Contraction Theory for Optimization, Control, and Neural Networks

Francesco Bullo

Center for Control,
Dynamical Systems & Computation
University of California at Santa Barbara
https://fbullo.github.io

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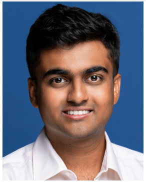
Acknowledgments

Veronica Centorrino
Scuola Sup Meridionale / ETH

Alexander Davydov
Rice University



Anand Gokhale
UC Santa Barbara

Emiliano Dall'Anese
Boston University

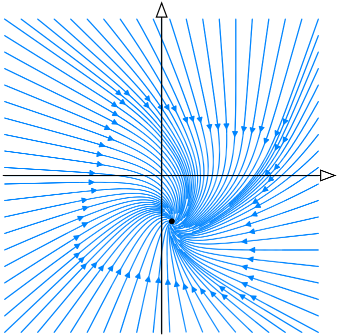
Anton Proskurnikov
Politecnico Torino

Giovanni Russo
Univ Salerno

AFOSR ARO ONR DTRA/ERDC

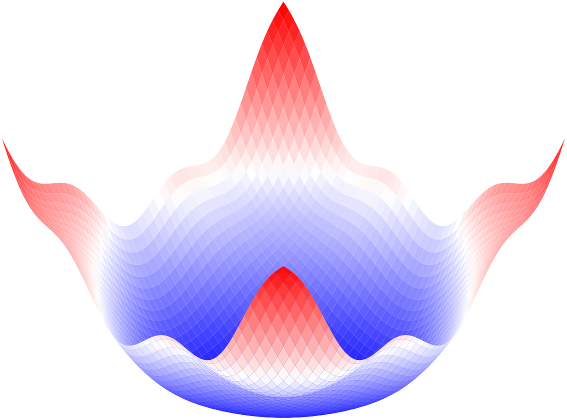
Frederick Leve @AFOSR FA9550-22-1-0059
Marc Steinberg @ONR N00014-22-1-2813
Donald Wagner @AFOSR FA9550-21-1-0203
Derya Cansever @ARO W911NF-24-1-0228

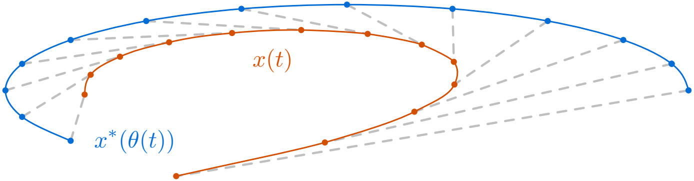
Outline

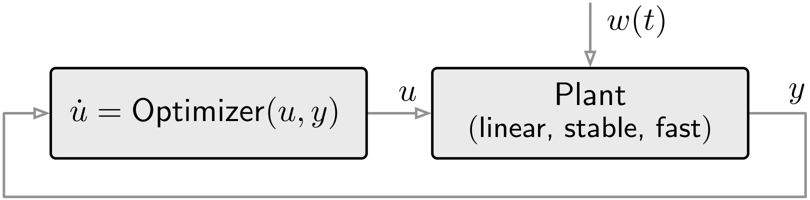


contractivity = robust computationally-friendly stability

fixed point theory + Lyapunov stability theory + geometry of metric spaces




$x(t)$
$x^*(\theta(t))$



optimization via dynamical systems

online time-varying optimization, optimization-based feedback control, ...

Contraction theory: historical notes

- **Origins**

  S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3(1):133–181, 1922. doi

- **Dynamics**:

  G. Dahlquist. *Stability and error bounds in the numerical integration of ordinary differential equations*. PhD thesis, (Reprinted in Trans. Royal Inst. of Technology, No. 130, Stockholm, Sweden, 1959), 1958

  S. M. Lozinskii. Error estimate for numerical integration of ordinary differential equations. I. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, 5:52–90, 1958. URL **http://mi.mathnet.ru/eng/ivm2980**. (in Russian)

- **Computation**:

  C. A. Desoer and H. Haneda. The measure of a matrix as a tool to analyze computer algorithms for circuit analysis. *IEEE Transactions on Circuit Theory*, 19(5):480–486, 1972. doi

- **Systems and control**:

  W. Lohmiller and J.-J. E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696, 1998. doi

- **Incomplete list of active scientists**
  Aminzare, Andrieu, Arcak, Astolfi, Chung, Coogan, Corless, Dall'Anese, Di Bernardo, Giesl, Kawano, Manchester, Margaliot, Martins, Ngoc, Pavel, Pavlov, Praly, Pham, Proskurnikov, Russo, Sepulchre, Slotine, Sontag, Tarbouriech, ...

