Deformation

We will now consider the issue of deformation for solid (or fluid) bodies. In our previous work on mechanics, we were introduced to the idea of strain as the change of length of a bar of material divided by the initial length and we found that material behavior can be understood from analysis of strain. We now wish to address strain in a body that is experiencing a very general deformation. Therefore we consider a line element that can be anywhere in the body and calculate the strain of this line element. Thus, consider a point P in the body that has position vector $\underline{x}$ and a neighboring point Q that is infinitesimally far away at $\underline{x}+d\underline{x}$. It follows that the line element PQ is $d\underline{x}$ as shown in Fig. 1.

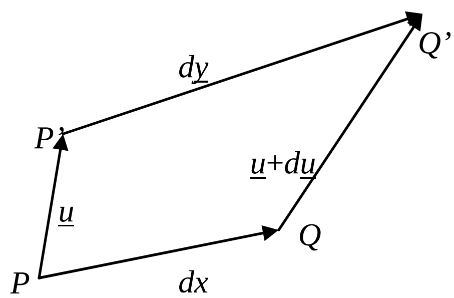

Figure 1

The body experiences a deformation and as a result, the point P moves to P' and Q moves to Q'. The displacement of point P is $\underline{u}$ and Q moves by the amount $\underline{u}+d\underline{u}$ since it is infinitesimally far from P. The line element $P'Q'$ is designated by $d\underline{y}$ and its length is $\sqrt{d\underline{y}.d\underline{y}}$ whereas PQ has length $dS=\sqrt{d\underline{x}.d\underline{x}}$. It follows that the strain ε of the line element PQ is given by

$$\varepsilon = \frac{\sqrt{d\underline{y}.d\underline{y}} - dS}{dS} \tag{1}$$

However, inspection of Fig. 1 shows us that

$$d\underline{y} = d\underline{x} + d\underline{u} = d\underline{x} + d\underline{x}.\nabla\underline{u} \tag{2}$$

where the final step in this result makes use of the properties of the identity tensor and the displacement gradient. As a consequence, the strain is

$$\varepsilon = \sqrt{1 + 2\frac{d\underline{x}.\nabla\underline{u}.d\underline{x}}{(dS)^2} + \frac{d\underline{x}.\nabla\underline{u}.(\nabla\underline{u})^T.d\underline{x}}{(dS)^2}} - 1 \qquad (3)$$

This calculation is quite valid but not very convenient for a number of reasons. There are a number of options that we could pursue to obtain more suitable formulations in general. However, we will not take this route, but instead consider the case of small or infinitesimal strain and look at this as a special case. This case is defined to be such that

$$\left|\frac{d\underline{x}.\nabla\underline{u}.(\nabla\underline{u})^T.d\underline{x}}{(dS)^2}\right| << \left|\frac{d\underline{x}.\nabla\underline{u}.d\underline{x}}{(dS)^2}\right| \qquad (4)$$

and

$$2\left|\frac{d\underline{x}.\nabla\underline{u}.d\underline{x}}{(dS)^2}\right| << 1 \qquad (5)$$

In this case, the expression in Eq. (3) can be expanded to give

$$\varepsilon \approx \frac{d\underline{x}.\nabla\underline{u}.d\underline{x}}{(dS)^2} \qquad (6)$$

where the omitted terms are negligible. Note that Eq. (3) is also correct when $\nabla\underline{u}$ is replaced by $(\nabla\underline{u})^T$ and *vice versa*. Thus we have

$$\varepsilon \approx \frac{d\underline{x}.(\nabla\underline{u})^T.d\underline{x}}{(dS)^2} \qquad (7)$$

as well and it follows that to first order

$$\frac{d\underline{x}.\left[\nabla\underline{u} - (\nabla\underline{u})^T\right].d\underline{x}}{(dS)^2} = 0 \qquad (8)$$

