

$$\begin{aligned} \big(C_1 + C_2\big) + \big(C_2 - C_1\big) = x_0 - j\frac{v_0}{\omega} &\implies \boxed{C_2 = \frac{1}{2}\big(x_0 - j\frac{v_0}{\omega}\big)} \\ C_1 = x_0 - C_2 &\implies \boxed{C_1 = \frac{1}{2}\big(x_0 + j\frac{v_0}{\omega}\big)} \end{aligned}$$

■

E2.4 **Inverted pendulum cart via Lagrangian mechanics.** Consider the inverted pendulum cart in Figure E2.3.

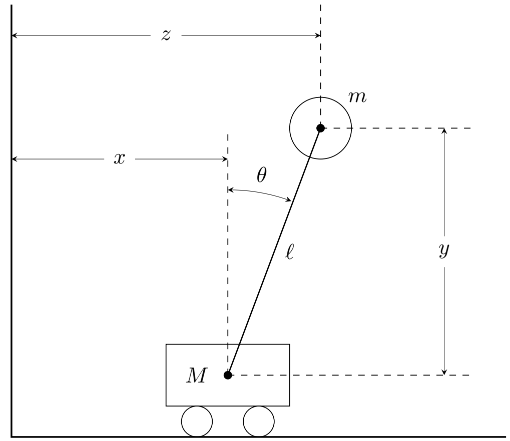


Figure E2.3: Inverted Pendulum Cart

(i) Write the kinetic energy of the cart, the kinetic energy of the pendulum, the potential energy of the cart, and the potential energy of the pendulum in terms of the variables x, $\dot{x}$, y, $\dot{y}$, z, and $\dot{z}$. (Assume that the rod has negligible mass and that the rotational kinetic energy is negligible.)

(ii) Now write the kinetic and potential energies you found in terms of only the variables x, $\dot{x}$, θ, and $\dot{\theta}$.

(iii) Write down the *Lagrangian* of the system $L := T - V$ where T is the total kinetic energy of the system and V is the total potential energy of the system.

(iv) Substitute the Lagrangian $L = L(x, \dot{x}, \theta, \dot{\theta})$ into the following *Euler-Lagrange equations*, taking care to compute the derivatives correctly:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}.$$

Simplify the resulting two equations.

Note: *It is a result in mechanics that the Euler-Lagrange equations are equivalent to Newton's law.*

(v) Now write down the dynamics of the system in the form

$$\ddot{x} = f(x, \dot{x}, \theta, \dot{\theta})$$
$$\ddot{\theta} = g(x, \dot{x}, \theta, \dot{\theta}).$$

Hint: You should have two equations and two unknowns for $\ddot{x}$ and $\ddot{\theta}$.

(vi) Do the dynamics (i.e., the functions f and g) depend on x or $\dot{x}$? What does this say about our system?

Note: *This alternative method of describing the dynamics is called* Lagrangian Mechanics. *Lagrangian mechanics allows us to find the equations of motion for a system without explicitly looking at the forces that elements exert on one another. For many systems (especially more complicated systems), this is the easiest way to find the equations of motion. Read more about this method at:* https://en.wikipedia.org/wiki/Lagrangian_mechanics

Answer:

(i) Kinetic energy of cart: $\frac{1}{2}M\dot{x}^2$, kinetic energy of pendulum: $\frac{1}{2}m(\dot{y}^2+\dot{z}^2)$, potential energy of cart: 0, potential energy of pendulum: mgy.

(ii) We have that $y = \ell\cos\theta$, and $z = x + \ell\sin\theta$. Taking derivatives accordingly and substituting yields the following. Kinetic energy of cart: $\frac{1}{2}M\dot{x}^2$, kinetic energy of pendulum: $\frac{1}{2}m(\ell^2\dot{\theta}^2+\dot{x}^2+2\ell\dot{x}\cos\theta\dot{\theta})$, potential energy of cart: 0, potential energy of pendulum: $mg\ell\cos\theta$.