- **Surveys and Perspectives**:

Z. Aminzare and E. D. Sontag. Contraction methods for nonlinear systems: A brief introduction and some open problems. In *IEEE Conf. on Decision and Control*, pages 3835–3847, Dec. 2014.

M. Di Bernardo, D. Fiore, G. Russo, and F. Scafuti. Convergence, consensus and synchronization of complex networks via contraction theory. In *Complex Systems and Networks*. Springer, 2016.

H. Tsukamoto, S.-J. Chung, and J.-J. E. Slotine. Contraction theory for nonlinear stability analysis and learning-based control: A tutorial overview. *Annual Reviews in Control*, 52:135–169, 2021.

P. Giesl, S. Hafstein, and C. Kawan. Review on contraction analysis and computation of contraction metrics. *Journal of Computational Dynamics*, 10(1):1–47, 2023.

A. Davydov and F. Bullo. Perspectives on contractivity in control, optimization and learning. *IEEE Control Systems Letters*, 8:2087–2098, 2024a.

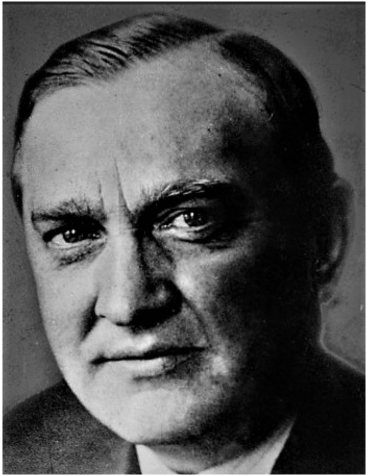
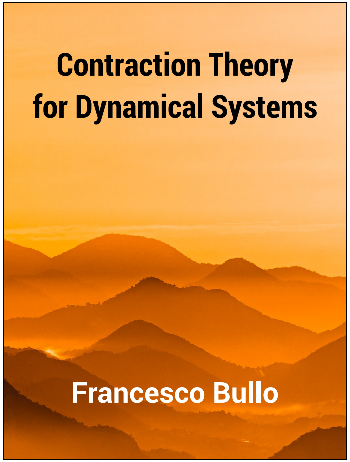


Figure: Stefan Banach (Krakow, 30 Mar 1892 – Lviv, 31 Aug 1945) was a self-taught Polish mathematician

1920: doctoral thesis on Banach spaces @ University of Lviv
1920-1922: Assistant Professor @ Lwow Polytechnic
1922: Full Professor @ Lwow Polytechnic
1924: Member of the Polish Academy of Arts and Sciences
1929: Founder, Lvov School of Mathematics
1931: first functional analysis: “Theory of Linear Operations”
1939-45: dark years

S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3(1):133–181, 1922.

Ongoing education and research on Contraction Theory



"Continuous improvement is better than delayed perfection" **Mark Twain**

- *Textbook*: Contraction Theory for Dynamical Systems, Francesco Bullo, rev 1.2, Aug 2024. (PDF freely available) **https://fbullo.github.io/ctds**
- *Tutorial slides*: **https://fbullo.github.io/ctds**
- *Youtube lectures*: "Minicourse on Contraction Theory" **https://youtu.be/FQV5PrRHks8** 6 lectures, total 12h
- *upcoming 2025 IEEE CDC Tutorial Session on "Contraction Theory in Control, Optimization, and Learning"*

Examples of contracting dynamics:

- V. Centorrino, A. Gokhale, A. Davydov, G. Russo, and F. Bullo. Euclidean contractivity of neural networks with symmetric weights. *IEEE Control Systems Letters*, 7:1724–1729, 2023. doi
- R. Delabays and F. Bullo. Semicontraction and synchronization of Kuramoto-Sakaguchi oscillator networks. *IEEE Control Systems Letters*, 7:1566–1571, 2023. doi

Applications to machine learning:

- S. Jafarpour, A. Davydov, A. V. Proskurnikov, and F. Bullo. Robust implicit networks via non-Euclidean contractions. In *Advances in Neural Information Processing Systems*, Dec. 2021. doi
- S. Jaffe, A. Davydov, D. Lapsekili, A. K. Singh, and F. Bullo. Learning neural contracting dynamics: Extended linearization and global guarantees. In *Advances in Neural Information Processing Systems*, 2024. doi

Application to neuroscience:

- V. Centorrino, A. Gokhale, A. Davydov, G. Russo, and F. Bullo. Positive competitive networks for sparse reconstruction. *Neural Computation*, 36(6):1163–1197, 2024. doi

Applications to optimization-based control:

- A. Davydov and F. Bullo. Exponential stability of parametric optimization-based controllers via Lur'e contractivity. *IEEE Control Systems Letters*, 8:1277–1282, 2024b. doi
- Z. Marvi, F. Bullo, and A. G. Alleyne. Control barrier proximal dynamics: A contraction theoretic approach for safety verification. *IEEE Control Systems Letters*, 8:880–885, 2024. doi
- Y. Chen, F. Bullo, and E. Dall'Anese. Sampled-data systems: Stability, contractivity and single-iteration suboptimal MPC. *IEEE Transactions on Automatic Control*, 2025. doi. Submitted
- **A. Davydov, V. Centorrino, A. Gokhale, G. Russo, and F. Bullo. Time-varying convex optimization: A contraction and equilibrium tracking approach. IEEE Transactions on Automatic Control, 70(11), 2025. doi. To appear**

Outline

Continuous-time dynamics and one-sided Lipschitz constants

$$\dot{x} = \mathsf{F}(x) \qquad \text{on } \mathbb{R}^n \text{ with norm } \|\cdot\| \text{ and induced log norm } \mu(\cdot)$$

One-sided Lipschitz constant (≈ **maximum expansion rate)**

$$\mathsf{osLip}(\mathsf{F}) \;=\; \sup_x \mu(D\mathsf{F}(x))$$

For **scalar map** f, $\mathsf{osLip}(f) = \sup_x f'(x)$

For **affine map** $\mathsf{F}_A(x) = Ax + a$

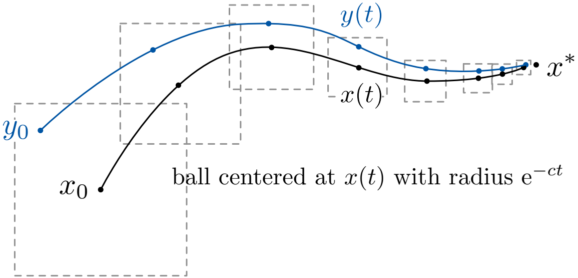
$$\mathsf{osLip}_{2,P}(\mathsf{F}_A) = \mu_{2,P}(A) \leq \ell \qquad \iff \qquad A^\top P + AP \preceq 2\ell P$$

$$\mathsf{osLip}_{\infty}(\mathsf{F}_A) = \mu_{\infty}(A) \leq \ell \qquad \iff \qquad a_{ii} + \sum_{j \neq i} |a_{ij}| \leq \ell$$

Banach contraction theorem for continuous-time dynamics

If $-c := \mathrm{osLip}(\mathsf{F}) < 0$, then

1. F is **infinitesimally contracting:** $\|x(t) - y(t)\| \leq e^{-ct}\|x_0 - y_0\|$
2. F has a unique, glob exp stable equilibrium x^*



Example contracting systems

1. **gradient descent flows** under strong convexity assumptions
   (proximal, primal-dual, distributed, Hamiltonian, saddle, pseudo, best response, etc)
2. **neural network dynamics** under assumptions on synaptic matrix
   (recurrent, implicit, reservoir computing, etc)
3. **Lur'e-type systems** under assumptions on nonlinearity and LMI conditions
   (Lipschitz, incrementally passive, monotone, conic, etc)
4. **interconnected systems** under contractivity and small-gain assumptions
   (Hurwitz Metzler matrices, network small-gain theorem, etc)

5. data-driven learned models (imitation learning)
6. feedback linearizable systems with stabilizing controllers
7. incremental ISS systems
8. nonlinear systems with a locally exponentially stable equilibrium
   are contracting with respect to appropriate Riemannian metric

Example #1: Gradient dynamics for strongly convex function

Given differentiable, strongly convex $f : \mathbb{R}^n \to \mathbb{R}$ with parameter $\nu > 0$, **gradient dynamics**

$$\dot{x} = \mathsf{F}_{\mathsf{G}}(x) := -\nabla f(x)$$

F_{G} is infinitesimally contracting wrt $\|\cdot\|_2$ with rate ν

unique globally exp stable point is global minimum

Convexity & Contractivity Theorem: For differentiable $f : \mathbb{R}^n \to \mathbb{R}$, equivalent statements:

1. f is **strongly convex** with parameter ν (and minimum x^*)
2. $-\nabla f$ is **ν-strongly infinitesimally contracting** (with equilibrium x^*)