Therefore, the skew part of the displacement gradient has nothing to do with the strain. To recognize this, we define the symmetric part of the displacement gradient to be the strain tensor $\underline{\varepsilon}$ and thus

$$\varepsilon = \underline{n}.\underline{\varepsilon}.\underline{n} \tag{9}$$

where

$$\underline{n} = \frac{d\underline{x}}{dS} \tag{10}$$

and is thus a unit vector parallel to $d\underline{x}$ and the strain tensor is

$$\underline{\varepsilon} = \frac{1}{2}\left[\underline{\nabla}\underline{u} + \left(\underline{\nabla}\underline{u}\right)^T\right] \tag{11}$$

Note that in Cartesian coordinates the components of the strain tensor are

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{12}$$

Thus to compute the strain of a line in the material with the strain defined in the usual way as the change of length divided by the original length, we take the strain tensor as defined in Eq. (11) and take it inner product twice with a unit vector parallel to the line of concern, as indicated by Eq. (9). This can be done for any infinitesimal line and it can have any orientation $\underline{n}$. The result is valid as long as ε<<1.

That $\underline{\varepsilon}$ is a tensor is confirmed by the fact that the displacement gradient is a tensor.

Now that we know how to compute axial strain, let us take a look at shear strain. Consider an infinitesimal square *PQRS* identified on any plane at any location in the material, as illustrated in Fig. 2. The edge *PQ* is the vector $\underline{n}dS$, where $\underline{n}$ is a unit vector and the edge *PS* is $\underline{m}dS$ where $\underline{m}$ is also unit. Of course, due to orthogonality, $\underline{n}.\underline{m}$=0. Due to a heterogeneous deformation, all of the points *P, Q, R* & *S* will move in general. However, we imagine a rigid body motion of uniform displacement superposed so that *P* remains stationary. Such a superposed motion will not alter the strain. As a result, the displacement of *P* is zero, *Q* moves to *Q'* by the amount $dS\underline{n}.\underline{\nabla}\underline{u}$, the point *S* moves to *S'* by $dS\underline{m}.\underline{\nabla}\underline{u}$ and *R* moves to *R'* by the displacement $dS\left(\underline{n}+\underline{m}\right).\underline{\nabla}\underline{u}$, all as shown in Fig. 2. Note that these displacements in general will have components orthogonal to the plane containing $\underline{n}$ and $\underline{m}$. However, these components are not visible in Fig. 2, which is viewed orthogonally to the plane containing $\underline{n}$ and $\underline{m}$. The resulting shear strain γ to first order, computed in the normal way as the shear angle, is the sum of the angles *S'PS* and *QPQ'* and when it is much smaller than unity, this can be approximated by the sine of the sum of these angles. This, however, is equal to the cosine of *Q'PS'* and we can use the inner product of the vectors *PQ'* and *PS'* divided by the product of the magnitudes of these vectors to compute this cosine. The vector *PQ'* is $\underline{n}dS + dS\underline{n}.\underline{\nabla}\underline{u}$ and *PS'* is $\underline{m}dS + dS\underline{m}.\underline{\nabla}\underline{u}$ and the magnitudes of each of these vectors to first order is *dS* because the strains are assumed to be infinitesimal and therefore, the lengths hardly change. Thus the shear strain is computed as

$$\gamma \approx \frac{\left(\underline{n}dS + dS\underline{n}.\underline{\nabla}\underline{u}\right).\left(\underline{m}dS + dS\underline{m}.\underline{\nabla}\underline{u}\right)}{\left(dS\right)^2} \tag{13}$$

where the denominator has been computed already to first order with higher order terms omitted. We can then make use of the orthogonality of $\underline{n}$ and $\underline{m}$ to obtain

$$\gamma \approx \underline{n}.\underline{\nabla}\underline{u}.\underline{m} + \underline{m}.\underline{\nabla}\underline{u}.\underline{n} = \underline{n}.\left[\underline{\nabla}\underline{u} + \left(\underline{\nabla}\underline{u}\right)^T\right].\underline{m} = 2\underline{n}.\underline{\varepsilon}.\underline{m} \tag{14}$$

showing that the infinitesimal shear strain of any infinitesimal element of material defined by the orthogonal vectors $\underline{n}$ and $\underline{m}$ can be computed from the strain tensor $\underline{\varepsilon}$. The strain tensor thus contains all the information needed at a material point to allow computation of the axial and shear strain in any orientation at that point.