(iii) The Lagrangian is given by

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\ell^2\dot{\theta}^2+\dot{x}^2+2\ell\dot{x}\cos\theta\dot{\theta}) - mg\ell\cos\theta.$$

(iv) After substituting and taking the derivatives, we have

$$\begin{aligned} m\ell^2\ddot{\theta} - m\ell\dot{x}\sin\theta\dot{\theta} + m\ell\ddot{x}\cos\theta &= -m\ell\dot{x}\sin\theta\dot{\theta} + mg\ell\sin\theta, \\ M\ddot{x} + m\ddot{x} + m\ell(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2) &= 0. \end{aligned}$$

Simplifying yields

$$\begin{aligned} \ell\ddot{\theta} + \ddot{x}\cos\theta - g\sin\theta &= 0, \\ (m+M)\ddot{x} + m\ell(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2) &= 0. \end{aligned}$$

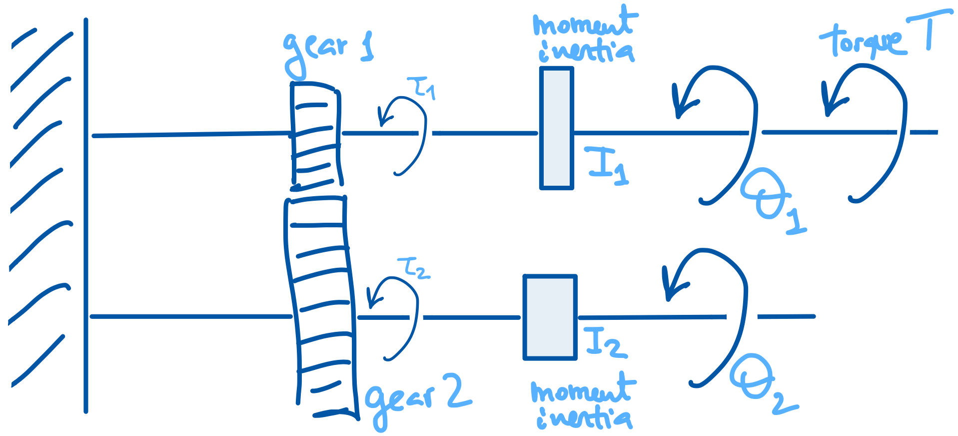
(v) We have two equations and two unknowns for $\ddot{x}$ and $\ddot{\theta}$. Substituting these equations into each other and isolating $\ddot{x}$ and $\ddot{\theta}$ yields the following equations of motion:

$$\begin{aligned} \ddot{x} &= \frac{m\ell\sin(\theta)\big(\dot{\theta}^2 - g\cos\theta\big)}{m+M-m\ell\cos^2\theta}, \\ \ddot{\theta} &= \frac{\sin\theta\big((m+M)g - m\ell\cos\theta\dot{\theta}^2\big)}{\ell(M+m\sin^2\theta)}. \end{aligned}$$

(vi) No, the dynamics do not depend on x or $\dot{x}$. This is because the dynamics are the same under all reference frames (which are either at rest or moving with an velocity). This means that the initial values that we choose for x and $\dot{x}$ are arbitrary – only their relative values matter.

■

E2.5 **Equations of motion for interconnected shafts.** Consider two parallel shafts with meshed gears as in figure.



Assume that

(i) the first shaft with angle θ_1 has moment of inertia I_1 and the second shaft with angle θ_2 has moment of inertia I_2;
(ii) the first shaft is subject to a external torque T, whereas no external torque is applied to the second shaft; and
(iii) the two shafts are interconnected via a pair of gears with n_1 teeth on the first gear and n_2 teeth on the second gear.

Show that the equations of motion are

$$\Big(I_1 + \frac{n_1^2}{n_2^2} I_2\Big)\ddot{\theta}_1 = T \tag{E2.8}$$

Hint: Write the two equations of motion for the two shafts, including torques τ_1 and τ_2 generated by the meshed gears. Then, using the equalities $n_1\dot{\theta}_1 = -n_2\dot{\theta}_2$ and $n_2\tau_1 = n_1\tau_2$, eliminate the intermediate variables $\dot{\theta}_2$, $\ddot{\theta}_2$, τ_1 and τ_2.

***Note:** The moment of inertia $I_1 + \frac{n_1^2}{n_2^2} I_2$ in equation (E2.8) is the equivalent moment of inertia of the interconnected shafts.*

Answer: When the shafts are not interconnected, we have

$$I_1\ddot{\theta}_1 = T \tag{E2.9}$$

$$I_2\ddot{\theta}_2 = 0 \tag{E2.10}$$

Add the gear interconnection and therefore two torques, call them τ_1 and τ_2:

$$I_1\ddot{\theta}_1 = T + \tau_1 \tag{E2.11}$$

$$I_2\ddot{\theta}_2 = \tau_2 \tag{E2.12}$$

Since $n_1\dot{\theta}_1 = -n_2\dot{\theta}_2$ and $n_2\tau_1 = n_1\tau_2$, we know

$$\dot{\theta}_2 = -\frac{n_1}{n_2}\dot{\theta}_1, \quad \ddot{\theta}_2 = -\frac{n_1}{n_2}\ddot{\theta}_1 \quad \text{and} \quad \tau_2 = \frac{n_2}{n_1}\tau_1 \tag{E2.13}$$