Euler Discretization Theorem for Contracting Dynamics

Given norm $\|\cdot\|$ and differentiable and Lipschitz $\mathsf{F} : \mathbb{R}^n \to \mathbb{R}^n$, equivalent statements

1. $\dot{x} = \mathsf{F}(x)$ is infinitesimally contracting
2. there exists $\alpha > 0$ such that $x_{k+1} = x_k + \alpha \mathsf{F}(x_k)$ is contracting

"The great watershed in optimization is not between linearity and nonlinearity, but convexity and nonconvexity." R.T. Rockafellar

R. I. Kachurovskii. Monotone operators and convex functionals. *Uspekhi Matematicheskikh Nauk*, 15(4):213–215, 1960

S. Jafarpour, A. Davydov, A. V. Proskurnikov, and F. Bullo. Robust implicit networks via non-Euclidean contractions. In *Advances in Neural Information Processing Systems*, Dec. 2021.

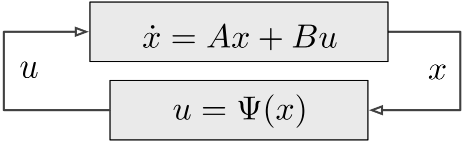
Example #2: Parametric convex optimization and contracting dynamics

Many convex optimization problems can be solved with contracting dynamics

$$\dot{x} = \mathsf{F}(x, \theta)$$

	Convex Optimization	**Contracting Dynamics**
Unconstrained	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$	$\dot{x} = -\nabla_x f(x, \theta)$
Constrained	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$ s.t. $\quad x \in \mathcal{X}(\theta)$	$\dot{x} = -x + \mathrm{Proj}_{\mathcal{X}(\theta)}(x - \gamma \nabla_x f(x, \theta))$
Composite	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta) + g(x, \theta)$	$\dot{x} = -x + \mathrm{prox}_{\gamma g_\theta}(x - \gamma \nabla_x f(x, \theta))$
Equality	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$ s.t. $\quad Ax = b(\theta)$	$\dot{x} = -\nabla_x f(x, \theta) - A^\top \lambda,$ $\dot{\lambda} = Ax - b(\theta)$
Inequality	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$ s.t. $\quad Ax \leq b(\theta)$	$\dot{x} = -\nabla f(x, \theta) - A^\top \nabla M_{\gamma, b(\theta)}(Ax + \gamma\lambda),$ $\dot{\lambda} = \gamma(-\lambda + \nabla M_{\gamma, b(\theta)}(Ax + \gamma\lambda))$

Example #3: Systems in Lur'e form



For $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times n}$, **nonlinear system in Lur'e form**

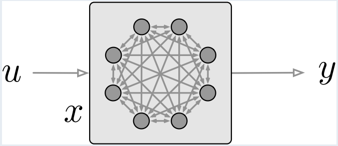
$$\dot{x} = Ax + B\Psi(x) \quad =: \mathsf{F}_{\mathsf{Lur'e}}(x)$$

where $\Psi : \mathbb{R}^m \to \mathbb{R}^m$ is described by an **incremental multiplier matrix** M

For $P = P^\top \succ 0$, following statements are equivalent:

1. $\mathsf{F}_{\mathsf{Lur'e}}$ infinitesimally contracting wrt $\|\cdot\|_{2,P}$ with rate $\eta > 0$ for each Ψ described by M,
2. $\exists \lambda \geq 0$ such that $\begin{bmatrix} PA + A^\top P + 2\eta P & PB \\ B^\top P & \mathbb{0}_{m \times m} \end{bmatrix} + \lambda M \preceq 0$

Example #4: Firing-rate networks for implicit ML via ℓ_∞



$$\dot{x} = -x + \Phi(Ax + Bu + b) \qquad \text{(recurrent NN)}$$

$$x = \Phi(Ax + Bu + b) \qquad \text{(implicit NN)}$$

$$x_{k+1} = (1-\alpha)x_k + \alpha\Phi(Ax_k + Bu + b) \qquad \text{(Euler discrt.)}$$

If

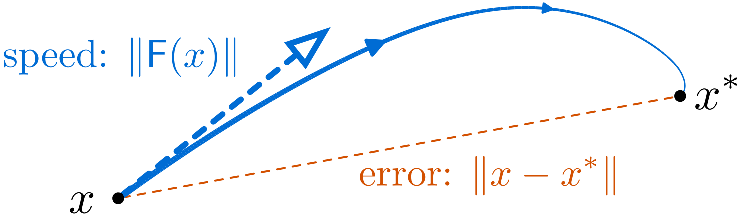
$$\mu_\infty(A) < 1 \qquad \Big(\text{i.e., } a_{ii} + \sum_{j\neq i} |a_{ij}| < 1 \text{ for all } i\Big)$$