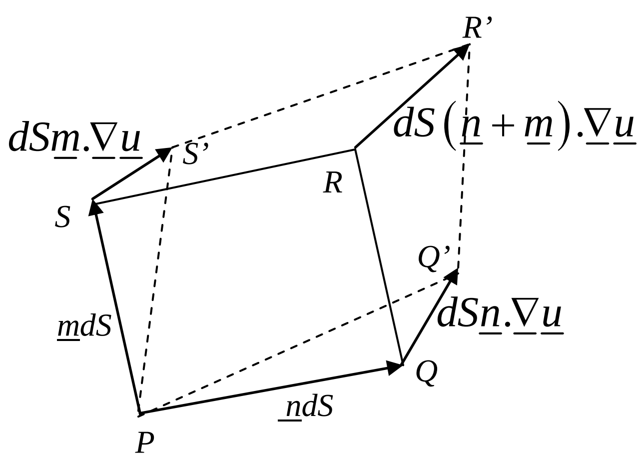

Figure 2.

Rotation We have established that the symmetric part of the displacement gradient gives us information about strain. What does the skew part tell us? The skew part is given by

$$\underline{\Omega} = \frac{1}{2}\left[\underline{\nabla}\underline{u} - \left(\underline{\nabla}\underline{u}\right)^T\right] \tag{15}$$

and therefore in Cartesian coordinates, the components are

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right) \tag{16}$$

It follows that

$$\underline{\nabla}\underline{u} = \underline{\varepsilon} + \underline{\Omega} \tag{17}$$

Eq. (8) tells us that the axial strain due to $\underline{\Omega}$ is zero for all line elements $d\underline{x}$ and therefore, $d\underline{x}.\underline{\Omega}$ is always orthogonal to $d\underline{x}$. It follows that

$$d\underline{x}.\underline{\Omega} = d\underline{x} \times \underline{\xi} \tag{18}$$

where $\underline{\xi}$ is some vector, since this is the only way that $d\underline{x}.\underline{\Omega}$ can be orthogonal to all $d\underline{x}$.

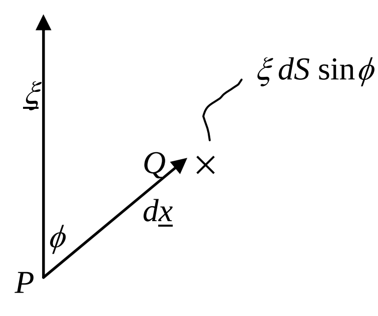

Figure 3

This situation is illustrated in Fig. 3 which is viewed orthogonally to the plane containing $\underline{\xi}$ and $d\underline{x}$. The cross product of these 2 vectors is therefore pointing towards us out of the plane of the paper, indicated by the cross representing a tip view of a vector arrow. From Eq. (18), we understand that the cross product represents a contribution to the displacement of one end of the line element $d\underline{x}$ relative to the other, *i.e.* the displacement of Q relative to P. Consideration of the geometric interpretation of the cross product indicates that the magnitude of this displacement is $\xi\, dS \sin\phi$, where ξ is the magnitude of $\underline{\xi}$, dS we know to be the length of $d\underline{x}$ and ϕ is the angle between the vectors $\underline{\xi}$ and $d\underline{x}$. Note that $dS \sin\phi$ is the radial distance from the vector $\underline{\xi}$ and to the point Q. Since this description is true for any infinitesimal line element $d\underline{x}$ emanating from P, we can conclude from study of Fig. 3 that in fact $d\underline{x}.\underline{\Omega}$ is a displacement due to a rigid rotation around the axis represented by $\underline{\xi}$, the magnitude of the angle of rotation is ξ and the direction of rotation is given by the left handed screw rule with the thumb pointing in the $\underline{\xi}$ direction. For this reason, the tensor $\underline{\Omega}$ is known as the rotation tensor (or more sloppily as the spin tensor – not very precise since spin suggests continuing movement). Thus we conclude that Eq. (17) represents a decomposition of the displacement gradient into the strain tensor that causes distortion of the material, and the rotation tensor that causes only a rigid rotation free of strain.