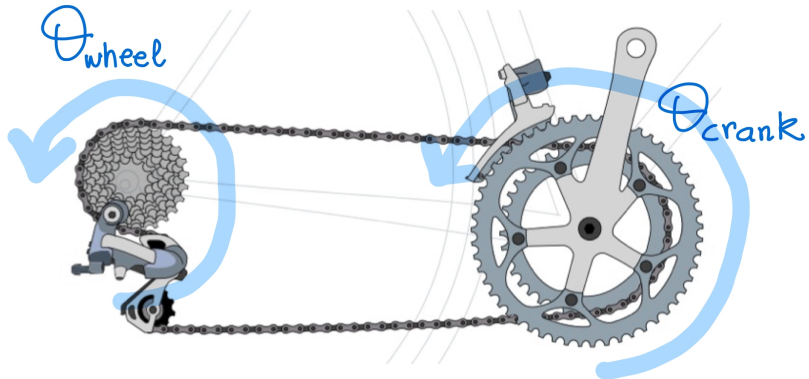
so that we can plug in and obtain

$$I_1\ddot{\theta}_1 = T + \tau_1 \tag{E2.14}$$

$$I_2\Big(-\frac{n_1}{n_2}\ddot{\theta}_1\Big) = \frac{n_2}{n_1}\tau_1 \tag{E2.15}$$

Equation (E2.8) follows from multiplying the second equation by $-n_1/n_2$ and summing the two equations, thereby eliminating the intermediate variable τ_1. ■

E2.6 **Sprockets and chains in bycicles.** In the image of a bycicle gear train below, the wheel angle θ_{wheel} and the crank angle θ_{crank} are measured counterclockwise (per convention in this text). In other words, while pedaling forward, both $\dot{\theta}_{\text{wheel}}$ and $\dot{\theta}_{\text{crank}}$ are negative, indicating counterclockwise motion.



(i) Do the wheel and crank sprockets satisfy the *equal tooth pitch* assumption? Justify your answer.

(ii) Write the no-slip condition for the interaction between the sprockets and the chain.

(iii) When biking uphill on a steep incline, explain which gear ratio is preferable. How does this compare to the preferable gear ratio when biking quickly on flat terrain?

Hint: Recall that: $\text{gear ratio} = \dfrac{\text{number of teeth on the output sprocket (rear wheel)}}{\text{number of teeth on the input sprocket (crank)}}$.

Notes: Many multi-speed bicycles feature a "50/34T crankset" coupled with an "11 speed 11-32 rear cassette." These numbers means:

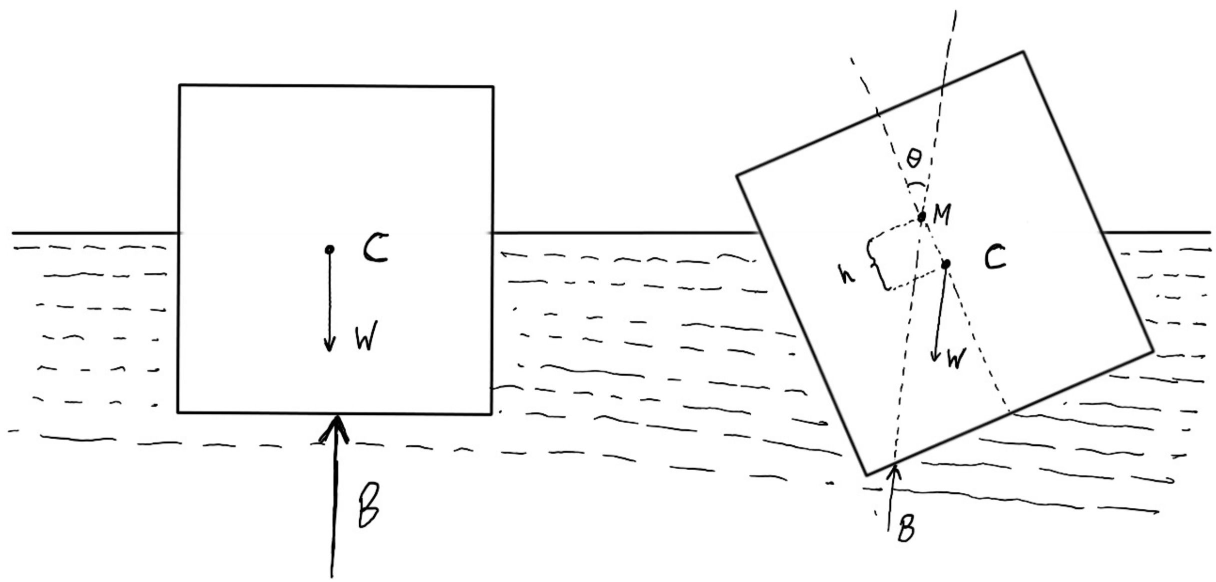
(i) the crankset has two "chainrings" with 50 and 34 teeth, respectively,

(ii) the rear cassette has 11 "cogs" ranging from 11 teeth up to 32 teeth (the exact number of teeth are 11/12/13/14/16/18/20/22/25/28/32).