- **recurrent NN is infinitesimally contracting** with rate $1 - \mu_\infty(A)_+$
- **implicit NN is well posed**
- **Euler discretization is contracting** at $\alpha^* = (1 - \min_i (a_{ii})_-)^{-1}$

Outline

Speed and error

osLip(F) $= -c < 0$ and x^* is equilibrium



The two canonical Lyapunov functions

Lyapunov functions. If $\mathsf{osLip}(\mathsf{F}) = -c < 0$ and $\mathsf{F}(x^*) = \mathbb{0}_n$, then

1. two global Lyapunov functions:

$$x \mapsto \|x - x^*\| \qquad \text{(error)}$$

$$x \mapsto \|\mathsf{F}(x)\| \qquad \text{(speed)}$$

2. for each $x(0) = x_0$ and $t \in \mathbb{R}_{\geq 0}$,

$$\|x(t) - x^*\| \leq \mathrm{e}^{-ct}\|x_0 - x^*\| \qquad \text{(error)}$$

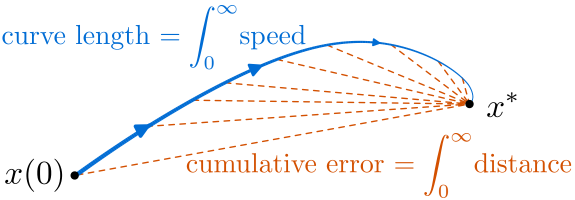
$$\|\mathsf{F}(x(t))\| \leq \mathrm{e}^{-ct}\|\mathsf{F}(x_0)\| \qquad \text{(speed)}$$

Additionally, $\mathsf{cost} : \mathbb{R}^n \to \mathbb{R}$ such that

$$x^* = \operatorname*{argmin}_x \mathsf{cost}(x) \qquad \text{and} \qquad \ell_{\mathsf{Cost}} = \mathsf{Lip}(\mathsf{cost})$$

Cumulative error and curve length

$\mathrm{osLip}(\mathsf{F}) = -c < 0$ and x^* is equilibrium



$$\begin{aligned}
\text{curve length}(x_{[0,\infty)}) &= \int_0^\infty \|\mathsf{F}(x(t))\| dt &&\leq \frac{1}{c}\|\mathsf{F}(x_0)\| \\
\text{cumulative error}(x_{[0,\infty)}) &= \int_0^\infty \|x(t) - x^*\| dt &&\leq \frac{1}{c}\|x_0 - x^*\| \\
\text{cumulative cost}(x_{[0,\infty)}) &= \int_0^\infty \mathsf{cost}(x(t)) - \mathsf{cost}(x^*) dt &&\leq \frac{\ell_{\mathsf{Cost}}}{c}\|x_0 - x^*\|
\end{aligned}$$

Outline

Parametric and time-varying convex optimization

$$\min \mathcal{E}(x) \quad \Longleftrightarrow \quad \dot{x} = \mathsf{F}(x) \qquad \rightsquigarrow \quad x^*$$

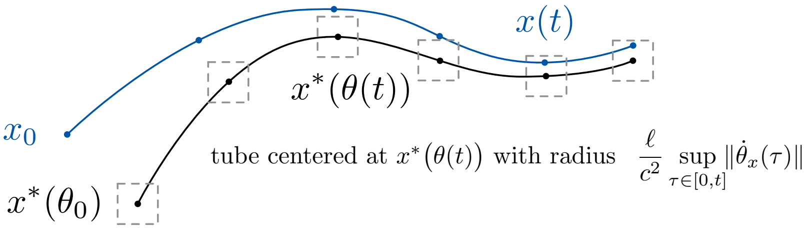
Parametric and time-varying convex optimization

1. **parametric contracting dynamics for parametric convex optimization**

$$\min \mathcal{E}(x,\theta) \quad \Longleftrightarrow \quad \dot{x} = \mathsf{F}(x,\theta) \qquad \rightsquigarrow \quad x^*(\theta)$$

2. **contracting dynamics for time-varying strongly-convex optimization**

$$\min \mathcal{E}\big(x,\theta(t)\big) \quad \Longleftrightarrow \quad \dot{x} = \mathsf{F}\big(x,\theta(t)\big) \qquad \rightsquigarrow \quad x^*\big(\theta(t)\big)$$



$$\dot{x}(t) = \mathsf{F}(x(t), \theta(t))$$
$$x^*(\theta(t)) = \text{equilibrium trajectory}$$