Since Eq. (18) can be rewritten in Cartesian components as

$$dx_i \Omega_{ij} = \varepsilon_{jkl} dx_k \xi_l \tag{19}$$

for any dx_i, we can deduce that

$$\Omega_{kj} = \varepsilon_{jkl} \xi_l \tag{20}$$

or in matrix notation

$$[\Omega] = \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix} \tag{21}$$

Thus the rotation tensor components tell us the vector around which the rigid rotation occurs and, of course, the angle of rotation, given by

$$\xi = \sqrt{\xi_i \xi_i} = \sqrt{-\frac{1}{2}\Omega_{ij}\Omega_{ji}} = \sqrt{-\frac{1}{2}\underline{\Omega} : \underline{\Omega}} \tag{22}$$

<u>Maximum Strain</u> Now that we know that the strain at a point in a body can be computed from the displacement gradient, we can ask the interesting question, “What is the largest strain at that point?” The reasons for wanting to do this are many. For example, some materials will fail if the strain exceeds a certain value. As we have seen, the axial strain ε at a point is computed by Eq. (9) and in general depends on the direction of $\underline{n}$. Thus, the axial strain is a function of orientation $\underline{n}$ and, in addition, it can vary with position in the body. For example, the strain is larger near the tip of a notch than it is elsewhere. In

general, we need to study the position dependence of the strain as well as the orientation dependence. We will defer studying the position dependence until we have solved some problems and instead consider the orientation dependence at a given point.

Thus we consider a stress tensor $\underline{\varepsilon}$ at a given point and take it as already having been established by some calculation or measurement. That is, all the components of the strain tensor are known and fixed. Therefore, we wish to compute the maximum value of ε from Eq. (9) in terms of the components of the unit vector $\underline{n}$. We can do this in the standard way and find all of the partial derivatives of ε with respect to each of the components n_i, set the partial derivatives to zero and thereby solve for the maximum value of ε. This is more or less what we will do. However, if we are not careful, we will end up actually minimizing the components of $\underline{n}$ and forcing them to disappear as a result (try it and see). This is inadmissible since we have already declared $\underline{n}$ to be a unit vector and it is its direction, not its magnitude that we want to vary to search out the largest value of ε. Therefore, we need to impose the constraint that $\underline{n}.\underline{n}$ must be equal to 1. This we will do by introducing a Lagrange multiplier λ. As a consequence, we will maximize a new version of ε given by

$$\varepsilon = \underline{n}.\underline{\varepsilon}.\underline{n} + \lambda\left(\underline{n}.\underline{n} - 1\right) \qquad (23)$$

This would seem to be a new quantity and not the axial strain. However, if we are successful, the maximum of ε will be found simultaneously with $\underline{n}.\underline{n}$-1 equal to zero and then as long as the Lagrange multiplier is finite, we will in fact have maximized the axial strain.

Instead of taking the partial derivatives with respect to the components of $\underline{n}$, we will inspect variations $\delta\varepsilon$ caused by variations $\delta\underline{n}$ and $\delta\lambda$. That is,

$$\delta\varepsilon = \delta\underline{n}.\underline{\varepsilon}.\underline{n} + \underline{n}.\underline{\varepsilon}.\delta\underline{n} + \delta\lambda\left(\underline{n}.\underline{n} - 1\right) + 2\lambda\underline{n}.\delta\underline{n} \qquad (24)$$

Variations are simply like infinitesimal increments and so the maximum strain is found by setting Eq. (24) to zero for all combinations of $\delta\underline{n}$ and $\delta\lambda$. In particular, $\delta\varepsilon$ must be zero when $\delta\underline{n}$ is zero and $\delta\lambda$ is non-zero. Thus,

$$\delta\lambda\left(\underline{n}.\underline{n} - 1\right) = 0 \qquad (25)$$

for $\delta\lambda$ non-zero and therefore requires $\underline{n}$ to be a unit vector. Similarly, $\delta\varepsilon$ must be zero when $\delta\lambda$ is zero and $\delta\underline{n}$ is non-zero. This leads to

$$\delta\underline{n}.\left(\underline{\varepsilon}.\underline{n} + \underline{\varepsilon}^T.\underline{n} + 2\lambda\underline{n}\right) = 0 \qquad (26)$$