The entries in the following table are all the possible gear ratios, calculated as the number of teeth on the front chainring divided by the number of teeth on the rear cog:

Chainring	11T	12T	13T	14T	16T	18T	20T	22T	25T	28T	32T
50T	$\frac{50}{11}=4.55$	$\frac{50}{12}=4.17$	$\frac{50}{13}=3.85$	$\frac{50}{14}=3.57$	$\frac{50}{16}=3.13$	$\frac{50}{18}=2.78$	$\frac{50}{20}=2.50$	$\frac{50}{22}=2.27$	$\frac{50}{25}=2.00$	$\frac{50}{28}=1.79$	$\frac{50}{32}=1.56$
34T	$\frac{34}{11}=3.09$	$\frac{34}{12}=2.83$	$\frac{34}{13}=2.62$	$\frac{34}{14}=2.43$	$\frac{34}{16}=2.13$	$\frac{34}{18}=1.89$	$\frac{34}{20}=1.70$	$\frac{34}{22}=1.55$	$\frac{34}{25}=1.36$	$\frac{34}{28}=1.21$	$\frac{34}{32}=1.06$

E2.7 **Equations of motion for ship on the ocean.** Consider an oceangoing ship. As shown in the left figure, when the ship is at rest, the balanced forces of weight W and buoyancy B act along the centerline of the ship, generating no torque. However, as shown in the right figure, when the ship is inclined at an angle θ (e.g., due to rough seas), the buoyancy force B shifts to the left, intersecting the centerline of the ship at a point called the *metacenter* and denoted as M. The hull of the ship is designed so that the metacenter M is above the ship's center of gravity C; this arrangement results in a restoring torque that stabilizes the ship. The vertical distance between the center of gravity C and the metacenter M is known as the *metacentric height* h.



(i) Derive the equation of motion for the inclined ship in terms of the inclination angle θ, the moment of inertia I_s, the weight W, and the metacentric height h. If no other forces are present, is there damping in this system?

(ii) Recall the small-angle approximations $\sin x \approx x$ and $\cos x \approx 1$ for x near zero. Assume a small inclination angle θ, and use the small-angle approximation to simplify the equation of motion from part (i). Identify the natural frequency ω_n of the system.

(iii) Write down the solution for the inclination angle θ as a function of time t, of the natural frequency ω_n and of the initial conditions $\theta(0), \dot{\theta}(0)$.

E2.8 **Mechanical modeling of a muscle.** A muscle connected to a fixed point and subject to a load force can be modeled by the equivalent mechanical system shown in the Figure E2.4. The key elements of the system are: (1) The muscle connects the fixed point to a mass m at position x. (2) The muscle is represented by the interconnection of two components, with the intermediate point at coordinate x_{interm}. (3) The muscle exerts a force F_{muscle} at the intermediate point. (4) A damper with damping coefficient b connects the intermediate point to the stationary point. (5) A spring with stiffness k connectes the intermediate point to the mass. (6) The mass m is subject to a load force F_{load}.

(i) Write a differential equation for the mass acceleration $\ddot{x}$ as a function of the load force F_{load}, the force generated by the muscle F_{muscle}, and the velocity of the intermediate point $\dot{x}_{\text{interm}}$.

Hint: First, use the free body diagram of the mass m. Next, consider the free body diagram of the intermediate point with zero mass; the net force on this intermediate point must be 0.

(ii) Determine the equilibrium condition that relates F_{load} and F_{muscle} such that the system is at rest (i.e., at an equilibrium, no motion).

(iii) Assuming the equilibrium condition holds, find the final length of the spring.