If $\|\dot{\theta}(t)\| \leq \delta$ for all t,

$$x(t) \quad \longrightarrow \quad \text{the tube with center } x^\star(\theta(t)) \text{ and radius } \frac{\ell\delta}{c^2}$$

Equilibrium tracking

For parameter-dependent vector field $\mathsf{F} : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ and differentiable $\theta : \mathbb{R}_{\geq 0} \to \Theta \subset \mathbb{R}^d$

$$\dot{x}(t) = \mathsf{F}(x(t), \theta(t))$$

- **contractivity wrt x**: $\quad \mathsf{osLip}_x(\mathsf{F}) \leq -c < 0$
- **Lipschitz wrt θ**: $\quad \mathsf{Lip}_\theta(\mathsf{F}) \leq \ell$

Theorem: Equilibrium tracking for contracting dynamics.

The equilibrium map $x^\star(\cdot)$ is Lipschitz with constant $\frac{\ell}{c}$ and

$$\text{speed}: \quad \|\mathsf{F}\big(x(t), \theta(t)\big)\| \;\leq\; \mathrm{e}^{-ct}\big\|\mathsf{F}\big(x_0, \theta_0\big)\big\| \;+\; \frac{\ell}{c} \sup_{\tau>0} \|\dot{\theta}(\tau)\|$$

$$\text{error}: \quad \|x(t) - x^\star(\theta(t))\| \;\leq\; \mathrm{e}^{-ct}\big\|x_0 - x^\star\big(\theta_0\big)\big\| \;+\; \frac{\ell}{c^2} \sup_{\tau>0} \|\dot{\theta}(\tau)\|$$

Exact equilibrium tracking with feedforward control

Time-varying contracting dynamics with feedforward prediction

$$\dot{x}(t) \;=\; \mathsf{F}(x(t),\theta(t)) \;-\; \big(D_x\mathsf{F}(x(t),\theta(t))\big)^{-1} D_\theta \mathsf{F}(x(t),\theta(t))\,\dot{\theta}(t)$$

Asymptotically exact equilibrium tracking

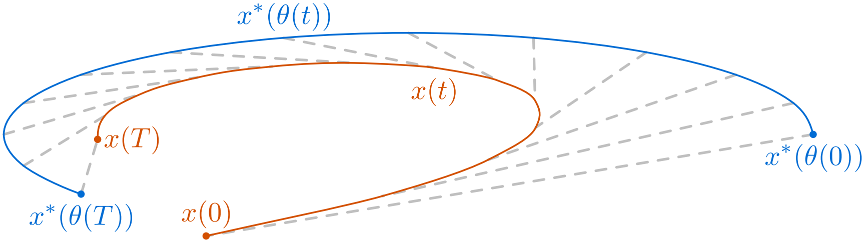
speed : $\big\|\mathsf{F}\big(x(t),\theta(t)\big)\big\| \;\leq\; \mathrm{e}^{-ct}\|\mathsf{F}\big(x_0,\theta_0\big)\|$

error : $\|x(t) - x^\star(\theta(t))\| \;\leq\; \dfrac{1}{c}\mathrm{e}^{-ct}\|\mathsf{F}(x_0,\theta_0)\| \;\overset{\ell_x = \mathsf{Lip}_x(\mathsf{F})}{\leq}\; \dfrac{\ell_x}{c}\mathrm{e}^{-ct}\|x_0 - x^\star(\theta_0)\|$

E.g., if $F = -\nabla_x f$, then $\quad \dot{x} = -\nabla_x f(x,\theta) + \big(\mathrm{Hess}\, f(x,\theta)\big)^{-1} D_\theta \nabla_x f(x,\theta)\,\dot{\theta}$

Outline

Dynamic regret



$$\text{cumulative tracking error}\big(x_{[0,T]}, \theta_{[0,T]}\big) = \int_0^T \big\| x(t) - x^*\big(\theta(t)\big) \big\| dt$$

$$\text{dynamic regret}\big(x_{[0,T]}, \theta_{[0,T]}\big) = \int_0^T \text{cost}\big(x(t), \theta(t)\big) - \text{cost}\big(x^*(\theta(t)), \theta(t)\big) dt$$

Dynamic regret

For parameter-dependent vector field $\mathsf{F} : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ and differentiable $\theta : \mathbb{R}_{\geq 0} \to \Theta \subset \mathbb{R}^d$

$$\dot{x}(t) = \mathsf{F}(x(t), \theta(t))$$

- **contractivity wrt x**: $\quad \mathrm{osLip}_x(\mathsf{F}) \leq -c < 0$
- **Lipschitz wrt θ**: $\quad \mathrm{Lip}_\theta(\mathsf{F}) \leq \ell$