where the definition of the transpose $\underline{\varepsilon}.\delta\underline{n} = \delta\underline{n}.\underline{\varepsilon}^T$ has been used. However, the strain tensor is symmetric and Eq. (26) is true for all $\delta\underline{n}$, so it can be rewritten as

$$\underline{\varepsilon}.\underline{n} + \lambda \underline{n} = 0 \tag{27}$$

This equation can be contracted with $\underline{n}$ to give

$$\underline{n}.\underline{\varepsilon}.\underline{n} + \lambda \underline{n}.\underline{n} = \varepsilon + \lambda = 0 \tag{28}$$

where the definition of ε and the fact that $\underline{n}$ is a unit vector have been used. Therefore, λ is $-\varepsilon$ and Eq. (27) then becomes

$$\left(\underline{\varepsilon} - \varepsilon \underline{I}\right).\underline{n} = 0 \tag{29}$$

This equation is an eigenvalue problem, showing that that the maximum axial strain is an eigenvalue of the strain matrix and the direction in which the maximum strain is oriented is the associated eigenvector. The eigenvalue problem requires for a solution

$$Det\left(\underline{\varepsilon} - \varepsilon \underline{I}\right) = 0 \tag{30}$$

and for a tensor the determinant is the same as one obtains for the matrix of components. Thus

$$\varepsilon^3 - I_1\varepsilon^2 - I_2\varepsilon - I_3 = 0 \tag{31}$$

where

$$\begin{aligned} I_1 &= trace\left(\underline{\varepsilon}\right) \\ I_2 &= \frac{1}{2}\left[\underline{\varepsilon} : \underline{\varepsilon} - I_1^2\right] \\ I_3 &= Det\left(\underline{\varepsilon}\right) \end{aligned} \tag{32}$$

or in Cartesian coordinates

$$\begin{aligned} I_1 &= \varepsilon_{kk} \\ I_2 &= \frac{1}{2}\left[\varepsilon_{ij}\varepsilon_{ji} - I_1^2\right] \end{aligned} \tag{33}$$

The parameters I_1, I_2 & I_3 are the strain invariants. They are named so because they are invariant under transformation of coordinate system. To see this, consider two orthonormal bases connected by the orthogonal transformation a_{ij} so that the coordinates of $\underline{\varepsilon}$ in one coordinate system are ε_{ij} and in the other are ε_{ij}' with the usual transformation

$$\varepsilon_{kl} = a_{ik}\varepsilon'_{ij}a_{jl} \tag{34}$$

If we substitute Eq. (34) into Eq. (33), we obtain

$$\begin{aligned} I_1 &= a_{ik}\varepsilon'_{ij}a_{jk} = a_{ik}a^T_{kj}\varepsilon'_{ij} = \delta_{ij}\varepsilon'_{ij} = \varepsilon'_{jj} \\ I_2 &= \frac{1}{2}\left[a_{ki}\varepsilon'_{kl}a_{lj}a_{mj}\varepsilon'_{mn}a_{ni} - I_1^2\right] = \frac{1}{2}\left[\delta_{kn}\varepsilon'_{kl}\delta_{lm}\varepsilon'_{mn} - I_1^2\right] \\ &= \frac{1}{2}\left[\varepsilon'_{nl}\varepsilon'_{ln} - I_1^2\right] \end{aligned} \tag{35}$$

Thus the 1st and 2nd invariants of strain have the same value whether they are computed in the unprimed or the primed coordinate system. A similar proof shows that the 3rd invariant also has the same value in both coordinate systems. These results are general and invariants are the same in any coordinate system. Furthermore this feature applies to any tensor, not just the strain tensor.

The invariants are sometimes written in different combinations. For example, the first and second of Eq. (32) may be combined to show that $\underline{\varepsilon}:\underline{\varepsilon}$ is also invariant. Thus we conclude that the trace, the sum of the squares of the components and the determinant of a symmetric tensor are all invariant to coordinate transformation.