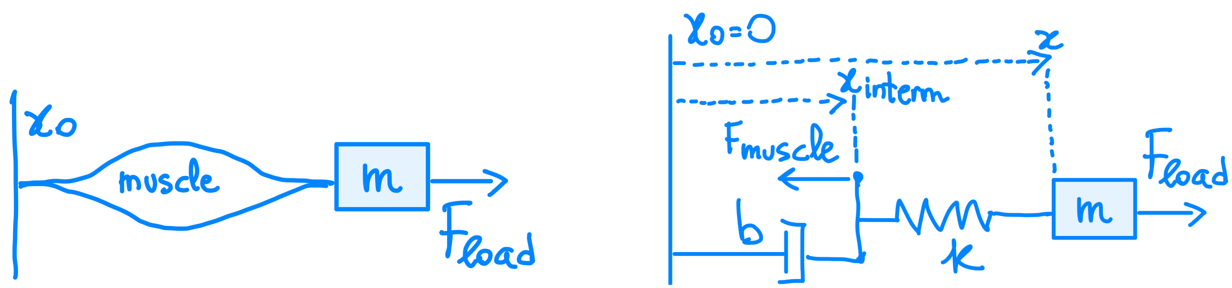
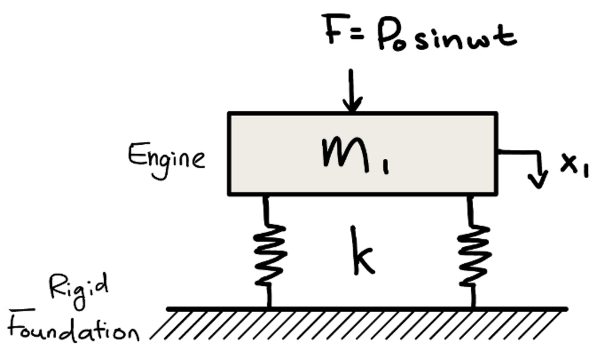


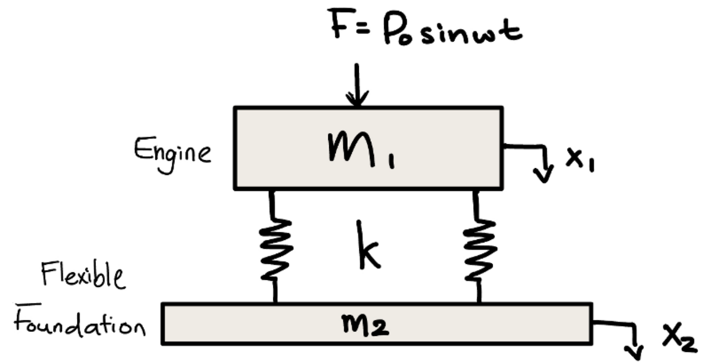
Figure E2.4: Left image: A muscle connected to a fixed point and subject to a load force. Right image: Equivalent mechanical system for the muscle excitation.

E2.9 **Rigid and flexible foundations in vibration isolation.** In many engineering applications, such as vehicles and machinery, controlling vibrations is critical to ensuring the stability and longevity of structures. Vibrating machinery is often mounted on structures, where it is necessary to reduce the transmission of vibrations. A common approach to isolating vibrations is by introducing a spring between the machine and the structure.

In this exercise, you will design a system where a vibrating machine is mounted on a structure. Our objective is to explore how the rigidity or flexibility of the foundation affects the system's dynamic properties.



(a) Rigid foundation



(b) Flexible foundation

Figure E2.5: Engine mounted on a rigid (left) and flexible (right) foundation, e.g., in a car. The parameter k is the total stiffness of the two springs. The engine mass is m_1, and, in the right image, m_2 is the mass of the foundation.

The system's motion is modeled using harmonic solutions of the form:

$$x_1 = x_{1m} \sin \omega t \tag{E2.16}$$

$$x_2 = x_{2m} \sin \omega t \tag{E2.17}$$

where x_{1m} and x_{2m} are the maximum oscillation amplitude of masses m_1 and m_2, respectively.

Rigid foundation system: Assume that the foundation m_2 is rigid.

(i) Write down the differential equations for the rigid foundation system as shown in Figure E2.5(a).

(ii) Find the squared natural frequency ω_n^2 of the rigid system as shown in Figure E2.5(a).

Flexible foundation system: Now, consider the flexible foundation system.

(iii) Write down the differential equations for the flexible foundation system in Figure E2.5(b).

(iv) Solve for the natural frequency squared ω_n^2 of the flexible system in Figure E2.5(b). Your final answer should be in terms of only the variables k, m_1, and m_2.

Comparative analysis: Finally, we perform a comparative analysis.

(v) If $m_1 = 10m_2$, what is the natural frequency of the system? How does it compare to the case where we assume the foundation is rigid?

(vi) Under what conditions is the natural frequency of the two systems equal?

Hint: Recall that the natural frequency is found by considering the solution to the un-forced system, i.e., the system with $F = 0$.
Hint: To find the natural frequency, you will need to substitute our assumed solution into the dynamical system and solve for the frequency.

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