Error and regret estimate

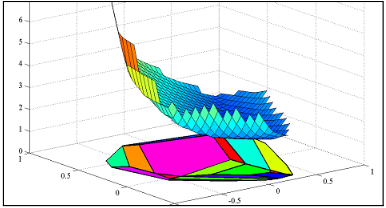
$$\text{cumulative tracking error}\big(x_0, \theta_{[0,T]}\big) \leq \frac{1}{c}\big\|x_0 - x^*(\theta_0)\big\| + \frac{\ell}{c^2}\,\text{curve length}\big(\theta_{[0,T]}\big)$$

$$\begin{aligned} \text{dynamic regret}\big(x_0, \theta_{[0,T]}\big) &\leq \ell_{\text{Cost}}\,\text{cumulative tracking error}\big(x_0, \theta_{[0,T]}\big) \\ &= \mathcal{O}\Big(1 + \text{curve length}\big(\theta_{[0,T]}\big)\Big) \end{aligned}$$

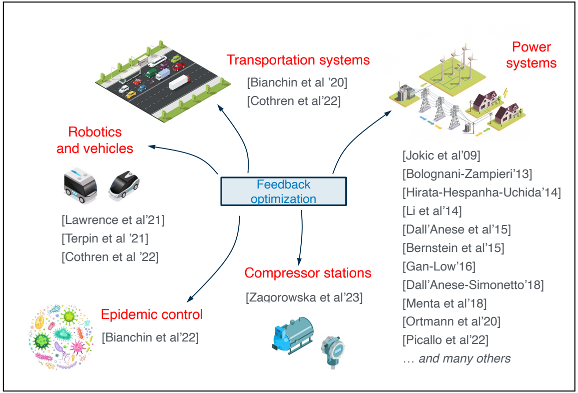
Outline

Motivation: Optimization-based control

1. **online feedback optimization**
2. **control barrier functions**
3. model predictive control
4. distributed optimization
5. imitation learning
6. ...

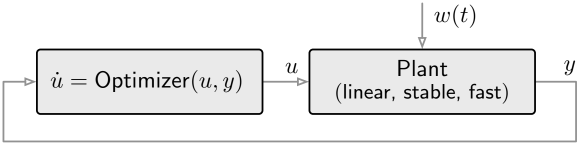


parametric QP. YALMIP + Multi-Parametric Toolbox



Online feedback optimization. Courtesy of Emiliano Dall'Anese.

Application: Online feedback optimization



$$\begin{cases} \min & \text{cost}_1(u) + \text{cost}_2(y) \\ \text{subj. to} & y = \text{Plant}\big(u, w(t)\big) \end{cases} \implies \begin{cases} \dot{u} = \text{Optimizer}(u, y) \\ y = \text{Plant}\big(u, w(t)\big) \end{cases}$$

Example #5: Gradient controller

Online feedback optimization

$$u^*\big(w(t)\big) \;:=\; \underset{u}{\operatorname{argmin}} \;\; \phi(u) + \psi(y(t)) \qquad \text{(c-strongly convex ϕ, convex ψ)}$$

$$\text{subj to} \;\; y(t) = Y_u u + Y_w w(t)$$

gradient controller

$$\dot{u} \;=\; \mathsf{F}_{\mathsf{GradCtrl}}(u, w) := -\nabla_u\big(\phi(u) + \psi(y(t))\big) \;=\; -\nabla\phi(u) - Y_u^\top \nabla\psi(Y_u u + Y_w w)$$

Contractivity of the gradient controller $\implies$ eq. tracking + regret estimate

1. $u(t) \quad \longrightarrow \quad$ tube with center $u^*\big(w(t)\big)$ and radius $\dfrac{\ell_w}{c^2} \sup_{\tau \leq t} \|\dot{w}(\tau)\|$
2. dynamic regret $\leq \dfrac{\ell_x}{c}\big\|u_0 - u^*\big(w_0\big)\big\| + \dfrac{\ell_x \ell_w}{c^2}$curve length$\big(w_{[0,T]}\big)$

Outline

Application to safety filters

Given $\dot{x} = F(x) + G(x)u$ with nominal controller $u_{\text{nom}}(x)$

Safe control design: render forward invariant safe set $\{x \in \mathbb{R}^n \mid h_i(x) \geq 0, \;\; i \in \{1, \dots, k\}\}$