Now we can return to Eq. (31), the characteristic equation for the eigenvalue problem, which we now realize is also invariant to coordinate transformation. Therefore, we do not have to worry that the results for maximizing ε will somehow be dependent on the coordinate system chosen. Clearly, Eq. (31) is cubic so in general there will be 3 solutions, which we will designate ε_{I}, $\varepsilon_{\mathrm{II}}$ and $\varepsilon_{\mathrm{III}}$ which we will define to be in descending order, so that ε_{I} is the largest of the solutions and therefore, the maximum axial strain. The next step is to substitute these values into Eq. (29) and solve for the eigenvectors $\underline{n}^{\mathrm{I}}$, $\underline{n}^{II}$ & $\underline{n}^{III}$ associated respectively with these 3 eigenvalues. The solutions for the eigenvectors are obtained by the usual manipulations of linear algebra for solving linear simultaneous equations. However, it is important to normalize the solutions so that they are indeed unit vectors.

So at this stage we have completed out mission of calculating the largest axial strain at a point, given by ε_{I} and the direct in which it is oriented, $\underline{n}^{\mathrm{I}}$, so that we can tell whether the material will fail and in what direction this failure will occur, if that is an appropriate criterion for the reliability of the material. However, it is interesting to do some further investigation and see if there is any relationship among the 3 eigenvectors for the 3 eigenvalues. To do this, we write down the 3 conditions satisfied by the eigenvalues and eigenvectors:

$$\begin{aligned} \underline{\varepsilon}.\underline{n}^{I} &= \varepsilon_{I}\underline{n}^{I} \\ \underline{\varepsilon}.\underline{n}^{II} &= \varepsilon_{II}\underline{n}^{II} \\ \underline{\varepsilon}.\underline{n}^{III} &= \varepsilon_{III}\underline{n}^{III} \end{aligned} \tag{36}$$

Take the inner product of the 1st equation with the 2nd eigenvalue and the inner product of the 2nd equation with the 1st eigenvalue to obtain

$$\begin{aligned} \underline{n}^{II}.\underline{\varepsilon}.\underline{n}^{I} &= \varepsilon_{I}\underline{n}^{II}.\underline{n}^{I} \\ \underline{n}^{I}.\underline{\varepsilon}.\underline{n}^{II} &= \varepsilon_{II}\underline{n}^{I}.\underline{n}^{II} \end{aligned} \tag{37}$$

Because of the symmetry of the strain tensor, the left hand sides of these 2 equations are the same and the inner products on the right are equal though the coefficients are in general different. Thus subtracting one from the other gives

$$\left(\varepsilon_{I} - \varepsilon_{II}\right)\underline{n}^{I}.\underline{n}^{II} = 0 \tag{38}$$

This can be repeated pair wise with all the equations to obtain

$$\begin{aligned} \left(\varepsilon_{I} - \varepsilon_{II}\right)\underline{n}^{I}.\underline{n}^{II} &= 0 \\ \left(\varepsilon_{II} - \varepsilon_{III}\right)\underline{n}^{II}.\underline{n}^{III} &= 0 \\ \left(\varepsilon_{III} - \varepsilon_{I}\right)\underline{n}^{III}.\underline{n}^{I} &= 0 \end{aligned} \tag{39}$$

The simplest case is where all the eigenvalues are distinct so that the terms in the parentheses in Eq. (39) are non-zero. It then follows that

$$\begin{aligned} \underline{n}^{I}.\underline{n}^{II} &= 0 \\ \underline{n}^{II}.\underline{n}^{III} &= 0 \\ \underline{n}^{III}.\underline{n}^{I} &= 0 \end{aligned} \tag{40}$$

showing that the eigenvectors are mutually orthogonal. They are also normalized so in fact they form an orthonormal basis. When the eigenvalues are indistinct, either in a pair or all 3, orthogonal eigenvectors can be arbitrarily defined in the same way even though the conditions from Eq. (39) become indefinite. This step is possible because any unit vector in a plane orthogonal to the eigenvector for the distinct eigenvalue (in the case of a pair of equal eigenvalues) or any unit vector at all (when all 3 eigenvalues are the same) can serve as eigenvectors. So, the bottom line is that Eq. (40) is always true (or can be made to be true in all cases). These directions (*i.e.* the eigenvector directions) are called the principal axes of the strain tensor and the eigenvalues of the strain tensor are called the principal strains. The principal strains are thus in mutually orthogonal directions, with one (ε_{I}) being the largest strain associated with the state of strain, one ($\varepsilon_{\mathrm{III}}$) being the smallest and one ($\varepsilon_{\mathrm{II}}$) being intermediate to the other 2, but having a stationary characteristic with respect to orientation. Note that the principal strains can have equal values, either pair-wise or all 3, and as noted above this is associated with some degeneracy in the principal directions which become arbitrary to some extent.

Furthermore, the right hand sides of Eq. (37) are zero, showing that in coordinate axes aligned with the eigenvectors, the shear stains are zero. This is true for all 3 components of shear strain relative to principal axes since Eq. (37) can be repeated for each pair of eigenvectors. Thus, the strain tensor can be written

$$\underline{\varepsilon} = \varepsilon_I \underline{n}^I \underline{n}^I + \varepsilon_{II} \underline{n}^{II} \underline{n}^{II} + \varepsilon_{III} \underline{n}^{III} \underline{n}^{III} \tag{41}$$

which is just dyadic notation for the tensor in the orthonormal basis formed by the eigenvectors.

In summary, the maximum and minimum axial strain are found in a state of strain with zero shear strain and are in mutually orthogonal directions. The maximum (and minimum) axial strain is an eigenvalue of the strain tensor and the direction in which it acts is an eigenvector. The eigenvalues of the strain tensor are also called the principal strains and their associated directions are called the principal directions or axes.

<u>Volume strain</u> We can now ask ourselves how we can compute the volume strain, which is the change in volume divided by the initial volume. This can be done readily in principal axes of strain because there is no shear strain. Therefore, a cube with edges parallel to the principal axes becomes a cuboid as shown in Fig. 3. The initial volume is $(dS)^3$ and the final volume is $(dS)^3\,(1+\varepsilon_{\mathrm{I}})\,(1+\varepsilon_{\mathrm{II}})\,(1+\varepsilon_{\mathrm{III}})$. Therefore, the volume strain is

$$\frac{\Delta V}{V} = \left(1+\varepsilon_I\right)\left(1+\varepsilon_{II}\right)\left(1+\varepsilon_{III}\right) - 1 = \varepsilon_I + \varepsilon_{II} + \varepsilon_{III} \tag{42}$$

where the last step is achieved by neglecting terms much smaller than 1. However, the sum of the principal strains is the trace of the strain tensor and this is invariant. Therefore, in any coordinate system

$$\frac{\Delta V}{V} = trace\,\underline{\varepsilon} \tag{43}$$

and in Cartesian coordinates, it is given by ε_{kk}.

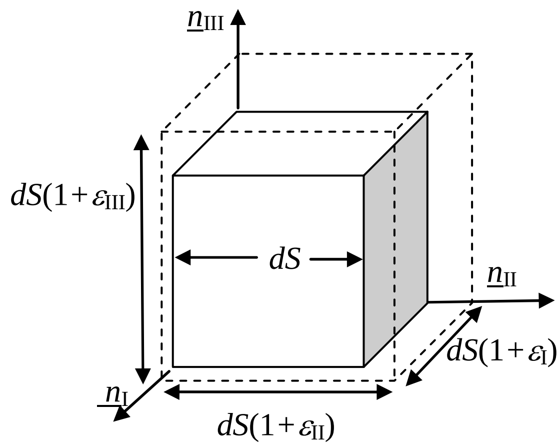

Figure 3

We can also get this result from use of the divergence theorem. Consider Fig. 4, where a volume *V* enclosed by surface *S* having outward unit normal $\underline{n}$ is depicted. The displacement of material points at the surface carries material outward normal to the surface by the amount $\underline{u}.\underline{n}$ as depicted. Therefore, the volume increase of the body due to material passing out through the surface element *dS* is $\underline{u}.\underline{n}\, dS$. Integrating this around the external surface of the body due compute the total change of volume gives

$$\Delta V = \int_S \underline{n}.\underline{u}\, dS \tag{44}$$

and dividing this by *V* gives the average volume strain of the body. However, the divergence theorem can be used to provide

$$\frac{\Delta V}{V} = \frac{1}{V}\int_V \underline{\nabla}.\underline{u}\, dV = \frac{1}{V}\int_V trace\,\underline{\varepsilon}\, dV \tag{45}$$

This result is true for any portion of any continuum body, including infinitesimal volumes in which the extent of integration becomes ΔV, providing the result in Eq. (43).