Safety filter (QP with linear inequalities)

$$
\begin{aligned}
u^\star(x) = \operatorname{argmin} \quad & \|u - u_{\text{nom}}(x)\|_2^2 \\
\text{s.t.} \quad & \dot{h}_i(x, u) \geq -\alpha(h_i(x)), \quad i \in \{1, \dots, k\} && \text{(safety constraints)} \\
& \|u\|_\infty \leq \overline{u} && \text{(actuator constraints)}
\end{aligned}
$$

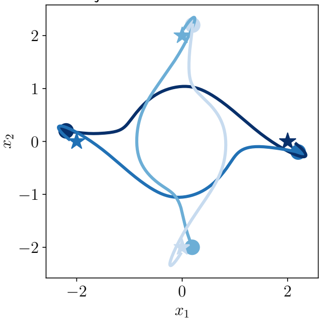
Relax safety constraints to log barriers + **adopt projected gradient dynamics**:

$$
\dot{u}(t) = -u(t) + \operatorname{Proj}_{\|u\|_\infty \leq \overline{u}}\Big(u(t) - \nabla_u \mathcal{E}_\eta\big(u(t), x(t)\big)\Big) + \operatorname{FeedForward}_\eta(u(t), x(t))
$$

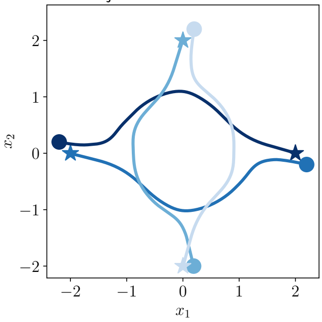
$$
\text{where} \quad \mathcal{E}_\eta(u, x) = \|u - u_{\text{nom}}(x)\|_2^2 - \eta \sum_{i=1}^{k} \log\Big(\nabla h_i(x)^\top \big(F(x) + G(x)u\big) + \alpha\big(h_i(x)\big)\Big)
$$

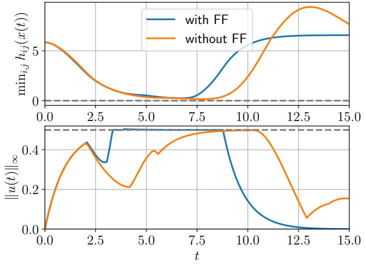
Collision avoidance with 4 robots

$$\dot{u} = \mathsf{F}_{\mathsf{prox}}(u, x)$$

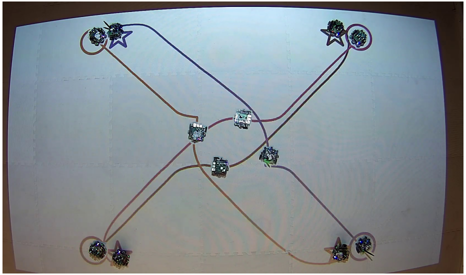


$$\dot{u} = \mathsf{F}_{\mathsf{prox}}(u, x) + \mathsf{FF}(u, x)$$

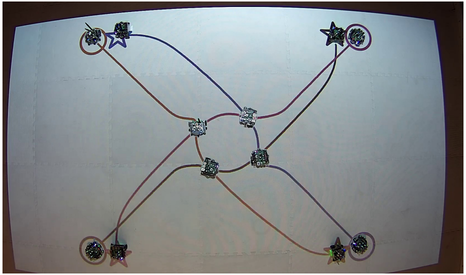


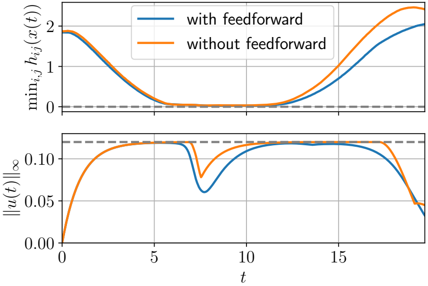


No feedforward



With feedforward





Videos:

- experiment without feedforward
- experiment with feedforward

Code: github link

contracting systems as controllers =
promising approach to optimization-based control

Outline

Conclusions

contractivity = robust computationally-friendly stability

fixed point theory + Lyapunov stability theory + geometry of metric spaces

Ongoing work

1. catalog of contracting dynamics with sharp constants
2. local, weak, k-, and other generalizations of contractivity
3. optimization-based control designs: MPC, CBFs, ...
4. ML and biologically-inspired neural networks

search for contraction properties
design engineering systems to be contracting
verify correct/safe behavior via known Lipschitz constants