<u>Deviatoric strain</u> Because the trace of the strain tensor is the volume strain, it is interesting to look at the rest of the strain tensor, since it is associated with distortion without volume change, whereas the trace is a volume change without distortion as can

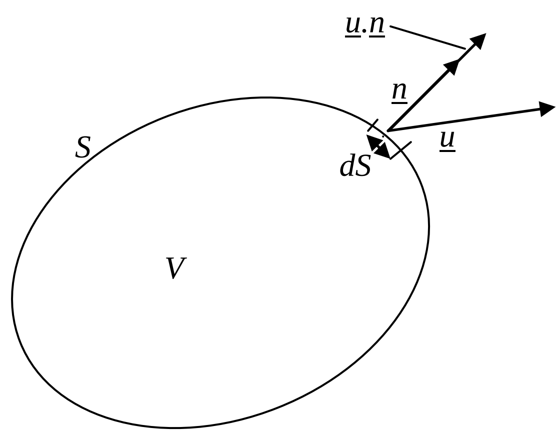

Figure 4

be deduced from Fig. 3. By subtracting out the trace times an identity tensor and dividing by 3, we obtain the deviatoric part $\underline{e}$ of the strain $\underline{\varepsilon}$. Thus

$$\underline{e} = \underline{\varepsilon} - \frac{1}{3}\underline{I}\, trace\, \underline{\varepsilon} \qquad (46)$$

or, more conveniently, in Cartesian coordinates

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} \qquad (47)$$

We can calculate the trace of the deviatoric strain as

$$e_{jj} = \varepsilon_{jj} - \frac{1}{3}\varepsilon_{kk}\delta_{jj} = 0 \qquad (48)$$

since it can easily be deduced that in 3-dimensions, $\delta_{jj} = 3$. Thus, the deviatoric strain is indeed free of volume change and therefore records the distortion in the material as opposed to its dilatation. In contrast the trace of the strain tensor represents a spherical dilatation (imagine a sphere increasing its diameter just enough to give the correct change of volume) and then the deviatoric strain can be imagined to be superposed on the dilated material to cause the shape distortion.

The deviatoric strain is important in cases where features of the material behavior are volume preserving such as is usually the case with plastic deformation and creep of metals.

Deformation rate and spin rate Now let us consider a truly infinitesimal displacement equal to a velocity $\underline{v}$ times an infinitesimal increment of time dt so that $\underline{u} = \underline{v}\, dt$. For this displacement, everything we have said about the infinitesimal strain in an approximate way becomes exact. When we divide everything by the infinitesimal time increment, we obtain some tensors that have an exact interpretation as rates of change. For example, the tensor

$$\underline{D} = \frac{1}{2}\left[\underline{\nabla}\,\underline{v} + \left(\underline{\nabla}\,\underline{v}\right)^T \right] \tag{46}$$

is called the rate of deformation tensor (sometimes the strain-rate tensor) and $D = \underline{n}.\underline{D}.\underline{n}$ is exactly the strain rate of an infinitesimal line element parallel to $\underline{n}$. The spin tensor is

$$\underline{\omega} = \frac{1}{2}\left[\underline{\nabla}\,\underline{v} - \left(\underline{\nabla}\,\underline{v}\right)^T \right] \tag{47}$$

and it represents a rigid body spin about an axis $\underline{\zeta}$ at a spin rate ζ (the magnitude of $\underline{\zeta}$) and the axis vector is calculated from $\underline{\omega}$ in exactly the same way that $\underline{\xi}$ is calculated from $\underline{\Omega}$. The largest strain rate in the material is an eigenvalue of $\underline{D}$ and its principal values and directions are found in exactly the same way that we found those for the strain tensor. The rate of deformation tensor is useful in fluid mechanics and in many nonlinear problems in solid mechanics such as plasticity, creep and viscoelasticity.